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General static axisymmetric solution of Einstein's vacuum field equations in prolate spheroidal coordinates

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A solution of Einstein's vacuum field equations is presented explicitly in Eqs. (22), (28), and (43). This completes our stationary metric presented previously and therefore gives the exact solution to the problem of describing the exterior field of rotating deformed bodies. The metric given here is equivalent to the general Weyl metric; however, we use prolate spheroidal coordinates for convenience. We investigate the Newtonian and the relativistic (Geroch-Hansen) multipole moments of the solution and conclude that it describes the exterior gravitational field of a static axisymmetric mass distribution.

I. INTRODUCTION

The problem of how to describe the gravitational field of astrophysical objects has long been of central importance in general relativity, both as an issue of principle and as a foundation for observational predictions. In a previous paper[1] we presented an outline of a stationary axisymmetric metric which can be used to describe the exterior gravitational field of a rotating axisymmetric mass distribution. In this paper we present *explicitly* all metric functions of the corresponding static solution.

If we consider astrophysical objects as axisymmetric bodies and neglect their rotation, then their exterior gravitational field can be described by static axisymmetric solutions of Einstein's vacuum field equations. Weyl[2] showed that all such solutions can be expressed in the form

$$ds^2=e^{2\psi}dt^2-e^{2(\gamma-\psi)}(d\rho^2+dz^2)-e^{-2\psi}d\phi^2 , \qquad (1)$$

where (t,ρ,z,ϕ) are the Weyl canonical coordinates, and ψ and γ are functions of the nonignorable coordinates ρ and z satisfying the field equations

$$\Delta\psi=\psi_{\rho\rho}+\frac{1}{\rho}\psi_\rho+\psi_{zz} , \qquad (2)$$

$$\gamma_\rho=\rho(\psi_\rho^2-\psi_z^2) , \qquad (3)$$

$$\gamma_z=2\rho\psi_\rho\psi_z . \qquad (4)$$

Here $\psi_\rho=\partial\psi/\partial\rho$, etc.

From Eqs. (3) and (4) we see that the metric function γ can be determined by quadratures from the explicit form of the function ψ. Thus a static axisymmetric solution of Einstein's vacuum field equations is given by a solution of the two-dimensional Laplace equation (2) for ψ. Hence, the general solution[3]

$$\psi=\sum_{n=0}^{\infty}a_n\frac{P_n(\cos\theta)}{r^{n+1}}, \quad r^2=\rho^2+z^2, \quad \cos\theta=\frac{z}{r} , \qquad (5)$$

$$\gamma=\sum_{l,n=0}^{\infty}\frac{(l+1)(n+1)}{l+n+2}\frac{a_na_l}{r^{l+n+2}}(P_{l+1}P_{n+1}-P_nP_l) \qquad (6)$$

contains *all* static axisymmetric asymptotically flat solutions of Einstein's vacuum field equations. Here a_n, $n=0,1,2,\ldots$, are constants and P_n represents the Legendre polynomial of order n.

The mass multipole moments represent the deviations from the spherical symmetry of a gravitational source. In general relativity they are different from the Newtonian moments because of the curvature of spacetime. Relativistic and coordinate-invariant definitions of multipole moments were proposed by Geroch and Hansen,[4,5] Thorne,[6] and Beig and Simon.[7] Although one is led to these definitions by different mathematical approaches, it can be shown that they are all physically equivalent.[8,9]

If we are looking for static axisymmetric vacuum solutions with mass multipole moments, then it is clear that all of them are contained in Eqs. (5) and (6) as special cases. In particular, the Schwarzschild metric can be obtained from Eq. (5) by choosing the constants a_n $(n=0,1,2,\ldots)$ appropriately. The choice can be fixed by calculating the respective multipole moments (cf. the Appendix) and demanding that all the moments higher than the monopole moment vanish. Thus, we get (m is the mass of the source)

$$a_0=-m, \quad a_1=0, \quad a_2=-\tfrac{1}{3}m^3, \quad \text{etc} . \qquad (7)$$

The resulting series (5) converges to the well-known expression[3]

$$\psi=\frac{1}{2}\ln\frac{r_++r_--2m}{r_++r_-+2m}, \quad r_\pm^2=\rho^2+(z\pm m)^2 , \qquad (8)$$

which represents the Schwarzschild metric in Weyl's canonical coordinates. This form of the metric already shows that one can use the prolate spheroidal coordinate $x=(r_++r_-)/2m$ for obtaining a more convenient representation of static metrics with multipole moments.

The following point needs to be emphasized: In the literature the solution (8) is interpreted as corresponding to the *Newtonian* potential of a strut of length $2m$. However, as Ehlers has shown,[10] this intepretation is erroneous because it is based on a nonrelativistic definition of the Newtonian limit. Ehlers put forward a *relativistic* and coordinate-invariant definition of the Newtonian limit which is leading to the conclusion that the metric (8) corresponds to the Newtonian potential of a spherically symmetric distribution of mass m according to the usual interpretation of the Schwarzschild metric. We obtain the same interpretation by analyzing the multipole moments of the metric (8) since it possesses only a monopole moment.

From a physical point of view, the next interesting metric contained in Eq. (5) is the metric possessing a mass *quadrupole* moment. Choosing a_0 and a_1 as given in Eq. (7), we get the following expression for the quadrupole moment M_2 of the metric (5):

$$M_2=-a_2-\tfrac{1}{3}m^3 \ . \tag{9}$$

Redefining the parameter a_2, for instance, as $a_2 \to a_2' = a_2 - m^3/3$, the relativistic quadrupole moment becomes arbitrary. Similarly, one can show that the metric (5) and (6) can be used in order to describe the gravitational field of static bodies with arbitrary multipole moments.

To study the gravitational field of realistic astrophysical bodies, we have to take their rotation into account. In this case, the corresponding metric, which describes the exterior field, has to be stationary. Such a solution was already presented by the author[1] [however, without the function γ which is explicitly given in this paper in Eq. (43)] by using the Hoenselaers-Kinnersley-Xanthopoulos[11] method of generating new solutions from known ones. The stationary metric was given in prolate spheroidal coordinates which are more convenient than the Weyl ones for the investigation of multipole moments. Therefore, we derive in this paper the general static solution in prolate spheroidal coordinates. This completes our previous investigation[1] and gives an exact solution to the problem of describing the exterior field of rotating deformed mass distributions in general relativity.

Erez and Rosen[12] presented a static solution with mass and *arbitrary* quadrupole moment in prolate spheroidal coordinates. Moreover, they extended the solution to the case of a mass with an arbitrary multipole moment. Finally, they asserted that a solution in prolate spheroidal coordinates with more than two different multipole moments has no physical significance because the solution does not lead to the respective potential (of a mass with two or more different moments) in the Newtonian limit. In this paper we show that the general axisymmetric solution in prolate spheroidal coordinates corresponds to the potential of an axisymmetric mass distribution in the Newtonian limit.

In Sec. II we calculate explicitly the general solution of the field equations (2)–(4) in prolate spheroidal coordinates. The solution contains an infinite number of *arbitrary* parameters which are interpreted in Sec. III as the Newtonian multipole moments of an axisymmetric mass distribution. The relativistic (Geroch-Hansen) multipole moments are also calculated and compared with the Newtonian ones. Some physical properties of the solution are discussed.

II. THE SOLUTION

Erez and Rosen[12] used prolate spheroidal coordinates (t,x,y,ϕ) to obtain a static axisymmetric vacuum solution with arbitrary quadrupole moment. These coordinates are related to the Weyl canonical coordinates by ($m=$const)

$$x=\frac{r_+ + r_-}{2m}, \quad y=\frac{r_+ - r_-}{2m}, \quad r_\pm^2=\rho^2+(z\pm m)^2$$
$$x\geq 1, \quad -1\leq y\leq 1 \ , \tag{10}$$

or

$$\rho^2=m^2(x^2-1)(1-y^2), \quad z=mxy \ . \tag{11}$$

The line element (1) takes the form

$$ds^2=e^{2\psi}dt^2-m^2e^{-2\psi}\left[e^{2\gamma}(x^2-y^2)\left[\frac{dx^2}{x^2-1}+\frac{dy^2}{1-y^2}\right]+(x^2-1)(1-y^2)dy^2\right] , \tag{12}$$

and the field equations (2)–(4) become

$$[\psi_x(x^2-1)]_x+[(1-y^2)\psi_y]_y=0 \ , \tag{13}$$

$$\gamma_x=\frac{1-y^2}{x^2-y^2}[x(x^2-1)\psi_x^2-x(1-y^2)\psi_y^2 -2y(x^2-1)\psi_x\psi_y] \ , \tag{14}$$

$$\gamma_y=\frac{x^2-1}{x^2-y^2}[y(x^2-1)\psi_x^2-y(1-y^2)\psi_y^2 +2x(1-y^2)\psi_x\psi_y] \ . \tag{15}$$

Before we begin to derive the solution, let us present some lemmas which can easily be proved by using the field equations (13)–(15).

Lemma 1. Let ψ and all its derivatives be free of singularities in y and $1-y^2$, i.e., $\psi,\psi_x,\psi_y,\psi_{xy},\ldots \neq 0$ and $<\infty$ for all values of y and $1-y^2$, and let γ be asymptotically flat, i.e.,

$$\lim_{x\to\infty}\gamma(x,y)=0 \ . \tag{16}$$

Then the function γ vanishes at the symmetry axis, i.e.,

$$\gamma(x,\pm 1)=0 \ . \tag{17}$$

Lemma 2. Let ψ be asymptotically flat, i.e.,

$$\lim_{x\to\infty}\psi(x,y)=0 \ , \tag{18}$$

then the asymptotically flat solution of the differential equations (14) and (15) can be calculated by

$$\gamma(x,y)=(x^2-1)\int_{-1}^{y}\frac{A(x,y)}{x^2-y^2}dy \ , \tag{19}$$

where

$$A(x,y)=y(x^2-1)\psi_x^2-y(1-y^2)\psi_y^2+2x(1-y^2)\psi_x\psi_y \ . \tag{20}$$

By introducing polar coordinates (t,r,θ,ϕ) into the line element (12),

$$x=\frac{r}{m}-1, \quad y=\cos\theta , \tag{21}$$

one can prove the following.

Lemma 3. The line element (12) reduces to the Minkowski metric in the limit $m\to 0$, if the metric functions are asymptotically flat, i.e.,

$$\lim_{r\to\infty}\psi(r,\theta)=0=\lim_{r\to\infty}\gamma(r,\theta) . \tag{22}$$

Now we calculate the general solution of Eqs. (13)–(15). Following Erez and Rosen, we make the ansatz $\psi(x,y)=A(x)B(y)$. Then, Eq. (13) becomes

$$[(x^2-1)A_x]_x-\nu A=0 , \tag{23a}$$

$$[(1-y^2)B_y]_y+\nu B=0 , \tag{23b}$$

where ν is a constant. Since these differential equations are linear, one can write their general solution as an infinite sum of Legendre P_ν polynomials and associated Legendre functions of second kind Q_ν (Ref. 13). Thus, we get

$$\psi=\sum_{\nu=0}^{\infty}[q_\nu Q_\nu(x)+p_\nu P_\nu(x)][b_\nu Q_\nu(y)+c_\nu P_\nu(y)] , \tag{24}$$

where q_ν, p_ν, b_ν, and c_ν are constants. If we restrict ourselves to physically relevant solutions, then solution (24) has to satisfy the conditions of *elementary* and *asymptotic* flatness.[14] Elementary flatness means that the functions ψ and γ must be regular at the symmetry axis $y=\pm 1$. To avoid logarithmic singularities of ψ at $y=\pm 1$, the constant ν must be integer positive or zero,[15] $\nu=n=0,1,2,\ldots$. Moreover, $Q_n(y=\pm 1)\sim-\infty$; therefore $b_n=0$ in Eq. (24). At infinity $(x\to\infty)$ we get[16] $Q_n(x)\sim 0$ and $P_n(x)\sim x^n$, then $p_n=0$. Thus, the *general elementary and asymptotically flat* solution of Eq. (13) is

$$\psi=\sum_{n=0}^{\infty}q_n Q_n(x)P_n(y) . \tag{25}$$

If we put

$$q_0=1, \quad q_1=0, \quad q_2\neq 0, \quad q_k=0 \quad (k>2) , \tag{26}$$

and let $x\to -x$, then from Eq. (25) we obtain the Erez-Rosen metric

$$\psi_{\rm ER}=\frac{1}{2}\ln\frac{x-1}{x+1}+\tfrac{1}{2}q_2(3y^2-1)\left[\tfrac{1}{4}(3x^2-1)\ln\frac{x-1}{x+1}+\frac{3}{2}x\right] . \tag{27}$$

The further simplification $q_2=0$ leads to the Schwarzschild metric which takes its usual form after the coordinate transformation (21). From the special cases given above, we see that it is useful to transform the solution (25) by $x\to -x$. Therefore, we write the general solution (25) in the form $[Q_n(-x)=(-1)^{n+1}Q_n(x)]$

$$\psi=\sum_{n=0}^{\infty}(-1)^{n+1}q_n Q_n(x)P_n(y) . \tag{28}$$

Now we solve the differential equations (14) and (15) for the function γ. According to lemma 2, the asymptotically flat solution for γ is given by the integral (19). Putting Eq. (28) into Eq. (19), we see that the term (x^2-y^2) in the denominator does not allow for an immediate integration. Therefore, we have to rewrite $A(x,y)$ in such a way that the expression (x^2-y^2) can be eliminated. We use the identities

$$1-y^2=1-x^2+(x^2-y^2)$$

and

$$\left[\sum_{n=0}^{\infty}q_nP_nQ_n\right]^2=\sum_{m,n=0}^{\infty}q_nq_mP_nP_mQ_nQ_m ,$$

to put Eq. (19) into the form

$$\gamma=\sum_{m,n=0}^{\infty}(-1)^{m+n}q_mq_n\int_{-1}^{y}\Gamma_y^{mn}dy , \tag{29}$$

where

$$\Gamma_y^{mn}=(x^2-1)P'_mQ_m(2xP_nQ'_n-yP'_nQ_n)+\frac{(x^2-1)^2}{x^2-y^2}[P_mQ'_m(yP_nQ'_n-xP'_nQ_n)+P'_mQ_m(yP'_nQ_n-xP_nQ'_n)] \tag{30}$$

with

$$P'_n=\frac{dP_n(y)}{dy} \quad\text{and}\quad Q'_n=\frac{dQ_n(x)}{dx} .$$

From Eq. (29) we see that only the symmetric part $\Gamma_y^{(mn)}$ of Γ_y^{mn} contributes to γ. The second term of Γ_y^{mn} is already symmetric with respect to the pair of indices (mn). We build the symmetric part of the first term and obtain

$$\Gamma_y^{(mn)}=(x^2-1)[x(P'_mQ_mP_nQ'_n+P'_nQ_nP_mQ'_m)-yP'_mQ_mP'_nQ_n]+\frac{(x^2-1)^2}{x^2-y^2}[P_mQ'_m(yP_nQ'_n-xP'_nQ_n)+P'_mQ_m(yP'_nQ_n-xP_nQ'_n)] . \tag{30'}$$

To eliminate the expression (x^2-y^2) in the second term of Eq. (30′), we use the following recurrence formulas[17] twice:

$$A_n\equiv yP_nQ'_n-xP'_nQ_n=-\frac{1}{n}(x^2-y^2)P'_nQ'_n-B_{n-1} \tag{31}$$

and

$$B_n\equiv yP'_nQ_n-xP_nQ'_n=(x^2-y^2)\mathcal{S}_n+\frac{\epsilon_nx+(1-\epsilon_n)y}{x^2-1} . \tag{32}$$

Here

$$\mathcal{S}_n=\sum_{k=0}^{[n/2]-1}\left[\frac{1}{n-2k}+\frac{1}{n-2k-1}\right]P'_{n-2k-1}Q'_{n-2k-1} \tag{33}$$

and

$$\epsilon_n=\begin{cases}1, & n=\text{even integer}\ ,\\ 0, & n=\text{odd integer}\ .\end{cases}$$

Accordingly we get, from Eq. (30′),

$$\begin{aligned}\Gamma_y^{(mn)}=&(x^2-1)[x(P'_mQ_mP_nQ'_n+P'_nQ_nP_mQ'_m)-yP'_mQ_mP'_nQ_n]\\&+(x^2-1)\left[(1-\epsilon_n)\mathcal{S}_m+\epsilon_n\mathcal{S}_{m+1}-\frac{\epsilon_n}{m+1}P'_mQ'_m\right]\\&+(x^2-1)^2\left[P'_mQ_m\mathcal{S}_n-P_mQ'_m\mathcal{S}_{n+1}+\frac{1}{n+1}P_mQ'_mP'_nQ'_n\right]+\frac{y+(\epsilon_n+\epsilon_m-2\epsilon_n\epsilon_m)(x-y)}{x^2-y^2}\ .\end{aligned} \tag{34}$$

In Eq. (34) there is only one term containing the expression (x^2-y^2); it can be integrated by means of elementary functions. For the integration of the other terms, we will use the recurrence relation[18]

$$\int_{-1}^{y}P_n\,dy=\frac{1}{2n+1}(P_{n+1}-P_{n-1})\quad (n\geq 0,\ P_{-1}=-1)\ . \tag{35}$$

In order to use this relation in Eq. (34), however, we have to express the products of the form P'_mP_n and P_mP_n as a sum of Legendre polynomials. This can be done by using the following relations, which can be proved by induction:[19]

$$P'_n=\sum_{k=0}^{[n/2]-1}(2n-4k-1)P_{n-2k-1}\ , \tag{36}$$

$$P_mP_n=\sum_{k=0}^{\mu(m,n)}K(m,n,k)P_{m+n-2k},\quad \mu(m,n)=\min(m,n)\ . \tag{37}$$

Here $K(m,n,k)$, $m,n,k=0,1,2,\ldots$, are the Clebsch-Gordan coefficients:

$$K(m,n,k)=\frac{2m+2n-4k+1}{2m+2n-2k+1}\,\frac{a_{m-k}a_ka_{n-k}}{a_{m+n-k}} \tag{38}$$

with

$$a_k=\frac{(2k-1)!!}{k!},\quad (2k-1)!!=(2k-1)(2k-3)\cdots,\quad (0)!!=(-1)!!=1\ . \tag{39}$$

Using Eqs. (35)–(37), one proves the relations

$$\begin{aligned}A_{n,m}&=-A_{m,n}+P_nP_m-(-1)^{m+n}\int_{-1}^{y}P_nP'_m\,dy\\&=\sum_{k=0}^{[(n-1)/2]}\sum_{l=0}^{\mu(n,m-2k-1)}\frac{(2m-4k-1)K(m-2k-1,n,l)}{2(m+n)-4(k+l)-1}[P_{m+n-2(k+l)}-P_{m+n-2(k+l+1)}]\ ,\end{aligned} \tag{40}$$

$$\begin{aligned}B_{m,n}=B_{n,m}&=\int_{-1}^{y}P'_nP'_m\,dy\\&=\sum_{j=0}^{[(m-1)/2]}\sum_{k=0}^{[(n-1)/2]}\sum_{l=0}^{\mu(m-2j-1,n-2j-1)}\frac{(2m-4j-1)(2n-4k-1)K(m-2j-1,n-2k-1,l)}{2(m+n)-4(j+k+l)-3}\\&\qquad\times(P_{m+n-2(j+k+l)-1}-P_{m+n-2(j+k+l)-3})\ ,\end{aligned} \tag{41}$$

$$C_{n,m}=C_{m,n}=\int_{-1}^{y}yP'_nP'_m\,dy=\int_{-1}^{y}P'_m[P'_{n+1}-(n+1)P_n]dy=-(n+1)A_{n,m}+B_{n+1,m}\ . \tag{42}$$

Using Eqs. (35) an (40)–(42), we can immediately integrate the function $\Gamma_y^{(mn)}$. According to Eq. (29), the result can be written as

$$\gamma=\sum_{m,n=0}^{\infty}(-1)^{m+n}q_mq_n\Gamma^{(mn)}\ , \tag{43}$$

where

$$\Gamma^{(mn)}=\frac{1}{2}\ln\left[\frac{x^2-1}{x^2-y^2}\right]+(\epsilon_n+\epsilon_m-2\epsilon_n\epsilon_m)\ln\left[\frac{x+y}{x-1}\right]$$
$$+(x^2-1)[x(A_{n,m}Q_n'Q_m+A_{m,n}Q_m'Q_n)-C_{n,m}Q_nQ_m]$$
$$+(x^2-1)\left[(1-\epsilon_n)S_m+\epsilon_nS_{m+1}-\frac{\epsilon_n}{m+1}[P_m-(-1)^m]Q_m'\right]$$
$$+(x^2-1)^2\left[Q_m\mathcal{B}_{m,n}-Q_m'\mathcal{A}_{m,n}+\frac{1}{n+1}A_{m,n}Q_m'Q_n'\right]. \tag{44}$$

Here we introduced the notations

$$\mathcal{A}_{m,n}=\int_{-1}^{y}P_m\mathcal{S}_{n+1}dy=\sum_{k=0}^{[(n-1/2)]}\left[\frac{1}{n-2k+1}+\frac{1}{n-2k}\right]\times A_{m,n-2k}Q_{n-2k}', \tag{45a}$$

$$\mathcal{B}_{m,n}=\int_{-1}^{y}P_m'\mathcal{S}_ndy=\sum_{k=0}^{[(n/2-1)]}\left[\frac{1}{n-2k}+\frac{1}{n-2k-1}\right]\times B_{m,n-2k-1}Q_{n-2k-1}', \tag{45b}$$

and

$$S_n=\int_{-1}^{y}\mathcal{S}_ndy=\sum_{k=0}^{[(n/2-1)]}\left[\frac{1}{n-2k}+\frac{1}{n-2k-1}\right]\times[P_{n-2k-1}+(-1)^{n+1}]Q_{n-2k-1}'. \tag{45c}$$

Thus, the general static axisymmetric vacuum metric in prolate spheroidal coordinates is given explicitly in Eqs. (28) and (43) by means of the Legendre polynomials and associated functions of second kind and their respective first-order derivatives. From the results given above, we can derive some physical properties of this solution.

(i) The general solution for ψ in Eq. (28) was obtained by using the conditions of *elementary* and *asymptotic* flatness.

(ii) For the calculation of γ, we used lemma 2. Thus, this function is asymptotically flat.

(iii) The function ψ has no singularities in y ($|P_n(y)|\leq 1$), from lemma 1 we deduce that the metric function γ has no singularities at the symmetry axis ($y=\pm 1$).

(iv) Putting $q_0=1$ and introducing polar coordinates by Eq. (21), an important property of this solution follows from lemma 3: The metric (28) and (43) leads to the Minkowski metric for vanishing mass ($m=0$) regardless of the value of the parameters q_n ($n=1,2,3,\ldots$). (In the following section we will show that m represents the total mass of the source.)

In the Newtonian theory of gravity, the multipole expansion of a mass distribution determines the gravitational potential uniquely. Since the multipole expansion is in general an infinite sum, it is necessary that this sum converges in a definite region around the source. (The points where the potential becomes singular are an exception.[20]) In other words, a divergent multipole expansion cannot be created by a limited mass distribution. In general relativity, the Newtonian potential is determined by the metric tensor component g_{tt}. Thus, the gravitational potential of a static axisymmetric solution converges if the metric function ψ does so. Using standard definitions and theorems of the theory of infinite series, one can prove the following.

Lemma 4. The infinite series

$$\psi=\sum_{n=0}^{\infty}(-1)^{n+1}q_nP_n(y)Q_n(x) \tag{28}$$

is uniformly convergent in the region $K=\{|x|>1, |y|\leq 1\}$, if the absolute values of the parameters q_n ($n=0,1,2,\ldots$) are limited:

$$|q_n|\leq q<\infty \quad \forall n\ . \tag{46}$$

The condition (46) does not limit the class of solutions contained in Eq. (28), and it has a very simple physical significance. In the next section we will show the parameters q_n determine the multipole moments of the mass distribution. Hence, condition (46) means that the multipole moments have to be limited, if the respective gravitational potential corresponds to the potential of a realistic mass distribution as that of an astronomical object.

III. THE MULTIPOLE MOMENTS OF THE SOLUTION

In this section we calculate the Newtonian multipole moments N_n and the relativistic ones M_n ($n=0,1,2,\ldots$) of the metric (28) and (43) in order to obtain the physical interpretation of the parameters q_n.

The Newtonian moments can be calculated by using the *coordinate-invariant* Ehlers definition[10]. According to Ehlers, the Newtonian potential Φ of a given static axisymmetric vacuum solution can be obtained from the limit

$$\Phi=\lim_{\lambda\to 0}\frac{1}{\lambda}\psi(\rho,z,\lambda)\ , \tag{47}$$

where $\lambda = c^{-2}$ (c=speed of the light) and $\psi(\rho,z,\lambda)$ is the metric function ψ in Weyl's canonical coordinates containing the parameter λ explicitly. That means that ψ has to be written in cgs units, for example. It is easy to see from Eq. (47) that the special case $q_0=1$, $q_k=0$ ($k>0$), i.e., the Schwarzschild metric (8), leads to the Newtonian potential of a spherically symmetric mass distribution $\Phi=-GM/(\rho^2+z^2)^{1/2}$, if we replace the parameter m of the coordinate transformation (10) by $MG\lambda$.

According to Eqs. (47) and (10), we obtain the Newtonian potential of the solution (28) by calculating the limit

$$\Phi=\lim_{\lambda\to 0}\frac{1}{\lambda}\sum_{n=0}^{\infty}(-1)^{n+1}q_n Q_n\left[\frac{r_+ + r_-}{2\lambda GM}\right]P_n\left[\frac{r_+ - r_-}{2\lambda GM}\right]. \tag{48}$$

For $\lambda\to 0$ we have

$$\lim_{\lambda\to 0} y=\lim_{\lambda\to 0}\frac{r_+ - r_-}{2\lambda GM}=\frac{z}{\sqrt{\rho^2+z^2}}, \tag{49a}$$

$$\lim_{\lambda\to 0} x=\lim_{\lambda\to 0}\frac{r_+ + r_-}{2\lambda GM}\to\infty . \tag{49b}$$

From Eq. (49a) and the property $|P_n(y)|\le 1$, we get

$$-1\le\lim_{\lambda\to 0}P_n\left[\frac{r_+ - r_-}{2\lambda GM}\right]=P_n\left[\frac{z}{\sqrt{\rho^2+z^2}}\right]\le 1 . \tag{50}$$

Equation (49b) means that the limit $\lambda\to 0$ is equivalent to the limit $x\to\infty$. Thus, it is useful to write the $Q_n(x)$ as a power series of x (Ref. 21):

$$Q_n\left[\frac{r_+ + r_-}{2\lambda GM}\right]=\sum_{l=0}^{\infty}b^n_{n+2l+1}\left[\frac{2\lambda GM}{r_+ + r_-}\right]^{n+2l+1}, \tag{51}$$

where

$$b^n_{n+2l+1}=\frac{(n+2l-1)(n+2l)}{2l(2n+2l+1)}b^n_{n+2l-1},$$

$$b^n_{n+1}=\frac{n!}{(2n+1)!!}.$$

From Eqs. (50) and (51) we see that the limit of each summand of Eq. (48) exists and converges. That means that the sum and the limit in Eq. (48) commute. Using Eqs. (49a) and (51), we obtain, from Eq. (48),

$$\Phi=\sum_{n=0}^{\infty}(-1)^{n+1}q_n P_n\left[\frac{z}{\sqrt{\rho^2+z^2}}\right]\times\left[\sum_{l=0}^{\infty}\lim_{\lambda\to 0}\frac{1}{\lambda}b^n_{n+2l+1}\left[\frac{2\lambda GM}{r_+ + r_-}\right]^{n+2l+1}\right]. \tag{52}$$

Introducing the parameters $\tilde{q}_n=q_n(G\lambda)^n$ $n=0,1,2,\ldots$, we get

$$\Phi=G\sum_{n=0}^{\infty}(-1)^{n+1}\frac{n!}{(2n+1)!!}\tilde{q}_n M^{n+1}\frac{P_n(\cos\theta)}{r^{n+1}},$$

$$\cos\theta=\frac{z}{r},\quad r=\sqrt{\rho^2+z^2} . \tag{53}$$

This is the Newtonian potential of an axisymmetric mass distribution with the multipole moments

$$N_n=(-1)^n\frac{n!}{(2n+1)!!}\tilde{q}_n M^{n+1},\quad n=0,1,2,\ldots, \tag{54}$$

or, in geometrized units $G=c=1$,

$$N_n=(-1)^n\frac{n!}{(2n+1)!!}q_n m^{n+1} . \tag{54'}$$

This coordinate-invariant calculation of the Newtonian moments leads us to an understanding of the physical significance of the parameters q_n in Eqs. (28) and (43). *The parameters q_n, $n=0,1,2,\ldots$ determine, modulo a constant factor, the Newtonian multipole moments of a static axisymmetric mass distribution whose exterior gravitational field is described by the vacuum solution* (28) and (43).

According to Eq. (54′), the monopole moment is the total mass m if $q_0=1$. The dipole moment is $N_1=-\frac{1}{3}q_1 m^2$; it can be made to vanish by means of a coordinate transformation which brings the center of mass to the origin of the coordinate system. The quadrupole moment is $N_2=\frac{2}{15}q_2 m^3$ and coincides with that of the Erez-Rosen metric. Taking higher multipole moments into account, we see that no "cross terms" will appear in the Newtonian potential. This result contradicts that of Erez and Rosen who used the limit $r\to\infty$ to calculate the Newtonian potential of the solution (28). This procedure obviously depends on the choice of the coordinate system.

For calculating the *relativistic, coordinate-invariant* Geroch-Hansen multipole moments M_n ($n=0,1,2,\ldots$), we use the procedure as given in the Appendix. The calculations can be simplified by using the following relation for the derivatives of the metric function ψ:

$$\left.\frac{d^k\psi(\tilde{z},1)}{d\tilde{z}^k}\right|_{\tilde{z}=0}=k!m^k\sum_{(n,l)^*}(-1)^{n+1}q_n b^n_{n+2l+1},$$

$$k=1,2,3,\ldots,$$

where $(n,l)^*$ means that the sum runs over all positive integer values of n and l which satisfy the constraint

$$n+2l+1=k .$$

The resulting relativistic moments can be written in the form

$$M_n=N_n+R_n,\quad n=0,1,2,\ldots . \tag{55}$$

Here N_n represent the Newtonian multipole moments given in Eq. (54′). The R_n can be expressed in terms of the N_n as

$$\begin{aligned}&R_0=R_1=R_2=0,\quad R_3=-\tfrac{2}{5}m^2N_1,\\ &R_4=-\tfrac{2}{7}m^2N_2-\tfrac{6}{7}mN_1^2,\\ &R_5=-\tfrac{2}{9}m^2N_3-\tfrac{48}{21}mN_2N_1-\tfrac{2}{7}N_1^3-\tfrac{4}{105}m^4N_1,\ldots,\\ &R_n=R_n(N_{n-2},N_{n-3},\ldots,N_0) .\end{aligned} \tag{56}$$

Here we put $q_0=1$ in order to obtain $N_0=m$. Thus the relativistic multipole moments can be given as a sum of the Newtonian moments plus some *relativistic* corrections R_n. In fact, introducing cgs units ($m \rightarrow MG/c^2$) and new parameters $\tilde{q}_n = q_n(G/c^2)^n$, $n=1,2,3,\ldots$ and calculating the limit $c \rightarrow \infty$, we see that the R_n vanish; i.e., these terms have a relativistic character.

We notice that the decomposition (55) is coordinate *invariant* because we used coordinate-invariant definitions to calculate both the Newtonian and the relativistic moments. From Eq. (56) we see that the first relativistic correction appears in the octupole moment provided a dipole (N_1) exists. Moving the origin of coordinates, the dipole can be made to vanish so that the first relativistic correction appears in the 16-pole moment. Thus, the parameters q_n, $n=0,1,2,\ldots$, of the general solution (28) and (43) determine the Newtonian and the relativistic multipole moments uniquely.

IV. CONCLUDING REMARKS

The general static axisymmetric vacuum solution given in Eqs. (28) and (43) describes the exterior gravitational field of a static axisymmetric mass distribution because (i) it leads exactly to the gravitational potential of such a mass distribution in the Newtonian limit, (ii) it possesses an infinite number of independent parameters q_n which determine the Newtonian and the relativistic multipole moments of the mass distribution, (iii) it is asymptotically flat, and (iv) it has no singularities outside a definite region which can be "filled" with matter [this can be shown by investigating the curvature scalars; there are two singularities: at $x=-1(r=0)$ and at $x=1(r=2m)$].

The direct relationship between the metrics (28) and (43) and (5) and (6) can be found by expressing the coordinate $r=m(x+1)$ of (5) in terms of $Q_n(x)$ and comparing with (28). These two metrics must be equivalent, modulo a redefinition of the parameters a_n. At spatial infinity, this can easily be seen by introducing the radial coordinate $r=m(x+1)$ into the metric function (28) and calculating the limit $r \rightarrow \infty$. In fact, the resulting metric function coincides with (5) after a redefinition of a_n.

Of course, in order to completely describe the gravitational field of a body, one must know the corresponding interior solution. This task is under consideration.

The results given in this paper have been checked by using heavily the computer algebra system REDUCE 3.3 (Ref. 22).

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APPENDIX: CALCULATION OF THE GEROCH-HANSEN MULTIPOLE MOMENTS

The explicit calculation of relativistic moments, by using the original method of Geroch[4] and Hansen,[5] is quite laborious. Hoenselaers[23] found a relation between the Ernst potential[24] of a given metric and its multipole moments. In this appendix we present this relation and derive a recurrence formula which can be very useful for the calculation of multipole moments of higher order.

For the sake of generality, we investigate *stationary* axisymmetric vacuum solutions. The Ernst potentials E and ξ are defined in terms of the metric functions by[24]

$$E=f+i\Omega, \quad \xi=\frac{1-E}{1+E} . \tag{A1}$$

For static metrics $f=\exp(2\psi)$ and $\Omega=0$. Let us consider $\xi=\xi(x,y)$ at the symmetry axis $y=1$ (Ref. 25). Introducing the inverse Weyl canonical coordinate $\bar{z}$ by

$$\bar{z}=\frac{1}{z}=\frac{1}{\sigma xy}, \quad \sigma=\text{const}=m \text{ for static metrics} , \tag{A2}$$

the "point at infinity" becomes $\bar{z} \rightarrow 0$. Furthermore, we define the conformally transformed potential $\tilde{\xi}$ by

$$\tilde{\xi}(\bar{z},1) \equiv \frac{1}{\bar{z}}\xi(\bar{z},1) . \tag{A3}$$

Hoenselaers showed that the mass multipole moments M_l and the current moments J_l of the source can be calculated by using the simple relation

$$M_l=\text{Re}(m_l+d_l), \quad J_l=\text{Im}(m_l+d_l) , \tag{A4}$$

where

$$m_l \equiv \frac{1}{l!}\frac{d^l\tilde{\xi}(\bar{z},1)}{d\bar{z}^l}\bigg|_{\bar{z}=0} . \tag{A5}$$

The second term d_l, $l=0,1,2,\ldots$, is determined by comparing Eq. (A4) with the original Geroch-Hansen definition. It can be expressed in terms of m_k with $k \leq l-1$: for example,

$$\begin{aligned} &d_0=d_1=d_2=d_3=0, \quad d_4=\tfrac{1}{7}m_0^*(m_1^2-m_2m_0) , \\ &d_5=\tfrac{1}{3}m_0^*(m_2m_1-m_3m_0)+\tfrac{1}{21}m_1^*(m_1^2-m_2m_0) . \end{aligned} \tag{A6}$$

Thus, the calculation of relativistic multipole moments M_l and J_l is equivalent to the calculation of m_l, i.e., the derivatives of the conformally transformed Ernst potential $\tilde{\xi}$.

Expanding the potential ξ in powers of $\bar{z}$,

$$\xi(\bar{z},1)=\sum_{k=1}^{\infty}\frac{d^k\xi(\bar{z},1)}{d\bar{z}^k}\bigg|_{\bar{z}=0}\frac{\bar{z}^k}{k!} , \tag{A7}$$

we obtain, from Eqs. (A5) and (A3),

$$m_l=\frac{1}{(l+1)!}\frac{d^{l+1}\xi(\bar{z},1)}{d\bar{z}^{l+1}}\bigg|_{\bar{z}=0} . \tag{A8}$$

Introducing E in Eq. (A8) by means of Eq. (A1), we get a useful recurrence formula for calculating m_l:

$$m_l=-\frac{h_{l+1}}{(l+1)!}\bigg|_{Y=1,\bar{z}=0} \tag{A9a}$$

with

$$h_1 = \frac{1}{2E}\frac{dE}{d\bar{z}} \tag{A9b}$$

and

$$h_l = \frac{dh_{l-1}}{d\bar{z}} + 2\xi h_1 h_{l-1} \quad \text{for } l \geq 2 \ . \tag{A9c}$$

To calculate the relativistic multipole moments of stationary or static axisymmetric solutions, we use the following procedure. (i) Calculate the Ernst potentials E and ξ according to Eq. (A1). (ii) Calculate the quantities m_l by using the recurrence relation (A9). (iii) Use Eq. (A4) to obtain the respective moments.

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