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Decoherence in quantum cosmology

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We discuss the manner in which the gravitational field becomes classical in quantum cosmology. This involves two steps. First, one must show that the quantum state of the gravitational field becomes strongly peaked about a set of classical configurations. Second, one must show that the system is in one of a number of states of a relatively permanent nature that have negligible interference with each other. This second step involves decoherence—destruction of the off-diagonal terms in the density matrix, representing interference. To introduce the notion of decoherence, we discuss it in the context of the quantum theory of measurement, following the environment-induced superselection approach of Zurek. We then go on to discuss the application of these ideas to quantum cosmology. We show, in a simple homogeneous isotropic model, that the density matrix of the Universe will decohere if the long-wavelength modes of an inhomogeneous massless scalar field are traced out. These modes effectively act as an environment which continuously "monitors" the scale factor. The coherence width is very small except in the neighborhood of a classical bounce. This means that one cannot really say that a classical solution bounces because the notion of classical spacetime does not apply. The coherence width decreases as the scale factor increases, which has implications for the arrow of time. We also show, using decoherence arguments, that the WKB component of the wave function of the Universe which represents expanding universes has negligible interference with the collapsing component. This justifies the usual assumption that they may be treated separately.

I. INTRODUCTION

It is one of the undeniable facts of our experience that the world about us is described by classical laws to a very high degree of accuracy. In classical mechanics, a system may be assigned a quite definite state and its evolution is described in a deterministic manner—given the state of the system at a particular time one can predict its state at later time with certainty. And yet, it is believed that the world is fundamentally quantum mechanical in nature. Phenomena on all scales up to and including the entire Universe are supposedly described by quantum mechanics. In quantum mechanics, because superpositions of interfering states are permissable, it is generally not possible to say that a system is in a definite state. Moreover, evolution is not deterministic but probabilistic—given the state of the system at a particular time, one can calculate only the probability of finding it in another state at a later time.

If quantum theory is to be reconciled with our classical experience, it is clearly essential to understand the sense in which, and the extent to which, quantum mechanics reproduces the effects of classical mechanics. This is an issue that assumes particular importance in the quantum theory of measurement.[1] There, one describes the measuring apparatus in quantum-mechanical terms; yet all such apparata behave in a distinctly classical manner when the experimenters eye reads the meter.

Early Universe cosmology provides another class of situations in which the emergence of classical behavior from quantum mechanics is a process of particular interest. In the inflationary universe scenario, for example, the classical density fluctuations required for galaxy formation supposedly originate in the quantum fluctuations of a scalar field, hugely amplified by inflation.[2] This is, in a sense, an extreme example of a quantum measurement process, in that the large-scale structure of the Universe we see today is a meter which has permanently recorded the quantum state of the scalar field at early times. The manner in which this quantum to classical transition comes about has been discussed by numerous authors.[3,4] A more fundamental situation of interest, and the one with which this paper is primarily concerned, is quantum cosmology, in which one attempts to apply quantum mechanics to closed cosmologies. Since this involves quantizing the gravitational field, one of the goals of this endeavor should surely be to predict the conditions under which the gravitational field may be regarded as classical.

The point of view we will take is that there are at least two requirements that must be satisfied before a system may be regarded as classical. The first requirement is that it must be possible to say that the system is in one of a number of definite states, where by definite we mean that the states are of a relatively permanent nature and that the interference between different states is exceedingly small. This involves the notion of *decoherence*—destruction of the off-diagonal terms in the density matrix, which represent interference terms. Note that this does not preclude the possibility that our knowledge of the system's state is not precise—we may have only probabilistic information, as in classical statistical mechanics. In fact, this is generally the case. The second requirement is that evolution should to a very good approximation be described by classical laws. This means that the wave function, or some distribution constructed from it, should be strongly peaked about a classical configuration, or a set of such configurations. One can show, in fact, that certain wave functions that are frequently described as "classical" exhibit a strong correlation between coordinates and momenta. For want of a concise word, and to draw the analogy with the quantum theory of measurement discussed below, we shall refer to this second requirement as *correlation.*

Certain previous discussions on interpretational issues in quantum cosmology, including that of the present author, have been lacking in that they consider only the second requirement, and possibly do not even recognize the need for the first.[5-7] It is the purpose of this paper to concentrate on the first requirement: that is, to discuss decoherence of spacetime in the context of quantum cosmology.

The quantum theory of measurement is a well-studied area in which, as noted above, the emergence of classical behavior from quantum mechanics is very important. We therefore begin, in Sec. II, by discussing it in some detail to introduce the notion of decoherence, following primarily the approach of Zurek.[8] The basic ideas are outlined here. As a result of interacting with the system it is measuring, the states of the measuring apparatus become correlated with the states of the system being measured. The apparatus-system pure-state density matrix, however, contains nonzero off-diagonal terms which represent interference between different possible outcomes of the measurement. It is only when these off-diagonal terms become small that one can say that the measuring apparatus has recorded a definite outcome. However, this means that the pure-state density matrix must evolve into a mixed-state diagonal density matrix, which cannot be achieved by unitary evolution. The resolution of this difficulty comes from the realization that the apparatus and system must necessarily be in interaction with the rest of the Universe, summarily referred to as "the environment." If one includes the state of the environment in the initial pure-state density matrix, then one finds that the reduced density matrix, obtained by tracing out the environment states, *can* evolve nonunitarily, taking an initial pure state to a final mixed state. The fact that the apparatus appears to be in a definite state, therefore, may be thought of as a consequence of the fact that it is continuously "monitored" by the environment. There is a further issue concerning the basis in which the density matrix becomes diagonal. That is, what are the states of a relatively permanent nature that the apparatus ends up in? Again the environment determines this—they are the states that are not disturbed by the interaction of the apparatus with the environment; that is, they are eigenstates of the operators which commute with the apparatus-environment interaction Hamiltonian.

In Sec. III we discuss the application of these ideas to quantum cosmology, drawing analogies with the quantum theory of measurement. Solutions to the Wheeler-DeWitt equation, the analogue of the Schrödinger equation, are typically oscillatory in certain regions of interest, of the form e^{iS}, where S is a solution to the Hamilton-Jacobi equation for the system. Such wave functions exhibit a strong correlation between coordinates and momenta of the form $p=\nabla S$. This relation between p and q defines a first integral to the field equations, and thus the wave function may be regarded as being peaked about a set of classical solutions. This however, is just one of the two requirements for the system to be classical: namely, correlation. We still need decoherence. To achieve this, we need to introduce an environment. Since the entire Universe is, by definition, a closed system, it has no external environment. However, one is never interested in observing more than a tiny fraction of the potentially measurable observables. One may, therefore, concentrate on just some of the variables describing the state of the Universe and regard the rest as the environment. For example, one may regard an inhomogeneous matter field as an environment for the homogeneous components of the metric.

A model of precisely this type is described in Sec. IV. We consider the de Sitter minisuperspace model, which is a homogeneous isotropic model described by a single scale factor a, and driven by a cosmological constant. For the environment, we take the inhomogeneous modes of a massless scalar field. We calculate the reduced density matrix obtained by tracing over the scalar field modes. Certain technical difficulties arise because there are an infinite number of such modes. We show, however, that to achieve decoherence, it is sufficient to trace over the long-wavelength modes only, which are finite in number.

In Sec. V we consider the case in which the wave function is a sum of two WKB components, representing a superposition of sets of collapsing and expanding solutions. We show, again using decoherence arguments, that the interference between these components is negligible. This justifies the usual assumption that they may be treated separately. Finally, in Sec. VI we discuss the implications of these considerations for the arrow of time and for classical solutions which undergo a nonsingular bounce. Connections with the work of other authors is also discussed.

II. THE QUANTUM THEORY OF MEASUREMENT AND THE ROLE OF THE ENVIRONMENT

To introduce the notion of decoherence, we begin by discussing it in a familiar context: namely, the quantum theory of measurement.[1] We follow primarily the approach of Zurek.[8]

Consider a system $\mathcal{S}$ described by a set of state vectors $\{|S_n\rangle\}$, in interaction with a measuring apparatus $\mathcal{A}$, with states vectors $\{|A_n\rangle\}$. Let the initial state of the system be a superposition of states with coefficients c_n, and the initial state of the apparatus be $|A_0\rangle$. Then the initial state of the combined system $\mathcal{S}\mathcal{A}$ is

$$|\Psi_i\rangle=\sum_n c_n|S_n\rangle|A_0\rangle \ . \tag{2.1}$$

The "first stage" of the measurement process involves bringing the system and apparatus into interaction, resulting in a correlation between states of the system and states of the apparatus. A typical such interaction will cause the initial state (2.1) to evolve into a final state of the form

$$|\Psi_f\rangle=\sum_n c_n|S_n\rangle|A_n\rangle\ . \tag{2.2}$$

One might wish to interpret (2.2) as indicating that the apparatus state $|A_n\rangle$ has become correlated with the system state $|S_n\rangle$—that the apparatus has "measured" the system and finds it to be in state $|S_n\rangle$ with probability $|c_n|^2$. However, the general formalism of quantum mechanics, which is assumed to apply to the measuring apparatus, allows for an arbitrary change of basis. In particular, one may introduce a new orthonormal basis for the apparatus, defined by

$$|\tilde{A}_m\rangle=\sum_n|A_n\rangle\langle A_n|\tilde{A}_m\rangle\ . \tag{2.3}$$

The final state of the combined system $\mathcal{SA}$ is then given by

$$\begin{aligned}|\Psi_f\rangle&=\sum_n c_n|S_n\rangle\sum_m|\tilde{A}_m\rangle\langle\tilde{A}_m|A_n\rangle\\&=\sum_m\tilde{c}_m|\tilde{S}_m\rangle|\tilde{A}_m\rangle\ ,\end{aligned} \tag{2.4}$$

where $|\tilde{S}_m\rangle$ are the *relative* states, and are defined by (2.4). In this new basis, it appears that the apparatus states $|\tilde{A}_m\rangle$ have become correlated with the system states $|\tilde{S}_m\rangle$—that the measuring apparatus finds itself in one of the states $|\tilde{A}_m\rangle$ and has measured the system to be in the state $|\tilde{S}_m\rangle$. So what has actually been measured? Measuring apparatuses are macroscopic objects which are not observed in superpositions. Clearly only one of the above is correct. But what is it that picks out a particular basis?

The difficulties are further highlighted if one considers the pure-state density matrix corresponding to the final state (2.2):

$$\rho_{\text{pure}}=|\Psi_f\rangle\langle\Psi_f|=\sum_{n,m}c_n c_m^*|S_n\rangle|A_n\rangle\langle A_m|\langle S_m|\ . \tag{2.5}$$

It involves nonzero off-diagonal terms. We are seeking to maintain, however, that the combined system's final state is a definite state in which each system state $|S_n\rangle$ is correlated with the apparatus state $|A_n\rangle$, with probability $|c_n|^2$ of finding the system in state $|S_n\rangle$. Such a situation can only be described by a diagonal mixed-state density matrix of the form

$$\rho_{\text{mixed}}=\sum_n|c_n|^2|S_n\rangle|A_n\rangle\langle A_n|\langle S_n|\ . \tag{2.6}$$

(2.6) and (2.5) differ by the presence of off-diagonal terms in (2.5), which represent interference between the different outcomes of the measurement. *It is only when these interference terms can be neglected that the combined system may be said to be in a definite state.*

There is no way that under unitary Schrödinger evolution the pure-state density matrix (2.5) will evolve into the mixed-state density matrix (2.6). It is for this reason that in conventional approaches to quantum mechanics one invokes the "second stage" of the measurement process: namely, the "collapse of the wave function." This is the process whereby one projects the state vector (2.2), a superposition of states, down onto just one of the states of the superposition. One thus obtains the state $|S_n\rangle|A_n\rangle$ with probability $|c_n|^2$. Since in this final situation the apparatus and system are assumed to have achieved definite, classical states, the ensemble of possibilities may be represented by the diagonal density matrix (2.6). The collapse of the wave function is thus essentially the transition from (2.5) to (2.6).

One can find many reasons for objecting to this process, but consider just two. First, it puts part of the act of measurement outside the realm of quantum mechanics. This is contrary to the hypothesis that quantum mechanics is universally applicable. The difficulties are particularly acute when one attempts to apply quantum mechanics to the entire Universe. Second, it does not single out a preferred basis, that is, it does not tell us what we have measured; yet in a realistic measurement situation, this is surely known. Enumerable papers have been written on this subject, and it is not the purpose of this paper to discuss all of them. There is, however, one particular suggestion, which is perhaps the most compelling, and is readily applicable to quantum cosmology. This is the environment-induced superselection approach pioneered by Zurek.[8]

The key point is that no macroscopic system can realistically be considered as closed and isolated from the rest of the Universe (with the possible exception of the entire Universe itself—see the next section). Laboratory measuring apparatuses interact with surrounding air molecules; even intergalactic gas molecules are not isolated because they interact with the microwave background. Let us refer to the rest of the Universe as "the environment." Then it can be argued that is is the inescapable interaction with the environment which leads to a continuous "measuring" or "monitoring" of a macroscopic system and it is this that causes the wave function to "collapse." More precisely, the environment causes the off-diagonal terms in ρ_{pure} to become exceedingly small, so that it is well-approximated by ρ_{mixed}. This is decoherence.

Let us study this in more detail. Consider once again the combined system $\mathcal{SA}$ considered above, but now take into account also the states $\{|\mathcal{E}_n\rangle\}$ of the environment $\mathcal{E}$. Let the initial state of the total system $\mathcal{SA}\mathcal{E}$ be

$$|\Phi_i\rangle=\sum_n c_n|S_n\rangle|A_0\rangle|\mathcal{E}_0\rangle\ . \tag{2.7}$$

As a result of the measurement interaction, this may evolve into a final state of the form

$$|\Phi_f\rangle=\sum_n c_n|S_n\rangle|A_n\rangle|\mathcal{E}_n\rangle\ . \tag{2.8}$$

In this state, not only have the system and apparatus become correlated with each other, but they have also become correlated with the environment. One is not interested, however, in the state of the environment. This is traced out in the calculation of any quantities of interest. The object of particular relevance, therefore, is the *reduced density matrix,* obtained by tracing over the environment states:

$$\tilde{\rho}=\mathrm{Tr}_{\mathcal{E}}|\Phi_f\rangle\langle\Phi_f|$$
$$=\sum_{n,m}c_n c_m^*\langle\mathcal{E}_m|\mathcal{E}_n\rangle|S_n\rangle|A_n\rangle\langle A_m|\langle S_m| \ . \qquad (2.9)$$

The density matrix $|\Phi_f\rangle\langle\Phi_f|$ of the total system evolves unitarily, of course. The reduced density matrix (2.9), however, does not. It, therefore, holds the possibility of evolving from an initial pure state to the final mixed state described by (2.6). In particular, if, as can be the case, the inner products $\langle\mathcal{E}_m|\mathcal{E}_n\rangle$ are very small when $n\neq m$, then (2.9) will be approximately of the desired form (2.6). If this happens then one may say that the environment has caused the density matrix to decohere. It has caused the system-apparatus combination to end up in one of a set of noninterfering states. We do not necessarily know exactly which of the states $|A_n\rangle$ the apparatus is in—we have only probabilistic information. But these are classical probabilities and the probabilistic element is no more mysterious than that of classical statistical mechanics.

This, however, is only part of the story. There is still the issue of finding the preferred basis. Decoherence occurs generically, in the sense that the entropy associated with the reduced density matrix, $S=-\mathrm{Tr}(\tilde{\rho}\ln\tilde{\rho})$, generally increases. The pertinent question to ask, however, is in which basis does $\tilde{\rho}$ become diagonal? That is, in which basis do the inner products $\langle\mathcal{E}_m|\mathcal{E}_n\rangle$ become very small for $n\neq m$? According to Zurek, this depends quite crucially on the particular form of the interaction between the system and the environment.[8] In particular, one may define a "pointer observable" to be any observable which commutes with the interaction Hamiltonian between the system and the environment. The preferred basis, the "pointer basis," is then the set of eigenstates of the pointer observable. Basically the idea is that if the system is in one of the eigenstates of the pointer observable, it is not disturbed by the interaction with the environment, so it may be regarded as being in a definite state. If, on the other hand, it is not in a pointer observable eigenstate, it may not be regarded as being in a definite state, because its state will change as a result of the interaction with the environment. Moreover, superpositions of pointer basis states will not be observed because, as we have already discussed, the environment has destroyed the interference between different such states—hence, the description "environment-induced superselection" coined by Zurek. We have found, therefore, the states that we called *definite states* in the Introduction—states of a relatively permanent nature that do not interfere. Note that the environment plays two roles in the measurement process: (i) it causes decoherence; (ii) it determines the preferred basis.

In a typical model of the measurement process, the environment will couple to the system through the position of some quantity, q, say. The system-environment interaction Hamiltonian will, therefore, commute with position, and thus the pointer basis is the position basis. The density matrix, therefore, diagonalizes in the position basis. (In fact, this will generally be true in many situations of interest, not just measurement situations. Fields generally couple to each other through their configuration-space coordinates, and not their momenta.) However, to say that something is classical, one would expect not only that the position takes reasonably sharp values, but the momentum also, to a degree consistent with the uncertainty principle. How can this come about? The answer lies in the self-Hamiltonian of the system. Recall that for the system to become classical, in addition to decoherence, one needs a strong correlation between coordinates and momenta q and p. If q becomes reasonably definite, therefore, p will become reasonably definite also, because it is correlated with it. Moreover, the uncertainly principle is not violated because the density matrix does not diagonalize exactly in position. No measurement can fix the position with arbitrary precision. The density matrix diagonalizes in q, but with a nonzero width—it is strongly peaked about $q=q'$ and very small for q far from q'. This nonzero width peak allows p to be reasonably sharp also, to a degree consistent with the uncertainty principle.

Since we have now argued that both q *and* p become reasonably definite, the claim above that the environment determines the preferred basis perhaps seems to be in question. The point is, however, that Zurek's claim is directed at quantum measurement situations in which the system evolves on a time scale much longer than the time scale of the system-environment interaction. The following example will hopefully clarify the situation. Unruh and Zurek considered a model involving a harmonic oscillator in interaction with an environment consisting of a scalar quantum field.[9] The oscillator coupled to the scalar field through the oscillator's position. The reduced density matrix was found to diagonalize in position very rapidly, in accordance with the claim above that position is the preferred basis. However, the subsequent evolution, due to the self-Hamiltonian of the oscillator, caused correlations to develop between position and momentum (on a much longer time scale), leading to the momentum becoming reasonably definite also. So although it is position that is initially measured by the environment, the oscillator Hamiltonian causes a rotation of the state in phase space, and the final preferred basis states are in fact some kind of coherent states.

III. QUANTUM COSMOLOGY

We now apply the ideas introduced in the previous section to quantum cosmology. First of all, a few remarks are in order concerning their applicability. One might get the impression from the way these ideas were presented that they are very much tied to the Hilbert-space structure of quantum mechanics. If this is the case it would represent an immediate difficulty because quantum cosmology is not at present known to possess such a structure—there is no notion of external time, of inner products on a surface of constant time, of normalizable wave functions, etc.[10] We are assuming, however, that the key features described in the previous section transcend the Hilbert-space language we used to describe them. Justification for this assumption could well come from a path-integral treatment, which is thought to apply to situations more general than those for which there exists a Hilbert-space structure.[10] Still, it may be that we are making some kind of logical jump. An example of the sort of problem that might arise is that since the wave functions in quantum cosmology are not normalizable in all of their arguments; the trace operation is difficult to define. We avoid this sort of difficult quite simply by not attempting to calculate quantities which are obviously ill defined.

We now briefly review the formalism of quantum cosmology.[11,12] The quantum state of the Universe is represented by a wave function $\Psi[h_{ij},\Phi]$, a functional on superspace, the space of three-metrics h_{ij}, and matter fields Φ on a three-surface. The wave function satisfies the Wheeler-DeWitt equation

$$(H_g+H_m)\Psi=0 \ , \tag{3.1}$$

where H_g and H_m are, respectively, the gravitational and matter Hamiltonians. Equation (3.1) is functional differential equation on superspace. Because of the difficulty in solving it, one is frequently interested in the minisuperspace approximation, in which one restricts the three-metric and matter fields so that they are described by a finite number of functions $q^{\alpha}(t)$ (Ref. 13). A purely gravitational minisuperspace model would then be described by a Wheeler-DeWitt equation of the form

$$H_g\Psi(q)=[-\tfrac{1}{2}\nabla^2+U(q)]\Psi(q)=0 \ , \tag{3.2}$$

where ∇^2 is the Laplacian on minisuperspace.

The solution to the Wheeler-DeWitt equation may be broadly described as either oscillatory or as exponential, depending on the region of minisuperspace. Whether or not the wave function is exponential or oscillatory in a given region will depend on the boundary conditions. The oscillatory regions are normally regarded as corresponding to classically allowed regions. One can show, in fact, that an oscillatory wave function is strongly peaked about particular classical configurations. This is not true of the exponential regions, which are regarded as classically forbidden. We will have no more to say about the exponential region.

Let us concentrate more on the oscillatory region. One can solve the Wheeler-DeWitt equation in this region using a WKB expansion, in which one writes

$$\Psi=Ce^{iS} \ , \tag{3.3}$$

where S is a rapidly varying phase and C is a slowly varying prefactor. S obeys the Hamilton-Jacobi equation

$$\tfrac{1}{2}(\nabla S)^2+U(q)=0 \tag{3.4}$$

and C satisfies

$$\nabla S\cdot\nabla(\ln C)+\tfrac{1}{2}\nabla^2 S=0 \ . \tag{3.5}$$

It may be shown that a wave function of the form (3.3) is strongly peaked about the region of phase space for which

$$p=\nabla S \ , \tag{3.6}$$

where p_{α} is the conjugate to q^{α} (Ref. 5). Using the fact that S satisfies (3.4), one may show that (3.6) defines a first integral to the classical field equations. It is in this sense that (3.3) is said to be strongly peaked about a set of classical solutions.

It is convenient to introduce the tangent vector to the set of solutions (3.6):

$$\frac{d}{dt}=\nabla S\cdot\nabla \ . \tag{3.7}$$

Here, t is nothing more than an affine parameter which labels the points along the trajectories about which the wave function is peaked. We will, of course, eventually deduce that it is "time." This may not be deduced, however, until it is established that the q^{α}, from which t is constructed, are definite classical quantities, whose different values are noninterfering.

The q^{α} describe a system analogous to the combined system $\mathcal{SA}$ in the previous section, and what we have achieved so far is closely analogous to the first stage of the measurement process—the process of correlation. We have shown that WKB wave functions (3.3) exhibit a strong correlation between certain variables, namely, p and q, through the relation (3.6). As should be clear from the discussion of the previous section, however, this is not sufficient. It is also necessary to demonstrate that the metric, represented by q^{α}, may be regarded as definite—that it is in a state of a relatively permanent nature which does not interfere with other such states.

Let us introduce, therefore, an environment which continuously monitors the metric. As pointed out in the Introduction, the entire Universe cannot by definition have anything external to it. It is, therefore, necessary to regard some of the variables describing the Universe as the system, and the rest as environment. The environment should be some kind of large reservoir into which information about correlations can be dissipated. It should, therefore, have a large number of modes. Since minisuperspace usually involves the homogeneous modes of the fields, the inhomogeneous modes, which have so far been ignored, are a natural candidate for the environment. One can use the inhomogeneous modes of either gravitational or matter fields. Here we shall use matter fields, although the case of a gravitational environment is very similar.[14]

Consider the Wheeler-DeWitt equation (3.1) with three-metric components q^{α} and inhomogeneous matter modes Φ:

$$[-\tfrac{1}{2}\nabla^2+U(q)+H_m(q,\Phi)]\Psi(q,\Phi)=0 \ . \tag{3.8}$$

Some models of this type are described in Refs. 15–17. We will assume that Φ constitutes a small perturbation on the minisuperspace background q^{α}. We, therefore, seek a solution to (3.8) of the form

$$\Psi(q,\Phi)=C(q)e^{iS(q)}\chi(q,\Phi). \tag{3.9}$$

Inserting (3.9) in (3.8), one once again obtains (3.4) and (3.5) for S and C. In addition, one obtains the following equation for χ:

$$i\nabla S\cdot\nabla\chi=H_m\chi \ . \tag{3.10}$$

From (3.7), one may see that (3.10) is a time-dependent Schrödinger equation along the trajectories (3.6) about which the wave function is peaked. Although solutions to the full Wheeler-DeWitt equation (3.8) are generally not normalizable, the solutions of interest to (3.10) usually are, and one may introduce an inner product

$$(\chi_1,\chi_2)=\int\mathcal{D}\Phi\,\chi_1^*(t,\Phi)\chi_2(t,\Phi) \ . \tag{3.11}$$

Equations (3.10) and (3.11) describe the familiar quantum field theory in curved spacetime for Φ, in the functional Schrödinger picture.[18]

Given the solution (3.9), we may proceed to construct the reduced density matrix obtained by tracing out the environment Φ:

$$\tilde{\rho}(q,q')=\int\mathcal{D}\Phi\,\rho(q,\Phi,q',\Phi) \ , \tag{3.12}$$

where ρ is the pure state density matrix for the total system, obtained from the wave function (3.9) (there is also the possibility that ρ may be a mixed state, but we do not consider that here[19]). There is, however, some ambiguity in how ρ should be defined. This is because there is some disagreement about the probability measure one should use in interpreting the wave function. Hawking and Page, for example, advocate that probabilities should be proportional to $|\Psi|^2dV$, where dV is the volume element on superspace.[20] Since the diagonal elements of a density matrix are probabilities, this corresponds to taking the density matrix to be

$$\rho(q,\Phi,q',\Phi')=\Psi^*(q,\Phi)\Psi(q',\Phi') \ . \tag{3.13}$$

This definition of the probability measure is analogous to that used for systems described by a time-dependent Schrödinger equation. The Wheeler-DeWitt equation, however, has the form of a Klein-Gordon equation, for which the appropriate probability measure is the "timelike" component of the conserved current:

$$J=i(\Psi\nabla\Psi^*-\Psi^*\nabla\Psi) \ . \tag{3.14}$$

This point of view has been advocated primarily by Vilenkin[21] (see also Ref. 22). It is not immediately obvious how one should construct a density matrix which reduces to this latter probability measure on the diagonal. One could perhaps consider a point-split version of (3.14). Since J is a vector, however, one would expect to encounter difficulties similar to those encountered when generalizing the Wigner function to curved backgrounds.[23] This technical problem is probably not insurmountable, but will not be pursued in this paper.

Because of the difficulty with the density matrix associated with the Klein-Gordon current, we will use the density matrix (3.13). In doing so, however, we do not wish to give the impression that we necessarily favor the probability measure $|\Psi|^2$ in preference to that constructed from (3.14). This choice is merely for mathematical convenience.

In terms of (3.9), the reduced density matrix is

$$\tilde{\rho}(q,q')=C^*(q)C(q')e^{-iS(q)+iS(q')}\times\int\mathcal{D}\Phi\,\chi^*(q,\Phi)\chi(q',\Phi) \ . \tag{3.15}$$

The trace over Φ can be made well defined because the wave functions $\chi(q,\Phi)$ are normalizable. The fact that Φ may involve an infinite number of modes will cause problems, but this is not an obstacle of principle, and is discussed in the next section. Note, however, that the trace over the metric components q is probably not well defined, because the wave function is not normalizable in q. We will not attempt to trace over q at any stage.

Given (3.15) the object is to show that the $\mathrm{Tr}\chi^*\chi$ term is very small if $q'\neq q$, and thus that the reduced density matrix decoheres. This would mean that there is no interference between different values of q. One should ask, however, whether this is the correct basis—that is, is q the variable in which one would expect the density matrix to decohere? The interaction Hamiltonian between the three-metric and its environment is the matter Hamiltonian, $H_m(q,\Phi)$. This is a function of q but not its conjugate; thus q commutes with $H_m(q,\Phi)$ and is, therefore, the "pointer observable." This means that the "initial" tendency is for the density matrix to diagonalize in q. The word initial is in quotes because it implies time evolution of the quantum state which does not exist in quantum cosmology, because there is no external time. Time is already in there among the metric components q^α. This makes it difficult to separate the processes of decoherence and correlation, as one can in ordinary quantum mechanics using the fact that they occur on different time scales, as described at the end of Sec. II. All that one can say here is that these processes occur in certain regions of minisuperspace. Again the fact that the density matrix does not diagonalize exactly in q means that p may become reasonably sharp also; and again one would expect the "final" basis states to be the analogue of coherent states, but this is probably much more difficult to justify explicitly in quantum cosmololgy, mainly because of the difficulty of constructing the analogue of coherent states. We hope to give a more satisfactory discussion of this awkward point in a future publication.

One can go further using the approach described here. Solutions to the Wheeler-DeWitt equation are not generally of the precise form (3.3) but are rather a sum of WKB components:

$$\Psi=\sum_k C_k e^{iS_k} \ , \tag{3.16}$$

where S_k and C_k are solutions to (3.4) and (3.5). A particular example is the Hartle-Hawking wave function which is claimed to be real:[24]

$$\Psi_{\mathrm{HH}}=Ce^{iS}+C^*e^{-iS} \ . \tag{3.17}$$

This wave function corresponds to a superposition of expanding and collapsing universes which are clearly quite distinct macroscopic states. In any interpretation of the wave function, it is always assumed that distinct WKB components may be treated separately; that is, it is assumed that they do not interfere. One would hope to show this explicitly using decoherence arguments. One thus considers solutions to (3.9) of the form

$$\Psi(q,\Phi)=\sum_k C_k(q)e^{iS_k(q)}\chi_k(q,\Phi) , \tag{3.18}$$

where χ_k is a solution of the Schrödinger equation (3.10) with Hamilton-Jacobi function S_k. The reduced density matrix then takes the form

$$\tilde{\rho}(q,q')=\sum_{k,j} C_k^*(q)C_j(q')e^{-iS_k(q)+iS_j(q')} \times \int \mathcal{D}\Phi \chi_k^*(q,\Phi)\chi_j(q',\Phi) . \tag{3.19}$$

The object here is to show not only that the $\mathrm{Tr}\chi_k^*\chi_j$ term is very small for $q\neq q'$ when $k=j$, but also that it is very small when $k\neq j$ for all q,q', including $q=q'$. If this is the case, then one might say that there is a superselection rule which forbids superpositions of distinct WKB wave functions.

It is notoriously difficult to offer general arguments concerning decoherence in quantum theory and the two examples above are no exception. However, it may be demonstrated explicitly in a particular model, and this is the subject of the next section.

IV. THE DE SITTER MINISUPERSPACE MODEL

To see how the density matrix diagonalizes, and the extent to which this occurs, it is of interest to study a particular model. The model we consider is the de Sitter minisuperspace model.[24,25] It is described by the metric

$$ds^2=l^2[-N^2dt^2+a^2(t)d\Omega_3^2] , \tag{4.1}$$

where $l^2=2/3\pi m_p^2$ and $d\Omega_3^2$ is the metric on the unit three-sphere. The Einstein-Hilbert action with cosmological constant for this metric is

$$I=\tfrac{1}{2}\int dt\, N\left[-\frac{a\dot{a}^2}{N^2}+a-H^2a^3\right] , \tag{4.2}$$

where $H^2=\Lambda/3$. The associated Wheeler-DeWitt equation is

$$H_g\Psi(a)=\frac{1}{2a}\left[\frac{\partial^2}{\partial a^2}+H^2a^4-a^2\right]\Psi(a)=0 . \tag{4.3}$$

The WKB approximation in the oscillatory region is valid for $Ha\gg 1$. The two linearly independent WKB solutions are of the form $e^{\pm iS}$, where

$$S=\frac{1}{3H^2}(H^2a^2-1)^{3/2} . \tag{4.4}$$

The wave function is, therefore, peaked about solutions to the field equations satisfying the first integral

$$-a\dot{a}\equiv p_a=\pm\frac{\partial S}{\partial a}=\pm a(H^2a^2-1)^{1/2} . \tag{4.5}$$

The solutions are expanding or collapsing components of de Sitter space $a(t)=H^{-1}\cosh(Ht)$. For the moment we will concentrate on the solution e^{-iS} which corresponds to the expanding half.

For the environment we will take certain modes of a massless minimally coupled inhomogeneous scalar field $\Phi(\mathbf{x},t)$ (we will be more precise below). The scalar field is most conveniently handled by expanding the spatial dependences in harmonics on the three-sphere. One thus writes

$$\Phi(\mathbf{x},t)=\sum_{nlm} f_{nlm}(t)Q_{lm}^n(\mathbf{x}) . \tag{4.6}$$

Here, the $Q_{lm}^n(\mathbf{x})$ are eigenfunctions of the Laplacian on the three-sphere with eigenvalue $-(n^2-1)$. In addition to the eigenvalue label n, the eigenfunctions are labeled by degeneracy labels lm which run over n^2 values. The homogeneous mode $n=1$ is problematic in that it is not normalizable, and will, therefore, be excluded from our considerations.

Following Ref. 15, for example, one may show that the Hamiltonian for Φ is

$$H_m=\tfrac{1}{2}a^{-3}\sum_{nlm}\left[-\frac{\partial^2}{\partial f_{nlm}^2}+(n^2-1)a^4f_{nlm}^2\right] . \tag{4.7}$$

The total Wheeler-DeWitt equation for the model is now

$$(H_g+H_m)\Psi(a,\Phi)=0 . \tag{4.8}$$

One might have thought that the scalar field modes would excite metric fluctuations, which may not consistently be neglected. We are assuming, however, that all the scalar modes are small, and that we are working to quadratic order in inhomogeneities in the Hamiltonian. At this level of approximation, the metric fluctuations do not couple to the scalar field and may, therefore, be neglected.[15]

Equation (4.8) may be solved by writing

$$\Psi(a,\Phi)=C(a)e^{-iS(a)}\chi(a,\Phi) , \tag{4.9}$$

where $S(a)$ is given by (4.4) and χ satisfies the time-dependent Schrödinger equation

$$H_m\chi=i\frac{\partial\chi}{\partial t} . \tag{4.10}$$

t is defined in terms of a through (4.5). We will use t and a interchangeably. Since the different modes do not interact, (4.10) may be solved by writing

$$\chi(a,\Phi)=\prod_{nlm}\chi_{nlm}(t,f_{nlm}) . \tag{4.11}$$

Each wave function χ_{nlm} thus satisfies the Schrödinger equation

$$\tfrac{1}{2}a^{-3}\left[-\frac{\partial^2}{\partial f_{nlm}^2}+(n^2-1)a^4f_{nlm}\right]\chi_{nlm}=i\frac{\partial\chi_{nlm}}{\partial t} . \tag{4.12}$$

We will look for solutions of the form

$$\chi_{nlm}=A_n(t)\exp[-B_n(t)f_{nlm}^2] . \tag{4.13}$$

$\mathrm{Re}(B_n)$ must be positive for (4.13) to be normalizable. Substituting (4.13) in the Schrödinger equation one finds that A_n and B_n satisfy the equations

$$ia^3\dot{A}_n = B_n A_n \ , \tag{4.14}$$

$$ia^3\dot{B}_n = -\tfrac{1}{2}(n^2-1)a^4 + 2B_n^2 \ . \tag{4.15}$$

The normalization condition $(\chi_{nlm}, \chi_{nlm}) = 1$ fixes A_n up to a phase:

$$A_n = \pi^{-1/4}(B_n + B_n^*)^{1/4} e^{i\alpha_n(t)} \ . \tag{4.16}$$

The phase α_n is then determined by (4.14).

The general solution to (4.15) may be given in terms of conformal time η, where $dt = a\,d\eta$ and $a(\eta) = (H\cos\eta)^{-1}$. It may be written

$$B_n = -\frac{ia^2}{2}\left[\frac{1}{v_n}\frac{dv_n}{d\eta} - \frac{1}{a}\frac{da}{d\eta}\right] , \tag{4.17}$$

where $v_n(\eta)$ is given by

$$v_n(\eta) = P_n e^{in\eta}\left[1 - \frac{i}{n}\tan\eta\right] + Q_n e^{-in\eta}\left[1 + \frac{i}{n}\tan\eta\right] \ . \tag{4.18}$$

P_n and Q_n are arbitrary constants. B_n depends only on the ratio Q_n/P_n. Choice of this ratio corresponds to choosing a particular vacuum state for the field. For simplicity we make the choice $Q_n = 0$, although it would be of interest to study other choices also (see Sec. VI). With this choice, B_n may be written

$$B_n = \tfrac{1}{2}(n^2-1)a^2 \frac{n + i(H^2a^2-1)^{1/2}}{n^2 + H^2a^2 - 1} \ . \tag{4.19}$$

We are interested in the reduced density matrix calculated using (4.9). This will be proportional to a product over nlm of terms of the form

$$\int df_{nlm}\, \chi^*(a, f_{nlm})\chi(a', f_{nlm}) \equiv \tilde{\rho}_{nlm}(a, a') \ . \tag{4.20}$$

Inserting (4.13) and (4.16), this yields

$$\tilde{\rho}_{nlm} = e^{-i\alpha_n(a) + i\alpha_n(a')} \times \left[\frac{[B_n(a) + B_n^*(a)][B_n(a') + B_n^*(a')]}{[B_n^*(a) + B_n(a')]^2}\right]^{1/4} . \tag{4.21}$$

Inserting the expression for B_n, (4.19), one may show, at some length, that (4.21) reduces to

$$\tilde{\rho}_{nlm} = \left[1 + \frac{(a^2 - a'^2)^2}{4a^2a'^2} + \frac{[a^2(H^2a'^2-1)^{1/2} - a'^2(H^2a^2-1)^{1/2}]^2}{4n^2a^2a'^2}\right]^{-1/4} , \tag{4.22}$$

where in (4.22), we have ignored phases.

Let us now be more precise about what we are taking to be the environment. One could take the full infinite number of modes of the scalar field as the environment, in which case the reduced density matrix is of the form (3.15), with q representing a. The trace part is then given by

$$\int \mathcal{D}\Phi\, \chi^*(a, \Phi)\chi(a', \Phi) = \prod_{n=2}^{\infty}\prod_{lm} \tilde{\rho}_{nlm} \ . \tag{4.23}$$

Since $|\tilde{\rho}_{nlm}| \le 1$ with equality only when $a = a'$, it is easy to see that (4.23) is equal to one if $a = a'$ and is equal to zero if $a \ne a'$. This means that the reduced density matrix has diagonalized perfectly. However, this is rather unsatisfactory, as should be clear from the discussion in Sec. II. It means that the scale factor has been perfectly measured, which is clearly unrealistic. It also means that the momentum conjugate to a must have an infinite spread, so the system cannot really be said to be classical. This has happened because due attention has not been paid to the field-theory aspects of the problem—to the fact that we have an infinite number of modes, and need to introduce a regularization scheme. The most naive way of regularizing is to introduce a cutoff in (4.23) at some very large value of n, $n = N$, say. The product will then be dominated by the very-short-wavelength modes, and a straightforward calculation yields

$$\prod_{n=2}^{N}\prod_{lm} \tilde{\rho}_{nlm} \approx \exp(-\tfrac{1}{3}\lambda^{-3}\bar{a}\Delta^2) \ , \tag{4.24}$$

where $\bar{a} = (a + a')/2$, $\Delta = |a - a'|/2$, and $\lambda = \bar{a}/N$ is the wavelength of the cutoff. Equation (4.24) is a Gaussian of width $\sigma \approx \lambda^{3/2}\bar{a}^{-1/2}$. Not surprisingly it depends on the cutoff. A reasonable value to choose for λ might be the Planck length ($\lambda = 1$ in our units), in which case the coherence width σ is much less than $\bar{a}$ when $\bar{a}$ is much greater than the Planck length, giving very effective decoherence. This should not be regarded as an approximation to the case of an infinite number of modes, however, because such noncovariant regularization schemes are well known to give misleading or incorrect results.[17] A more sophisticated approach is to use covariant field-theory methods and these yield peaks of nonzero width about the diagonal, independent of any cutoff.[26]

The approach that we will take here, however, is to note that there is a natural cutoff in the sum over modes at a much more modest scale than the Planck scale. Each mode has a wavelength a/n. There are a finite number of modes whose wavelength is outside the horizon, i.e., $a/n > 1/H$. Since these will be unobservable, it is natural to try tracing over these modes only. Consider, therefore, the quantity of the form (4.23), but with the product over n not from $n=2$ to $n=\infty$, but from $n=2$ to a number of order $H\bar{a}$. With this choice of environment, the reduced density matrix is

$$\tilde{\rho}(a,\Phi_s,a',\Phi_s')=e^{+iS(a)-iS(a')}\chi_s^*(a,\Phi_s)\times\chi_s(a',\Phi_s')F_{11}(a,a') , \qquad (4.25)$$

where χ_s is the wave function of the short-wavelength part of the scalar field Φ_s, with Φ_s defined to be (4.6) with n summed from $H\bar{a}$ to infinity. F_{11} is the trace over the long-wavelength modes Φ_l, with Φ_l defined to be (4.6) summed from $n=2$ to $H\bar{a}$. F_{11} is given by

$$\begin{aligned}F_{11}(a,a')&=\int d\Phi_l\chi_l^*(a,\Phi_l)\chi_1(a',\Phi_l)\\&=\prod_{n=2}^{H\bar{a}}\prod_{lm}\tilde{\rho}_{nlm}\\&\approx\prod_{n=2}^{H\bar{a}}\left[1+\frac{(a^2-a'^2)^2}{4a^2a'^2}+\frac{H^2(a-a')^2}{4n^2}\right]^{-n^2/4} ,\end{aligned} \qquad (4.26)$$

$$F_{11}\approx\exp\left[-\frac{1}{12}H^3\bar{a}^3\left[\ln\left[\frac{\bar{a}^2+\Delta^2}{\bar{a}^2}-\Delta^2\right]+\text{const}\right]\right] . \qquad (4.27)$$

In the region $0<\Delta\ll\bar{a}$, F_{11} has the form

$$F_{11}\approx\exp(-\tfrac{7}{12}H^3\bar{a}\Delta^2) . \qquad (4.28)$$

We see from the above that near the diagonal, the reduced density matrix has a Gaussian peak of width σ, with $\sigma/\bar{a}\approx(H\bar{a})^{-3/2}$. Away from the diagonal, it decays more effectively than a Gaussian. The off-diagonal terms will be exponentially smaller than the diagonal terms for $H\bar{a}\gg1$. The reduced density matrix, therefore, decoheres in this region, and the scale factor becomes reasonably sharp. Since we have already shown that there is a strong correlation between p_a and a in this region, we may conclude that p_a becomes sharp also. We may now safely conclude that the scale factor behaves classically in the region $H\bar{a}\gg1$.

Note that the density matrix does not diagonalize in Φ_s. This is because our chosen environment does not interact with the short-wavelength modes of the scalar field. These modes, therefore, remain quantum mechanical in this model.

V. THE INTEFERENCE BETWEEN EXPANDING AND COLLAPSING COMPONENTS

We have shown that the scale factor decoheres in a simple model for the case in which the wave function consists of just one WKB component (4.9). As discussed in Sec. III, however, a typical solution to the Wheeler-DeWitt equation will be a sum of such components. We would like to show that the interference between such components is very small. This will justify considering each component separately, which is what one normally does. To study this problem in generality, one would need the general solution S_k to the Hamilton-Jacobi equation and then needs to solve the Schrödinger equation along the trajectories to which each solution corresponds. This would be rather difficult. Here we will study just one particularly simple case.

Consider the model of the previous section. Since the Wheeler-DeWitt operator is real, if the WKB wave function (4.9) is a solution, then its complex conjugate is a solution also. We will, therefore, study the real solution

$$\Psi(a,\Phi)=C(a)e^{-iS(a)}\chi(a,\Phi)+C^*(a)e^{+iS(a)}\chi^*(a,\Phi) . \qquad (5.1)$$

The Hartle-Hawking proposal supposedly picks out a wave function of this form (see, however, Ref. 25). The first term represents an ensemble of expanding universes and the second, a set of collapsing ones. We seek to show that these components do not interfere. We do this by repeating the calculation of the previous section for the wave function (5.1).

It is not difficult to show that the reduced density matrix is a sum of four terms. It is

$$\begin{aligned}\tilde{\rho}(a,\Phi_s,a',\Phi_s')={}&e^{+iS(a)-iS(a')}\chi_s^*(a,\Phi_s)\chi_s(a',\Phi_s')F_{11}(a,a')+e^{+iS(a)+iS(a')}\chi_s^*(a,\Phi_s)\chi_s^*(a',\Phi_s')F_{21}(a,a')\\&+e^{-iS(a)-iS(a')}\chi_s(a,\Phi_s)\chi_s(a',\Phi_s')F_{12}(a,a')+e^{-iS(a)+iS(a')}\chi_s(a,\Phi_s)\chi_s^*(a',\Phi_s')F_{22}(a,a') ,\end{aligned} \qquad (5.2)$$

where F_{11} $(=F_{22}^*)$ is given by (4.26), and F_{12} $(=F_{21}^*)$ is given by

$$F_{12}(a,a')=\int d\Phi_l\chi_l(a,\Phi_l)\chi_l(a',\Phi_l) . \qquad (5.3)$$

F_{12} is the object of interest, because it represents the interference between expanding and collapsing components of the wave function.

Following the details of Sec. IV, one may show that

$$F_{12}(a,a')=\prod_{n=2}^{Ha}\prod_{lm}e^{+i\alpha_n(a)+i\alpha_n(a')}\left[\frac{[B_n(a)+B_n(a)][B_n(a')+B_n(a')]}{[B_n(a)+B_n(a')]^2}\right]^{1/4} . \qquad (5.4)$$

Inserting the explicit form for $B_n(a)$, one obtains

$$F_{12}(a,a')=\prod_{n=2}^{Ha}\prod_{lm}\left[1+\frac{(a^2-a'^2)^2}{4a^2a'^2}+\frac{[a^2(H^2a'^2-1)^{1/2}+a'^2(H^2a^2-1)^{1/2}]^2}{4n^2a^2a'^2}\right]^{-1/4}, \tag{5.5}$$

where we have ignored phases. Each term in the product differs from (4.22) in that the minus sign between the two square-root terms in (4.22) has become a plus sign in (5.5). This has the crucial consequence that, in the region $a,a'\gg H^{-1}$, each term in the product in (5.5) is strictly less than one, *even when* $a=a'$. This means that the interference terms $F_{12}(a,a'),F_{21}(a,a')$ are smaller than the diagonal terms $F_{11}(a,a),F_{22}(a,a)$. In particular, one finds that

$$F_{12}\approx\exp\left\{-\frac{1}{12}H^3\bar{a}^3\left[\ln\left[\frac{\bar{a}^2+\Delta^2}{\bar{a}^2-\Delta^2}\right]+\text{const}\right]\right\}. \tag{5.6}$$

This expression is valid for all ranges of Δ, including $\Delta=0$. The cross terms are, therefore, exponentially smaller than the diagonal terms. One may, therefore, ignore the interference between expanding and collapsing parts of the wave function.

VI. DISCUSSION

We have discussed the essential requirements for the emergence of classical behavior in quantum cosmology. These are correlation and decoherence, in close analogy with the quantum theory of measurement. We first discussed how the WKB wave function predicts a strong correlation between p_a and a, showing that the wave function is strongly peaked about sets of classical solutions. It was then shown, in a simple model, that the long-wavelength modes of a massless scalar field provide an environment which destroys the quantum-mechanical interference between different values of the scale factor. This environment also destroys the interference between the expanding and collapsing WKB components of the wave function. We again emphasize that both correlation *and* decoherence are necessary before one can say that a system is classical. Many previous discussions of the emergence of classical behavior from quantum theory in a cosmological setting concentrated only on the first element—correlation.[3-7] This has been rightly criticized by Unruh and Zurek.[9]

To what extent does the way in which gravity becomes classical depend on cosmological boundary conditions? Boundary conditions enter in two ways: (i) they enter in the solution to the background Wheeler-DeWitt equation, determining the regions in which the solution is oscillatory or exponential, and thus determine the extent to which there is a correlation between p and q; (ii) they enter in the solution to the Schrödinger equation for the quantum state of the environment, which in turn determines the coherence width of the density matrix. It is possible to investigate this second point in greater detail.

In Sec. IV we calculated the reduced density matrix after having made a particular choice for the quantum state of the environment. In particular, we took $Q_n=0$ in (4.18). This is obviously not the only choice. One could calculate the reduced density matrix with arbitrary Q_n. However, the resulting algebra turns out to be rather messy, and it is easier to do the following. The reduced density matrix is proportional to a product of terms of the form (4.21). Close to the diagonal,

$$B_n(a')\approx B_n(a)+(a-a')\frac{dB_n}{da}(a),$$

and the derivative with respect to a may be calculated using (4.15). Writing $B=B_1+iB_2$ (dropping the lable n), where B_1 and B_2 are real, one finds that, close to the diagonal, (4.21) is given by

$$\rho_{nlm}\approx\left\{1+\frac{(a-a')^2}{4H^4B_1^2}\left[\frac{16B_1^2B_2^2}{a^8}+\left[\frac{n^2-1}{2}-\frac{2(B_1^2-B_2^2)}{a^4}\right]^2\right]\right\}^{-1/4}. \tag{6.1}$$

The point now is that B_1 and B_2 are essentially arbitrary. The solutions to the equation for B, (4.15), are parametrized by a single complex parameter, which may be taken to be the value of B at a particular value of a. From (6.1) it is easily seen that the coherence width of the reduced density matrix will depend quite crucially on B_1 and B_2; thus the extent to which the scale factor becomes classical may depend quite crucially on the boundary conditions. This conclusion is in consonance with the suggestion of Gell-Mann, Hartle, and Telegdi that the fact that the Universe is described so well by classical laws is a consequence of a law of initial conditions.[27]

We showed in Sec. IV that the reduced density matrix has a coherence width $\sigma\approx\bar{a}(H\bar{a})^{-3/2}$. This width is much less than the scale factor when $\bar{a}\gg H^{-1}$. When $\bar{a}$ is close to H^{-1}, however, the width will not be small and there will be interference between different values of the scale factor. Now the classical solutions about which the wave function is peaked are de Sitter space, which has a bounce at $a = H^{-1}$. The point to be made, however, is that one cannot say whether or not a classical bounce occurs, because the scale factor may not be regarded as classical in the neighborhood of H^{-1}. This is perhaps not a surprising conclusion, and possibly not new—one might well have guessed it from looking at the wave function alone. However, one can be more confident in drawing this conclusion from the behavior of the reduced density matrix. Note that it depends on the fact that we have taken only the long-wavelength modes to be the environment. Including some shorter-wavelength modes would have localized a to a much smaller width.

There is another feature of interest concerning the coherence width. This is that it decreases as a increases. This has possible implications for the thermodynamic arrow of time. One might wish to associate entropy with the degree of coherence of the scale factor. In particular, one could associate increasing entropy, and hence the thermodynamic arrow of time, with the direction of decreasing coherence. This means that entropy increases when the Universe expands and decreases when the Universe contracts, in this model. This idea was first discussed in the context of quantum cosmology by Hawking.[28] Hawking claimed that the wave function defined by the Hartle-Hawking proposal is peaked about sets of classical solutions which have the property that entropy decreases during the collapsing phase. It was pointed out by Page, however, that this claim relies on the assumption that the classical trajectories about which the wave function is peaked are time symmetric.[29] While the set of solutions is time symmetric, most individual members of the set are not. Page's objection does not obviously apply here in that the classical trajectories are de Sitter space, which are time symmetric. Moreover, Hawking and Page were concerned with models which have an expanding phase followed by a collapsing phase, which is not the case here. Hawking's suggestion may, therefore, make sense in the simple model considered here. In their original discussions of the arrow of time in quantum cosmology, Hawking and Page discussed entropy in a very heuristic manner—they did not discuss the reduced density matrix. The ideas concerning decoherence discussed in this and other papers cited below could make the arguments of Hawking and Page much more concrete, opening the way for a more precise discussion of the arrow of time in quantum cosmology. Some preliminary steps in this direction have been made by Fukuyama and Morikawa.[30]

A difficulty that occurs in a more precise discussion is the definition of entropy. In ordinary quantum mechanics, decoherence is a process which occurs *in time*. One may define an entropy for the reduced density matrix at time t:

$$S(t) = -\mathrm{Tr}(\tilde{\rho} \ln \tilde{\rho}) \ . \tag{6.2}$$

$S(t)$ increases when the variables over which one has traced in (6.2) decohere. The direction in which this occurs is the thermodynamic arrow of time. It is characteristic of quantum cosmology, however, that there is no external time. Time is an intrinsic variable which is already contained among the components of the three-metric or its conjugate momentum. In certain circumstances, one may take time to be the scale factor, for example. In discussing the decoherence of three-metric, therefore, we are discussing the decoherence *of time itself*—we are showing that there exists a variable which labels the order of events and whose different values do not interfere, as one implicitly assumes about t in quantum mechanics. It is not obvious, therefore, what the analogue of (6.2) should be in quantum cosmology. That is, on which side of Eq. (6.2) does the scale factor belong—does it play the role of time on the left-hand side, or is it traced over on the right-hand side? We have given a heuristic discussion of entropy increase by examining the behavior of the coherence width of a as a function of a itself; but a more concrete discussion cannot be given in the absence of an expression of the form (6.2). For the purposes of discussing the arrow of time it would be useful to have a model which possesses more degrees of freedom—e.g., a homogeneous mode of a scalar field, or extra scale factors. One could then try regarding one of the degrees of freedom as "time" and trace over the rest in an expression of the form (6.2). We hope to return to this point in a future publication.

Finally, we discuss the related work of other authors. Decoherence of spacetime was discussed in a reasonably heuristic fashion, without using quantum cosmology, by Joos.[31] Decoherence in quantum cosmology appears first to have been discussed by Kiefer[14] and by Zeh.[32] Kiefer considered a massive scalar field minisuperspace model and took the environment to be the infinite number of inhomogeneous modes of the metric and scalar field. He also considered a model in which the environment was fermionic.[33] Mellor and Moss considered a Kaluza-Klein model and took the environment to be the perturbation modes on the internal space.[34] Morikawa[26] and Fukuyama and Morikawa[30] considered similar models but gave a more careful treatment of the infinite number of modes of the environment, using field-theory methods. They also discussed the arrow of time. The paper most similar to this one is that of Padmanabhan, who took a massless scalar field as an environment, but in a general spacetime background.[35] A very general discussion involving decoherence in cosmology is that of Gell-Mann, Hartle, and Telegdi.[27] Sakagami has discussed the destruction of quantum coherence of scalar field fluctuations in inflationary universe models[36] Other discussions of the emergence of classical properties through decoherence in nonrelativisitic quantum mechanics are those of Joos and Zeh[37] and Unruh and Zurek.[9] Unruh and Zurek also discuss scalar fields in inflationary universe models. The papers of Zurek[8] have been particularly influential in the present work.

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[1]See, for example, the collection of papers in *Quantum Theory and Measurement,* edited by J. A. Wheeler and W. H. Zurek (Princeton University Press, Princeton, New Jersey, 1983).

[2]A. H. Guth and S. Y. Pi, Phys. Rev. Lett. **49**, 1110 (1982); S. W. Hawking, Phys. Lett. **115B**, 295 (1982).

[3]A. M. Guth and S. Y. Pi, Phys. Rev. D **32**, 1899 (1985); M. Sasaki, Prog. Theor. Phys. **76**, 1036 (1986); D. Lyth, Phys. Rev. D **31**, 1931 (1985).

[4]J. J. Halliwell, Phys. Lett. B **185**, 341 (1987).

[5]J. J. Halliwell, Phys. Rev. D **36**, 3626 (1987).

[6]J. B. Hartle, in *Gravitation in Astrophysics (Cargese 1986),* proceedings of the NATO Advanced Study Institute, Cargese, France, 1986, edited by B. Carter and J. B. Hartle (NATO ASI Series B, Vol. 156) (Plenum, New York, 1986).

[7]S. Wada, Mod. Phys. Lett. A **3**, 645 (1988).

[8]W. H. Zurek, Phys. Rev. D **24**, 1516 (1981); **26**, 1862 (1982); in *Quantum Optics, Experimental Gravitation and Measurement Theory,* proceedings of the NATO Advanced Study Institute, edited by P. Meystre and M. O. Scully (NATO ASI Series B, Vol. 94) (Plenum, New York, 1982); in *Proceedings of the Osgood Hill Conference on Conceptual Problems in Quantum Gravity,* edited by A. Ashtekar and J. Stachel (Birkhäuser, Boston, in press).

[9]W. Unruh and W. H. Zurek, Phys. Rev. D (to be published).

[10]J. B. Hartle, Phys. Rev. D **38**, 2985 (1988); **37**, 2818 (1988).

[11]Some introductory and/or review papers on quantum cosmology are J. B. Hartle, in *High Energy Physics 1985,* proceedings of the Yale Summer School, New Haven, Connecticut, edited by M. J. Bowick and F. Gursey (World Scientific, Singapore, 1985); in *Gravitation in Astrophysics (Cargese 1986)* (Ref. 6); J. J. Halliwell, Report No. NSF-ITP-88-131, 1988 (unpublished). An extensive list of papers on quantum cosmology may be found in Ref. 12.

[12]J. J. Halliwell, Report No. NSF-ITP-88-132, 1988 (unpublished).

[13]For the older work on minisuperspace, see C. W. Misner, in *Relativity,* edited by M. Carmeli, S. Fickler, and L. Witten (Plenum, San Francisco, 1970), and references therein. References to more modern work may be found in Ref. 12.

[14]C. Keifer, Class. Quantum Gravit. **4**, 1369 (1987).

[15]J. J. Halliwell and S. W. Hawking, Phys. Rev. D **31**, 1777 (1985).

[16]T. Banks, W. Fischler, and L. Susskind, Nucl. Phys. **B262**, 159 (1985); W. Fischler, B. Ratra, and L. Susskind, *ibid.* **B259**, 730 (1985); A. Vilenkin, Phys. Rev. D **37**, 888 (1986); S. Wada, Nucl. Phys. **B272**, 729 (1986).

[17]P. D. D'Eath and J. J. Halliwell, Phys. Rev. D **35**, 1100 (1987).

[18]Scalar field theory in curved spacetime in the functional Schrödinger picture is described in R. Brandenberger, Nucl. Phys. **B245**, 328 (1984); C. J. C. Burges, *ibid.* **B244**, 533 (1984); R. Floreanini, C. T. Hill, and R. Jackiw, Ann. Phys. (N.Y.) **175**, 345 (1987); B. Ratra, Phys. Rev. D **31**, 1931 (1985).

[19]S. W. Hawking, Phys. Scr. **T15**, 151 (1987); D. N. Page, Phys. Rev. D **34**, 2267 (1986).

[20]S. W. Hawking and D. N. Page, Nucl. Phys. **B298**, 789 (1988).

[21]A. Vilenkin, Phys. Rev. D **39**, 1116 (1989); **33**, 3560 (1986).

[22]J. J. Halliwell, in *Proceedings of the Osgood Hill Conference on Conceptual Problems in Quantum Gravity* (Ref. 8).

[23]See, for example, E. Calzetta, S. Habib, and B. L. Hu, Phys. Rev. D **37**, 2901 (1988).

[24]J. B. Hartle and S. W. Hawking, Phys. Rev. D **28**, 2960 (1983); S. W. Hawking, Nucl. Phys. **B239**, 257 (1984).

[25]J. J. Halliwell and J. Louko, Phys. Rev. D **39**, 2206 (1989).

[26]M. Morikawa, Kyoto Report No. KUNS 923, 1988 (unpublished).

[27]M. Gell-Mann, J. B. Hartle, and V. Telegdi (unpublished).

[28]S. W. Hawking, Phys. Rev. D **32**, 2489 (1985).

[29]D. N. Page, Phys. Rev. D **32**, 2496 (1985).

[30]T. Fukuyama and M. Morikawa, Phys. Rev. D **39**, 462 (1989).

[31]E. Joos, Phys. Lett. A **116**, 6 (1986).

[32]H. D. Zeh, Phys. Lett. A **116**, 9 (1986); **126**, 311 (1988).

[33]C. Kiefer, Heidelberg Report No. HD-TVP-88-10, 1988 (unpublished).

[34]F. Mellor and I. G. Moss, Newcastle report, 1988 (unpublished).

[35]T. Padmanabhan, Phys. Rev. D **39**, 2924 (1989).

[36]M. Sakagami, Prog. Theor. Phys. **79**, 442 (1988).

[37]E. Joos and H. D. Zeh, Phys. B **59**, 223 (1985).