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Analysis of spatially inhomogeneous perturbations of the FRW cosmologies

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We use Bardeen's gauge-invariant formalism to analyze the behavior of, and relationship between, various geometric and physical quantities of cosmological interest at the linear level. This leads to a cosmologically oriented gauge-invariant characterization of the different perturbation modes that can arise. In particular a link is made between the existence of gravitational-wave modes and the conformal curvature of hypersurfaces in spacetime. We indicate how these results can be useful in the analysis of exact solutions of the Einstein field equations.

I. INTRODUCTION

Simply stated, the cosmological perturbation problem is concerned with solutions of the linearized Einstein field equations (EFE's) when the background model is an appropriate Friedmann-Robertson-Walker (FRW) solution. The first relativistic treatment was given by Lifshitz[1] and since then (particularly between the years 1967 and 1980) the literature has grown steadily. The last major contribution was the comprehensive paper of Bardeen.[2] The problem is routinely covered in graduate and undergraduate texts such as Weinberg,[3] Landau and Lifshitz,[4] Raychaudhuri,[5] and Peebles.[6]

The main application of cosmological perturbation theory has been in attempts to solve the galaxy formation problem within the framework of general relativity theory and to the study of the propagation of the microwave background radiation in a spatially inhomogeneous universe. In the galaxy formation problem the quantity of primary physical interest is the so-called *density contrast* (or *relative density perturbation*) $\delta\mu$ defined by

$$\delta\mu=\frac{\mu-\mu_0}{\mu_0}\ , \tag{1.1}$$

where μ and μ_0 denote the perturbed and background energy densities, respectively. It is fairly well known that the behavior of $\delta\mu$ (and other quantities) suffers from ambiguities due to the freedom to perform infinitesimal coordinate transformations.

Our interest in linear perturbations of the FRW models stems from their possible help in interpreting the behavior of exact inhomogeneous cosmologies, in particular in determining when an exact solution of the EFE's can be considered as approximating an FRW solution over some epoch. In this paper we use Bardeen's gauge-invariant formalism to study the behavior of several geometric and physical quantities of cosmological interest. This enables us to derive a cosmologically oriented gauge-invariant characterization of the different perturbation types that can arise. In particular we link the existence of gravitational-wave modes to the conformal curvature of hypersurfaces in spacetime.

The outline of this paper is as follows. Sections II and III set up the mathematical equations, assumptions, notations, and terminology that are used throughout the paper. In Secs. IV and V we use Bardeen's gauge-invariant formalism to derive a characterization of the different perturbation types that can arise. Sections VI and VII contain a discussion of the relationship between various geometric and physical quantities of cosmological interest. In the concluding section we indicate how the derived results can be useful in the analysis and interpretation of exact solutions of the EFE's.

II. THE BASIC EQUATIONS

In this section we set up the standard mathematical framework for studying cosmological perturbations. This approach was first introduced by Lifshitz[1] (see also Lifshitz and Khalatnikov[7]) although it will be useful for the later development to use the notation of Bardeen.[2]

The background spacetime is taken to be one of the FRW models so that, relative to comoving coordinates, the background line element can be written in the form (here and elsewhere greek letters assume the values 1,2,3, whereas latin letters assume the values 0,1,2,3)

$$ds^2=S^2(\eta)[-d\eta^2+{}^3g_{\alpha\beta}(x^\lambda)dx^\alpha dx^\beta]\ , \tag{2.1}$$

where ${}^3g_{\alpha\beta}$ is the metric of a three-space of constant curvature ($K=\pm1$, 0 determines the sign of the curvature in the usual manner). We write the line element of the perturbed spacetime in the form

$$ds^2=S^2(\eta)[-(1+h_{00})d\eta^2+2h_{0\alpha}d\eta\, dx^\alpha +({}^3g_{\alpha\beta}+h_{\alpha\beta})dx^\alpha dx^\beta]\ , \tag{2.2}$$

where $h_{ab}(x^c)$ represent the perturbation in the gravitational field. We make the following simplifying assumptions as to the source in the EFE's: (1) We restrict our considerations to the case of a perfect fluid source; (2) we assume that the equation of state of the fluid in the background and the perturbed solution is $p=(\gamma-1)\mu$ for the *same* value of $\gamma(1\le\gamma\le2)$. More general situations have been considered in the literature (see, for example, Weinberg,[3] Bardeen,[2] Press and Vishniac[8]). However, more complicated physical assumptions do not alter the *mathematics* of the problem significantly, and so for our purposes the above situation will suffice. The EFE's for the background spacetime reduce to the two equations

$$\left[\frac{\dot{S}}{S}\right]^2=\frac{2M}{S^{3\gamma-2}}-K\ , \tag{2.3}$$

$$\mu_0=\frac{6M}{S^{3\gamma}}\ , \tag{2.4}$$

where M is an arbitrary constant, an overdot denotes $d/d\eta$, and the subscript on μ_0 refers to a background quantity.

The energy density μ and the fluid four-velocity u^a in the perturbed model are written in the form

$$\mu=\mu_0(1+\delta\mu)\ , \tag{2.5}$$

$$u^\alpha=U^0V^\alpha,\quad u^0=S^{-1}(1-\tfrac{1}{2}h_{00})\ , \tag{2.6}$$

where $\delta\mu$ and V^α are first-order quantities and u^0 is obtained from $u^a u_a=-1$.

The spatial homogeneity and spatial isotropy of the background spacetime imply (see Lifshitz[1] and Lifshitz and Khalatnikov[7] Appendix J for more details) that an arbitrary perturbation of the gravitational field can be expressed as a linear combination of the eigenfunctions (harmonics) of the Laplace operator in the three-space of constant curvature with line element (in fact, Lifshitz made an explicit choice of spatial coordinates and only considered the case of positive and negative spatial curvature; for the spatially flat background see, for example, Weinberg[3])

$$ds^2_{(3)}={}^3g_{\alpha\beta}(x^\lambda)dx^\alpha dx^\beta\ ,$$

that is, solutions of

$$Q_{\alpha\beta\cdots\gamma|\mu}{}^{|\mu}+k^2Q_{\alpha\beta\cdots\gamma}=0\ ,$$

where a vertical bar denotes covariant differentiation with respect to ${}^3g_{\alpha\beta}$. For the current problem we only need three types of eigenfunctions: namely, the scalar, vector, and second-rank tensor harmonics. Below we have listed the definitions of the appropriate harmonics using the notation of Bardeen.[2]

(1) Scalar harmonics. Harmonics constructed from solutions of the scalar Helmholtz equation:

$$Q^{(0)}{}_{|\alpha}{}^{|\alpha}+k^2Q^{(0)}=0\ .$$

Eigenvalues:

$$k^2=n^2-K\quad\begin{cases}n>0 & \text{if } K=-1,0\ ,\\ n \text{ integral} & \text{if } K=+1\ .\end{cases}$$

Corresponding vector and tensor:

$$Q^{(0)}{}_\alpha\equiv-k^{-1}Q^{(0)}{}_{|\alpha},$$

$$Q^{(0)}{}_{\alpha\beta}\equiv k^{-2}Q^{(0)}{}_{|\alpha\beta}+\tfrac{1}{3}{}^3g_{\alpha\beta}Q^{(0)},\quad Q^{(0)}{}_\alpha{}^\alpha=0\ .$$

(2) Vector harmonics. Harmonics constructed from divergenceless solutions of the vector Helmholtz equation:

$$Q^{(1)\alpha|\beta}{}_{|\beta}+k^2Q^{(1)\alpha}=0,\quad Q^{(1)}{}_\alpha{}^{|\alpha}=0\ .$$

Eigenvalues:

$$k^2=n^2-2K\quad\begin{cases}n>0 & \text{if } K=-1,0\ ,\\ n=2,3,\ldots & \text{if } K=+1\ .\end{cases}$$

Corresponding tensor:

$$Q^{(1)\alpha\beta}\equiv-\tfrac{1}{2}k^{-1}[Q^{(1)\alpha|\beta}+Q^{(1)\beta|\alpha}],\quad Q^{(1)}{}_\alpha{}^\alpha=0\ .$$

(3) Tensor harmonics. Traceless, divergenceless solutions of the tensor Helmholtz equation:

$$\begin{aligned}&Q^{(2)\alpha\beta}{}_{|\gamma}{}^{|\gamma}+k^2Q^{(2)\alpha\beta}=0\ ,\\ &Q^{(2)}{}_\alpha{}^\alpha=0,\quad Q^{(2)}{}_{\alpha\beta}{}^{|\beta}=0\ .\end{aligned} \tag{2.7}$$

Eigenvalues:

$$k^2=n^2-3K\quad\begin{cases}n>0 & \text{if } K=-1,0\ ,\\ n=3,4,\ldots & \text{if } K=+1\ .\end{cases}$$

Remark. The separation constant (wave number) k determines the spatial scale of the perturbation relative to the comoving background coordinates. The proper wavelength associated with the perturbation is $\lambda=(2\pi S/k)$. [The assumption $n\neq0$ in the $K=0$ case means that we restrict our attention to finite-wavelength perturbations $(k\neq0)$.]

A general perturbation is then expressed as a linear combination of these harmonics *with time-dependent coefficients.* The three harmonic types are completely decoupled and so lead to the following terminology: "scalar perturbations" are perturbations constructed only from scalar harmonics; "vector perturbations" are perturbations constructed only from vector harmonics; "tensor perturbations" are perturbations constructed only from tensor harmonics. The fact that the three harmonic types are decoupled allows the scalar, vector, and tensor perturbations to be discussed independently.

III. BARDEEN'S GAUGE-INVARIANT FORMALISM

The coordinates chosen in (2.2) are not unique since we are free to perform infinitesimal gauge transformations of the form

$$x^{a'}=x^a+\xi^a,\quad |\xi^a|\ll1\ . \tag{3.1}$$

We will refer to this freedom as gauge freedom. If T is any tensor field and ΔT denotes a first-order perturbation in T then it is a fairly standard result (see, for example, Sachs[9]) that the transformation (3.1) induces a change in ΔT given by

$$\Delta T'=\Delta T-L_\xi T\ , \tag{3.2}$$

where L_ξ denotes the Lie derivative in the background spacetime. If the perturbation in a tensor field is unchanged by an infinitesimal gauge transformation then that quantity is called a *gauge invariant.* It follows from (3.2) that the only gauge-invariant quantities are those that are zero in the background spacetime (or, for scalar quantities, those that are constant). In the particular case of interest to us, namely, when the background spacetime is an FRW model, it follows that the shear tensor, and acceleration and vorticity vectors of the fluid congruence are gauge invariants, as well as the electric and magnetic parts of the Weyl tensor, since all of these quantities are zero in the background spacetime. However the density contrast $\delta\mu \equiv \Delta\mu/\mu_0$ is gauge dependent. The differences in the values of $\delta\mu$ resulting from the use of different gauge conditions has led to much ambiguity within the perturbation literature over the years.

In an effort to overcome the gauge difficulties inherent in the cosmological perturbation problem Bardeen[2] has introduced a formalism based entirely on gauge-invariant quantities. We find this formalism the most transparent for the present analysis and hence we describe it briefly below. The full formalism includes the possibility of stress and entropy perturbations. However, as stated previously, we will restrict our considerations to the case of a perfect fluid with γ-law equation of state.

Bardeen's gauge-invariant method involves three distinct steps: (1) expand all metric and fluid quantities in terms of the "spherical harmonics" defined in the previous section; (2) form combinations of the metric and fluid quantities so as to obtain gauge invariants; (3) take combinations of the EFE's and conservation equations so as to obtain "simple" relationships between the gauge invariants.

The results of the first two steps are given below. We refer the reader to Bardeen's paper for full details:

Scalar perturbations

$$\begin{aligned}
&h_{00}=2A(\eta)Q^{(0)}, \quad h_{0\alpha}=-B^{(0)}(\eta)Q^{(0)}{}_{\alpha} ,\\
&h_{\alpha\beta}=2H_L(\eta)Q^{(0)}\,{}^3g_{\alpha\beta}+2H_T{}^{(0)}(\eta)Q^{(0)}{}_{\alpha\beta} ,\\
&V^{\alpha}=V^{(0)}(\eta)Q^{(0)\alpha}, \quad u^0=S^{-1}[1-A(\eta)Q^{(0)}] ,\\
&\delta\mu=\delta(\eta)Q^{(0)} .
\end{aligned}$$

Gauge-invariant metric perturbations

$$\begin{aligned}
\phi_A &\equiv A+\frac{1}{k}\dot{B}^{(0)}+\frac{1}{k}\left[\frac{\dot{S}}{S}\right]B^{(0)}\\
&\quad -\frac{1}{k^2}\left[\ddot{H}_T{}^{(0)}+\left[\frac{\dot{S}}{S}\right]\dot{H}_T{}^{(0)}\right] ,\\
\phi_H &\equiv H_L+\tfrac{1}{3}H_T{}^{(0)}+\frac{1}{k}\left[\frac{\dot{S}}{S}\right]B^{(0)}-\frac{1}{k^2}\left[\frac{\dot{S}}{S}\right]\dot{H}_T{}^{(0)} .
\end{aligned} \tag{3.3}$$

Gauge-invariant matter perturbations

$$\begin{aligned}
&V_s{}^{(0)}\equiv V^{(0)}-\frac{1}{k}\dot{H}_T{}^{(0)} ,\\
&\epsilon_m\equiv\delta+\frac{3\gamma}{k}\left[\frac{\dot{S}}{S}\right](V^{(0)}-B^{(0)}) ,\\
&\epsilon_g\equiv\delta-\frac{3\gamma}{k}\left[\frac{\dot{S}}{S}\right]\left[B^{(0)}-\frac{1}{k}\dot{H}_T{}^{(0)}\right] .
\end{aligned}$$

Vector perturbations

$$\begin{aligned}
&h_{00}=0, \quad h_{0\alpha}=-B^{(1)}(\eta)Q^{(1)}{}_{\alpha} ,\\
&h_{\alpha\beta}=2H_T{}^{(1)}(\eta)Q^{(1)}{}_{\alpha\beta}, \quad V^{\alpha}=V^{(1)}(\eta)Q^{(1)\alpha} ,\\
&\delta\mu=0 .
\end{aligned}$$

Gauge-invariant metric perturbation

$$\Psi\equiv B^{(1)}-k^{-1}\dot{H}_T{}^{(1)} .$$

Gauge-invariant matter perturbation

$$V_s{}^{(1)}\equiv V^{(1)}-k^{-1}\dot{H}_T{}^{(1)}$$

or (3.4)

$$V_c\equiv V^{(1)}-B^{(1)}=V_s{}^{(1)}-\Psi .$$

Tensor perturbations

$$\begin{aligned}
&h_{00}=h_{0\alpha}=0 ,\\
&h_{\alpha\beta}=2H_T{}^{(2)}(\eta)Q^{(2)}{}_{\alpha\beta}, \quad V^{\alpha}=0 ,\\
&\delta\mu=0 .
\end{aligned}$$

Gauge-invariant metric and matter perturbation

In this case $H_T{}^{(2)}$ is gauge invariant.

We briefly discuss the gauge invariants ϵ_m, ϵ_g. These quantities were introduced by Bardeen in order to give a gauge-invariant measure of the density perturbation, thereby eliminating the gauge problems associated with the nonuniqueness of the density contrast $\delta\mu$. However, there is obviously no unique way of defining such a quantity. The criterion that Bardeen used was that the gauge-invariant quantity reduce to the density contrast amplitude δ as soon as the perturbation comes inside the particle horizon [as defined by the condition $k^{-1}(\dot{S}/S) \ll 1$]. Both ϵ_m and ϵ_g satisfy this criterion. We will restrict our attention to ϵ_m since we find it the most natural gauge-invariant variable to use when describing scalar perturbations (see below). Notice that in any gauge with $B^{(0)}=V^{(0)}$, ϵ_m reduces to δ, and, since it is a gauge invariant, we can say that ϵ_m measures the density perturbation in any such gauge.

The third step in Bardeen's approach is to take linear combinations of the EFE's and the conservation equations, thereby obtaining, for each perturbation type, a basic differential equation, and a set relationships between the gauge invariants. The equations that we shall need are the following:

Scalar perturbations

Basic differential equation

$$\begin{aligned}
&(\mu_0 S^3\epsilon_m)^{\cdot\cdot}+(3\gamma-2)\frac{\dot{S}}{S}(\mu_0 S^3\epsilon_m)^{\cdot}\\
&\quad+\left[(\gamma-1)(k^2-3K)-\frac{\gamma}{2}\mu_0 S^2\right](\mu_0 S^3\epsilon_m)=0 .
\end{aligned} \tag{3.5}$$

Relationships

$$\phi_H = \frac{\mu_0 S^2}{2(k^2-3K)}\epsilon_m \ , \tag{3.6}$$

$$\phi_A = -\phi_H \ , \tag{3.7}$$

$$V_s^{(0)} = -\frac{k(\mu_0 S^3 \epsilon_m)^{\cdot}}{\gamma(k^2-3K)\mu_0 S^3} \ . \tag{3.8}$$

Here and elsewhere we will be assuming $k^2 \neq 3K$ in the scalar perturbation case. According to Lifshitz[1] this mode is *not* physical and so there is no essential loss in generality:

Vector perturbations

Basic differential equation

$$\dot{V}_c - (3\gamma-4)\frac{\dot{S}}{S}V_c = 0 \ . \tag{3.9}$$

Relationship

$$\psi = \frac{2\gamma\mu_0 S^2}{k^2-2K}V_c \ . \tag{3.10}$$

The $k^2=2K$ mode is unphysical.

Tensor perturbations

Basic differential equation

$$\ddot{H}_T^{(2)} + 2\frac{\dot{S}}{S}\dot{H}_T^{(2)} + (k^2+2K)H_T^{(2)} = 0 \ . \tag{3.11}$$

The above equations reduce the number of independent gauge invariants to three, one for each perturbation type. They can be obtained by specializing Bardeen's equations to the case of a perfect fluid with γ-law equation of state.

IV. GEOMETRIC AND KINEMATIC QUANTITIES

In this section, in order to shed light on the physical interpretation of the three types of perturbation, we give the first-order expressions for various geometric and kinematic quantities in terms of the metric perturbations and gauge invariants defined previously.

Direct calculation yields the following expressions for the kinematic quantities of the perturbed fluid congruence, the trace and trace-free parts of the Ricci tensor of the hypersurfaces $\{\eta=\text{const}\}$, the Cotton-York tensor of these hypersurfaces, and the electric and magnetic parts of the Weyl tensor, respectively:

$$\sigma_\alpha{}^\beta = S^{-1}(-kV_s^{(0)}Q^{(0)}{}_\alpha{}^\beta - kV_s^{(1)}Q^{(1)}{}_\alpha{}^\beta + \dot{H}_T^{(2)}Q^{(2)}{}_\alpha{}^\beta) \ , \tag{4.1}$$

$$\theta = 3S^{-1}\left[\frac{\dot{S}}{S} + \left[\dot{H}_L - \frac{\dot{S}}{S}A + \frac{k}{3}V^{(0)}\right]Q^{(0)}\right] , \tag{4.2}$$

$$\dot{u}_\alpha = \left[\dot{V}_s^{(0)} + \frac{\dot{S}}{S}V^{(0)} - k\phi_A\right]Q_\alpha{}^{(0)} + \left[\dot{V}_c + \frac{\dot{S}}{S}V_c\right]Q^{(1)}{}_\alpha \ , \tag{4.3}$$

$$\omega_\alpha{}^\beta = S^{-1}V_c W_\alpha{}^\beta, \quad W_\alpha{}^\beta \equiv \tfrac{1}{2}(Q^{(1)}{}_\alpha{}^{|\beta} - Q^{(1)\beta}{}_{|\alpha}) \ , \tag{4.4}$$

$$R = 3S^{-2}[2K + \tfrac{4}{3}(k^2-3K)(H_L + \tfrac{1}{3}H_T^{(0)})Q^{(0)}] \ , \tag{4.5}$$

$$S_\alpha{}^\beta = S^{-2}[-k^2(H_L + \tfrac{1}{3}H_T^{(0)})Q^{(0)}{}_\alpha{}^\beta + (k^2+2K)H_T^{(2)}Q^{(2)}{}_\alpha{}^\beta] \ , \tag{4.6}$$

$$C^*{}_\alpha{}^\beta = 2(k^2+2K)H_T^{(2)}\eta^{\beta\gamma\delta}Q^{(2)}{}_{\alpha\gamma|\delta} \ , \tag{4.7}$$

$$E_\alpha{}^\beta = \frac{S^{-2}}{2}\{k^2(\phi_A - \phi_H)Q^{(0)}{}_\alpha{}^\beta + k\Psi Q^{(1)}{}_\alpha{}^\beta - [\ddot{H}_T^{(2)} - (k^2+2K)H_T^{(2)}]Q^{(2)}{}_\alpha{}^\beta\} \ , \tag{4.8}$$

$$H_{\alpha\beta} = -S^{-4}(\tfrac{1}{2}\Psi Q^{(1)\gamma}{}_{|(\alpha}{}^{|\delta}\eta_{\beta)0\gamma\delta} + \dot{H}_T^{(2)}Q^{(2)}{}_{(\alpha}{}^{\gamma|\delta}\eta_{\beta)0\gamma\delta}) \ . \tag{4.9}$$

We also recall the definition of the gauge invariant ϵ_m: namely,

$$\epsilon_m = \delta + \frac{3\gamma}{k}\frac{\dot{S}}{S}(V^{(0)} - B^{(0)}) \ . \tag{4.10}$$

The expressions for $\omega_\alpha{}^\beta$, R, and $S_\alpha{}^\beta$ and the scalar and vector contributions to $\sigma_\alpha{}^\beta$ have been given by Bardeen.[2] We note that the nonspatial components of the kinematical quantities and the electric and magnetic parts of the Weyl tensor are zero to first order. This follows from the orthogonality of these quantities with the fluid velocity. We also note that $C^*{}_\alpha{}^\beta$ *is* a gauge invariant (this follows since it contains no scalar or vector contributions and so is unaffected by a gauge transformation), but that $S_\alpha{}^\beta$ and R are, in general, gauge-dependent quantities (see the discussion below).

We see from formulas (4.1)–(4.10) that not all perturbation types (i.e., scalar, vector, tensor) contribute to each of these quantities. The presence or absence of particular perturbation types is summarized in Table I. In particular, *it is only* $\sigma_\alpha{}^\beta$ *and* $E_\alpha{}^\beta$ *that contain all three perturbation types* to first order. It is interesting to note that there are no scalar perturbation contributions to $H_\alpha{}^\beta$, and no contribution to $S_\alpha{}^\beta$ from the vector perturbations.

TABLE I. Contribution to various geometric and kinematic quantities from the different perturbation types.

Quantity	Scalar modes	Vector modes	Tensor modes
ϵ_m	Yes	No	No
$\sigma_\alpha{}^\beta$	Yes	Yes	Yes
$\omega_\alpha{}^\beta$	No	Yes	No
$\dot{u}_\alpha$	Yes	Yes	No
$E_\alpha{}^\beta$	Yes	Yes	Yes
$H_\alpha{}^\beta$	No	Yes	Yes
R	Yes	No	No
$S_\alpha{}^\beta$	Yes	No	Yes
$C^*{}_\alpha{}^\beta$	No	No	Yes

We see from Table I that only the scalar perturbations contribute to ϵ_m and that only vector perturbations contribute to $\omega_\alpha{}^\beta$. For this reason the scalar perturbations are often referred to as "*matter*" perturbations and the vector perturbations are called "*rotational*" perturbations. The tensor perturbations do not directly affect the matter distribution or impact rotation into the fluid. They are usually referred to as pure "*gravitational wave*" perturbations. It has been conjectured by Berger, Eardley, and Olson[10] that there is a link between the Cotton-York tensor of hypersurfaces in spacetime and the presence of gravitational waves. Since only tensor perturbations contribute to the Cotton-York tensor, this conjecture is consistent with the above interpretation of the tensor perturbation modes.

There is a possibility of confusion in the interpretation of the spatial Ricci tensor $R_\alpha{}^\beta$. We first consider its trace-free part $S_\alpha{}^\beta$. In the background spacetime, $S_\alpha{}^\beta=0$, and it might be expected that $S_\alpha{}^\beta$ is gauge invariant. However, $S_\alpha{}^\beta$ is not a *uniquely* defined geometric quantity in *spacetime*; it refers to the family of hypersurfaces $\{\eta=\text{const}\}$ in spacetime, and any change in this family of hypersurfaces (in the perturbed spacetime) will change $S_\alpha{}^\beta$. In the Bardeen formalism, the general infinitesimal gauge transformation can be expressed in the form

$$\eta'=\eta+T(\eta)Q^{(0)} ,$$

$$x'^\alpha=x^\alpha+L^{(0)}(\eta)Q^{(0)\alpha}+L^{(1)}(\eta)Q^{(1)\alpha} .$$

It follows that $S_\alpha{}^\beta$ is gauge invariant under spatial coordinate transformations, but *not* under transformations involving a change in η. This also applies to R. Since the vector and tensor perturbations do not contribute to the η-coordinate transformations, there will be no ambiguity in these cases, and so $S_\alpha{}^\beta$ and R are gauge invariants. In the scalar perturbation case the η-coordinate freedom can be uniquely specified by choosing, for example, a gauge with $B^{(0)}=V^{(0)}$. This choice can always be made and it does indeed specify the η coordinate uniquely. The geometrical significance of this choice is that when the vector perturbations are absent [or if we just consider scalar (tensor) perturbations], the hypersurfaces $\{\eta=\text{const}\}$ are orthogonal to the fluid congruence (since in this case $\omega_\alpha{}^\beta=0=u_\alpha$). However in this gauge it follows from (3.3) and (3.4) that (Bardeen denotes this quantity by ϕ_m [see his Eq. (5.19)])

$$H_L+\tfrac{1}{3}H_T{}^{(0)}=\phi_H-\frac{1}{k}\frac{\dot{S}}{S}V_s{}^{(0)} ,$$

and so Eqs. (4.5) and (4.6) imply that *in this gauge* the scalar contribution to R and $S_\alpha{}^\beta$ is

$$R=3S^{-2}\left[2K+\tfrac{4}{3}(k^2-3K)\left[\phi_H-\frac{1}{k}\frac{\dot{S}}{S}V_s{}^{(0)}\right]Q^{(0)}\right] ,$$

$$S_\alpha{}^\beta=-k^2S^{-2}\left[\phi_H-\frac{1}{k}\frac{\dot{S}}{S}V_s{}^{(0)}\right]Q^{(0)}{}_\alpha{}^\beta ,$$

respectively. We now *define* the quantities $R^*, S^*{}_\alpha{}^\beta$ *in all gauges* by

$$R^*=3S^{-2}\left[2K+\tfrac{4}{3}(k^2-3K)\left[\phi_H-\frac{1}{k}\frac{\dot{S}}{S}V_s{}^{(0)}\right]Q^{(0)}\right] , \tag{4.11}$$

$$S^*{}_\alpha{}^\beta=-k^2S^{-2}\left[\phi_H-\frac{1}{k}\frac{\dot{S}}{S}V_s{}^{(0)}\right]Q^{(0)}{}_\alpha{}^\beta+(k^2+2K)S^{-2}H_T{}^{(2)}Q^{(2)}{}_\alpha{}^\beta . \tag{4.12}$$

In any gauge with $B^{(0)}=V^{(0)}$, R^* and $S^*{}_\alpha{}^\beta$, coincide with R and $S_\alpha{}^\beta$. However, $S^*{}_\alpha{}^\beta$ is a gauge invariant and so in general (i.e., in *all* gauges) gives the trace-free part of the Ricci tensor of the hypersurfaces which are orthogonal to the fluid congruence when the vector perturbations are absent. Thus we have a gauge-invariant measure of the perturbation in the spatial curvature of the $\{\eta=\text{const}\}$ hypersurfaces. Notice that for $K\neq0$, R^* is only gauge invariant under spatial coordinate transformations. When discussing the spatial curvature of the $\{\eta=\text{const}\}$ slices we will always use R^* and $S^*{}_\alpha{}^\beta$.

Remark. Bardeen[2] discusses the *scalar* contribution to the perturbation in a number of families of hypersurfaces including the comoving hypersurfaces which we are considering.

V. GAUGE-INVARIANT CHARACTERIZATION OF THE DIFFERENT PERTURBATION TYPES

We have shown in Sec. III that the EFE's with perfect fluid source reduce the number of independent gauge invariants to three, one for each of the perturbation types. The choice of which independent gauge invariants are used is somewhat arbitrary. We find it convenient to take ϵ_m as our basic gauge invariant in the scalar perturbation case, and V_c in the vector case. The tensor perturbations are described by the gauge invariant $H_T{}^{(2)}$.

We now use the EFE's to express the geometric and kinematic quantities introduced in the previous section in terms of the basic gauge invariants. This enables us to give a gauge-invariant characterization of the different perturbation types. The results of this section are proved under the assumptions of a perfect-fluid source with γ-law equation of state. For brevity this will not be explicitly stated in each theorem. We note, however, that the results are also valid in the case when the equation of state, in both the background and perturbed models assumes the form $p = p(\mu)$ with $\mu + p \neq 0$. It is convenient to consider the three perturbation types separately.

A. Scalar perturbations

In the case of scalar perturbations all of the gauge-invariant geometric and kinematic quantities introduced in the previous section can be expressed in terms of ϵ_m as

$$\sigma_\alpha{}^\beta = \frac{k^2(\mu_0 S^3 \epsilon_m)^\cdot}{(k^2-3K)(\gamma\mu_0 S^4)} Q^{(0)}{}_\alpha{}^\beta , \tag{5.1}$$

$$\dot{u}_\alpha = \frac{k(\gamma-1)}{\gamma} \epsilon_m Q^{(0)}{}_\alpha , \tag{5.2}$$

$$E_\alpha{}^\beta = -\frac{k^2\mu_0}{2(k^2-3K)} \epsilon_m Q^{(0)}{}_\alpha{}^\beta , \tag{5.3}$$

$$R^* = 3S^{-2}[2K + \tfrac{4}{3}(k^2-3K)\phi_m] Q^{(0)} , \tag{5.4}$$

$$S^*{}_\alpha{}^\beta = -k^2 S^{-2} \phi_m Q^{(0)}{}_\alpha{}^\beta , \tag{5.5}$$

$$H_\alpha{}^\beta = 0, \quad \omega_\alpha{}^\beta = 0, \quad C^*{}_\alpha{}^\beta = 0 , \tag{5.6}$$

where

$$\phi_m \equiv \frac{1}{k^2-3K} \left\{ \frac{1}{\gamma} \frac{\dot{S}}{S} \dot{\epsilon}_m + \frac{3}{2} \left[K + \left[\frac{\dot{S}}{S} \right]^2 \frac{2-\gamma}{\gamma} \right] \epsilon_m \right\} . \tag{5.7}$$

Equation (5.1) is an immediate consequence of Eqs. (3.8) and (4.1). The expression for $\dot{u}_\alpha$ can be obtained in a straightforward manner from Eqs. (4.3), (3.5)–(3.8), and Eq. (2.4). Equation (5.3) follows from Eqs. (4.8), (3.6), and (3.7), whereas the expressions for R^* and $S^*{}_\alpha{}^\beta$ are derived using Eqs. (4.11), (4.12), (3.5), (3.7), and (2.3) and (2.4).

We see immediately that ϵ_m has definite geometric significance in spacetime since it completely determines the time dependence of the Weyl tensor. The importance of ϵ_m for scalar perturbations is contained in the following theorem.

Theorem 5.1. There is a scalar perturbation contribution to the perturbed model if and only if $\epsilon_m \neq 0$.

Proof. A direct consequence of Eqs. (5.1)–(5.7).

The following corollary to Theorem 5.1 will be needed later.

Corollary 5.1. If $\sigma_\alpha{}^\beta \equiv 0$, there are no scalar perturbations.

Proof. If $\sigma_\alpha{}^\beta \equiv 0$, Eq. (5.1) implies that $(\epsilon_m \mu_0 S^3)^\cdot = 0$. Combining this with Eq. (3.5) implies that $\epsilon_m \equiv 0$ since $\mu_0 S^3$ is nonconstant [see Eq. (2.4)], and so from Theorem 5.1, there are no scalar perturbations.

B. Vector perturbations

The geometric and kinematic quantities can be expressed in terms of V_c as

$$\sigma_\alpha{}^\beta = -\frac{k}{k^2-2K} S^{-1}[(k^2-2K) - 2\gamma\mu_0 S^2] V_c Q^{(1)}{}_\alpha{}^\beta , \tag{5.8}$$

$$\dot{u}_\alpha = 3(\gamma-1)\frac{\dot{S}}{S} V_c Q^{(1)}{}_\alpha , \tag{5.9}$$

$$\omega_\alpha{}^\beta = S^{-1} V_c W_\alpha{}^\beta , \tag{5.10}$$

$$E_\alpha{}^\beta = -\frac{2k}{k^2-2K}\mu_0 \frac{\dot{S}}{S} V_c Q^{(1)}{}_\alpha{}^\beta , \tag{5.11}$$

$$H_\alpha{}^\beta = -\frac{\gamma\mu_0}{k^2-2K} S^{-2} V_c Q^{(1)\gamma}{}_{|(\alpha}{}^{|\delta} \eta_{\beta)0\gamma\delta} , \tag{5.12}$$

$$\epsilon_m = R^* = 0, \quad S^*{}_\alpha{}^\beta = 0 = C^*{}_\alpha{}^\beta .$$

The essential property that the vector perturbations introduce is *vorticity.* Indeed the presence of vorticity is the characterizing property of the vector perturbations as the following theorem shows.

Theorem 5.2. There is a vector perturbation contribution to the perturbed model if and only if $\omega_\alpha{}^\beta \neq 0$.

Proof. An immediate consequence of Eqs. (5.8)–(5.12) and the fact that the scalar and tensor perturbations do not contribute to $\omega_\alpha{}^\beta$.

The following corollary is an immediate consequence of the previous theorem, Eq. (5.8), and the fact that $\mu_0 S^3$ is nonconstant.

Corollary 5.2. If $\sigma_\alpha{}^\beta \equiv 0$ then there are no vector perturbations.

C. Tensor perturbations

The tensor perturbation contributions to the geometric and kinematic quantities (4.1)–(4.12) are already expressed in terms of the basic gauge invariant $H_T{}^{(2)}$. The only "new" quantity that the tensor perturbations introduce is the Cotton-York tensor. In order that a corresponding result to Theorems 5.1 and 5.2 can be proved we need the following lemma.

Lemma 5.1. The tensor harmonics $Q^{(2)}{}_\alpha{}^\beta$ satisfy

$$\eta^{\alpha\beta\gamma} Q^{(2)}{}_{\mu\beta|\gamma} \neq 0 ,$$

provided

$$k^2 + 3K \neq 0 .$$

Proof. It follows from (2.7) that the tensor harmonics satisfy

$$Q^{(2)}{}_{\alpha\beta|\mu}{}^{|\mu} + k^2 Q^{(2)}{}_{\alpha\beta} = 0 , \tag{5.13}$$

$$Q^{(2)}{}_{\alpha\beta|}{}^\beta = 0 . \tag{5.14}$$

Assume that

$$\eta^{\alpha\beta\gamma} Q^{(2)}{}_{\mu\beta|\gamma} = 0 . \tag{5.15}$$

Contraction of this expression with $\eta_{\sigma\rho\alpha}$ yields

$$Q^{(2)}{}_{\mu\sigma|\rho} - Q^{(2)}{}_{\mu\rho|\sigma} = 0 ,$$

which implies, on taking a covariant derivative and contracting,

$$Q^{(2)}{}_{\mu\sigma|\rho}{}^{|\rho} - Q^{(2)}{}_{\mu\rho|\sigma}{}^{|\rho} = 0 .$$

We now use (5.13) and interchange the order of the covariant derivatives in the second term to obtain

$$k^2Q^{(2)}{}_{\mu\sigma}+(Q^{(2)}{}_{\mu\rho}{}^{|\rho}{}_{|\sigma}+R_{\mu}{}^{\upsilon\rho}{}_{\sigma}Q^{(2)}{}_{\upsilon\rho}+R_{\rho}{}^{\upsilon\rho}{}_{\sigma}Q^{(2)}{}_{\mu\upsilon})=0\ ,$$

where $R_{\alpha\beta\gamma\delta}$ denotes the Riemann tensor in the three-space of constant curvature. Use of (5.14) together with the standard expression for $R_{\alpha\beta\gamma\delta}$ yields

$$(k^2+3K)Q^{(2)}{}_{\mu\sigma}=0\ . \tag{5.16}$$

Thus, provided that $(k^2+3K)\neq 0$, (5.16) only has the trivial solution $Q^{(2)}{}_{\mu\sigma}=0$, and hence all nontrivial tensor harmonics satisfy

$$\eta^{\alpha\beta\gamma}Q^{(2)}{}_{\mu\beta|\gamma}\neq 0\ .$$

We can now prove the following.

Theorem 5.3. There is a tensor perturbation contribution to the perturbed model if and only if $C^*{}_{\alpha}{}^{\beta}\neq 0$.

Proof. For tensor modes it follows from Sec. II that $k^2+3K\neq 0$. The result now follows from the previous lemma, Eq. (4.7), and the fact that the vector and tensor harmonics do not contribute to the Cotton-York tensor.

Remark. The above result certainly lends support to the conjecture of Berger, Eardley, and Olson mentioned in the previous section.

The following theorem is also easily proved.

Theorem 5.4. If $\sigma_{\alpha}{}^{\beta}\equiv 0$ there are no tensor perturbations.

Proof. Suppose that $\sigma_{\alpha}{}^{\beta}\equiv 0$. Then Eq. (4.1) implies that $\dot{H}_T{}^{(2)}\equiv 0$, and so from Eq. (3.11), $H_T{}^{(2)}\equiv 0$, since $k^2+2K\neq 0$ for tensor perturbations (see Sec. II).

The results in Secs. V A–V C above can be combined into the following two theorems, the first of which shows the importance of the rate of shear tensor in the perturbed model, and the second giving a gauge-invariant characterization of the perturbation types.

Theorem 5.5. There is a nontrivial perturbation to first order if and only if the rate of shear tensor of the perturbed congruence is nonzero.

Proof. The result follows directly from Corollaries 5.1, 5.2, and Theorem 5.4.

Remarks. In the above theorem, "nontrivial perturbation" means a true, physical perturbation (as opposed to a pure gauge solution).

We note that the previous theorem holds with the shear tensor replaced by the electric part of the Weyl tensor. However, since none of the other geometric and kinematic quantities considered here have contributions from all three perturbation types, it follows that only $\sigma_{\alpha}{}^{\beta}$ and $E_{\alpha}{}^{\beta}$ can be used to characterize a nontrivial perturbation.

In the exact theory the FRW models can be characterized by either of the following conditions (Ellis[11]): (i) $\sigma_a{}^b=0$, $\omega_a{}^b=0$, $\dot{u}_a=0$; (ii) $E_a{}^b=H_a{}^b=0$, assuming an equation of state of the form $p=p(\mu)$. Theorem 5.5 and the remarks of the previous paragraph show how these characterizations can be weakened in the linearized theory.

Finally, we give a gauge-invariant characterization of the different perturbation types.

Theorem 5.6. The different perturbation types can be characterized in the following gauge-invariant manner: (i) Purely scalar perturbations if and only if $\omega_{\alpha}{}^{\beta}=C^*{}_{\alpha}{}^{\beta}=0$; (ii) purely vector perturbations if and only if $\epsilon_m=C^*{}_{\alpha}{}^{\beta}=0$; (iii) purely tensor perturbations if and only if $\epsilon_m=\omega_{\alpha}{}^{\beta}=0$.

Proof. Immediate consequence of Theorems 5.1–5.3.

Remarks. Different gauge-invariant conditions characterizing the tensor perturbations have been given by other authors (see, for example, Hawking[12] and Niedra[13]).

VI. EXACT SOLUTIONS OF THE PERTURBATION EQUATIONS WHEN $K=0$

The solutions of the perturbation equations have been given by a number of authors using various gauges and under a number of different assumptions regarding the sign of the background curvature or the particular γ-law equation of state. When the background curvature is nonzero, the general solutions of these equations for general γ are not known; usually some sort of approximation has to be used in solving the equations (see, for example, Lifshitz,[1] Lifshitz and Khalatnikov[7]). However, if $K=0$, then the solutions of the perturbation equations are known for general γ-law equation of state.

In this section we give the solutions of the basic differential equations (3.5), (3.9), and (3.11), for the $K=0$ background model. Thus, except where stated otherwise, we will assume that *our background model is a $K=0$ FRW model with γ-law equation of state.* Some discussion is given of the contribution to the various geometric and kinematic quantities considered previously.

The solutions of (3.5), (3.9), and (3.11) in the case of a $K=0$ background model with γ-law equation of state are as follows.

1. Scalar perturbations

Equation (3.5) has the general solution

$$\epsilon_m=\tau^{2-\nu}[C_+J_\nu(c_s\tau)+C_-N_\nu(c_s\tau)],\quad \gamma\neq 1\ , \tag{6.1}$$

$$\epsilon_m=C_+\eta^2+C_-\eta^{-3},\quad \gamma=1\ , \tag{6.2}$$

where

$$c_s{}^2\equiv\gamma-1,\quad \tau\equiv k\eta,\quad \nu=\frac{3\gamma+2}{2(3\gamma-2)}\ , \tag{6.3}$$

and C_+,C_- are arbitrary constants. Here and elsewhere J_ν and N_ν denote the Bessel functions of the first and second kinds, of order ν.

2. Vector perturbations

Equation (3.9) has the general solution

$$V_c=C_1S^{(3\gamma-4)}\ , \tag{6.4}$$

where C_1 is an arbitrary constant and $S\propto\eta^{2/(3\gamma-2)}$ [see Eq. (3.3) with $K=0$].

3. Tensor perturbations

Equation (3.11) has the general solution

$$H_T{}^{(2)}=\tau^{1-\nu}[C_+J_{\nu-1}(\tau)+C_-N_{\nu-1}(\tau)]\ , \tag{6.5}$$

where C_+, and C_- are arbitrary constants (we note that the $C_\pm$ arising in the scalar case and the $C_\pm$ arising in the tensor case are independent) and τ and ν are given in (6.3).

The solutions (6.1) and (6.2) have been given by Bardeen.[2] Solutions of the vector and tensor perturbation equations (as well as the scalar case) have been given, for example, by Lifshitz,[1] Lifshitz and Khalatnikov,[7] and Weinberg.[3] Most attention has been concentrated on the *scalar* perturbations.

The solutions (6.1)–(6.5) together with the expressions given in the previous section allow us to write down the explicit time dependence of the geometric and kinematic quantities considered. In the scalar ($\gamma \neq 1$) and tensor cases, due to the behavior of the Bessel functions, this time dependence will in general be oscillatory, although for small values of their argument the oscillatory behavior of the Bessel functions can be approximated by a power law. In the next section we will discuss the behavior of the geometric and kinematic quantities in this power-law regime. However we first comment on their general behavior.

In general the contribution from the scalar and tensor perturbations to the quantities given in Sec. IV will contain two independent time modes arising from the constants C_+, and C_-. An exception arises in the scalar case regarding the spatial Ricci tensor of the hypersurfaces orthogonal to the fluid flow. When $\gamma=1$ we find that only the C_+ mode in ϵ_m contributes to R^* and $S^*{}_\alpha{}^\beta$. In fact,

$$R^* = 40C_+S^{-2}Q^{(0)}, \quad S^*{}_\alpha{}^\beta = 10C_+S^{-2}Q^{(0)}{}_\alpha{}^\beta , \qquad \gamma=1 . \tag{6.6}$$

Thus in the case of *dust*, only one mode of the scalar perturbation contributes to the spatial curvature. We note that the C_+ mode in Eq. (6.2) increases into the future, so that the expressions (6.6) support the conclusion reached by Liang[14] (based on the results of Eardley, Liang, and Sachs,[15] and Liang[16]) that the increasing mode in $\delta\mu$ arises from primordial curvature fluctuations. Using elementary properties of the Bessel functions it is straightforward to show that in general the scalar $\gamma \neq 1$) and tensor perturbation contributions to $S^*{}_\alpha{}^\beta$ are, respectively,

$$S^*{}_\alpha{}^\beta = \frac{2k^2(\gamma-1)}{\gamma(3\gamma-2)} S^{-2}\tau^{1-\nu}[C_+J_{\nu-1}(c_s\tau) + C_-N_{\nu-1}(c_s\tau)]Q^{(0)}{}_\alpha{}^\beta , \qquad \gamma \neq 1 ,$$

$$S^*{}_\alpha{}^\beta = k^2S^{-2}\tau^{1-\nu}[C_+J_{\nu-1}(\tau) + C_-N_{\nu-1}(\tau)]Q^{(2)}{}_\alpha{}^\beta .$$

Thus, in general there *will* be two independent contributions from each perturbation type. It should be noticed that although the order of the Bessel functions is the same in the above expressions, the arguments of these functions are different, so that the time dependence does not coincide in general. The exceptional case is when there is a stiff equation of state ($c_s{}^2=1$).

We remark that in the stiff case the time dependence of the *tensor* contribution to σ/θ coincides with the time dependence of ϵ_m; again this is a curious result. Finally we also note that the time dependence of the vector perturbation modes is independent of k (Lifshitz[1] was the first to note that the time dependence of the metric perturbations in the vector perturbation case are independent of k) and power law in *general* as follows from Eq. (6.4).

VII. TIME DEPENDENCE OF GAUGE-INVARIANT GEOMETRIC AND KINEMATIC QUANTITIES WHEN $K=0$ AND $\tau \ll 1$

As noted in the previous section, the Bessel functions arising in Eqs. (6.1) and (6.5) have power-law behavior for small values of their argument. Specifically,

$$J_\rho(\tau) \propto \tau^\rho, \quad N_\rho(\tau) \propto \tau^{-\rho} , \tag{7.1}$$

for $\tau \ll 1$. The corresponding contributions to the geometric and kinematic quantities from the scalar and tensor perturbations will also be power law. (Note that we are again restricting our attention to the $K=0$ background model.) Rather than give the time dependence of the geometric and kinematic quantities themselves, we have formed dimensionless ratios of (scalars formed from) these quantities with the rate of expansion scalar. This enables the relative dynamical significance of the quantities to be compared. The time dependence of the scalars themselves can be recovered using

$$\theta_0 \propto \eta^{-3\gamma/(3\gamma-2)} ,$$

where θ_0 ($=3\dot{S}/S^2$) denotes the background value of θ. (We only need the background value since all of the scalars considered are first-order quantities.) The results of a straightforward calculation are given in Table II. The various scalars appearing in this table are defined as

$$F = \tfrac{1}{2}\sqrt{F^{i\cdots j}F_{i\cdots j}} .$$

Before discussing the results we make some important remarks regarding the table.

In the scalar and tensor cases, a typical time dependence is of the form $\{C_+t^n, C_-t^m\}$. By this we mean that the time dependence of the corresponding quantity (in the scalar or tensor case) is a linear combination of the two given modes, the constant coefficients in this linear combination being independent of the separation constant k. The constants C_+ and C_- serve only to identify the origin of the corresponding terms [cf. Eqs. (6.1) and (6.5)]. For example, when $\tau \ll 1$, the scalar perturbation contribution to σ/θ is, to leading order in τ,

$$\sigma \approx \frac{1}{\sqrt{2}} \left| -\frac{1}{3\gamma}C_+\tau^2 + \frac{3}{2}C_-\tau^{3(\gamma-2)/(3\gamma-2)} \right| \sqrt{Q^0{}_{\alpha\beta}Q^{(0)\alpha\beta}} ,$$

where C_+ and C_- have been rescaled over their values in Eq. (6.1). We thus write this time dependence in Table II as $\{C_+\tau^2, C_-\tau^{3(\gamma-2)/(3\gamma-2)}\}$. We note that in the scalar case the results given in Table II hold for $c_s\tau \ll 1$ [see Eqs. (6.1) and (7.1)], whereas in the tensor case the results are valid for $\tau \ll 1$ [cf. Eqs. (6.5) and (7.1)]. It follows in particular that for any fixed k the results will be valid for η sufficiently small.

TABLE II. Behavior of geometric and kinematic quantities when $\tau \ll 1$. Here

$$p \equiv \frac{3(2-\gamma)}{3\gamma-2}, \quad q \equiv \frac{2(3\gamma-4)}{3\gamma-2}, \quad r \equiv \frac{9\gamma-10}{3\gamma-2}, \quad s \equiv \frac{12(\gamma-1)}{3\gamma-2}.$$

Quantity	Scalar contribution	Vector contribution	Tensor contribution
ϵ_m	$\{C_+\tau^2, C_-\tau^{-p}\}$	0	0
$\frac{\sigma}{\theta}$	$\{C_+\tau^2, C_-\tau^{-p}\}$	$\{\tau^{-p}\}$	$\{C_+\tau^2, C_-\tau^{-p}\}$
$\frac{\omega}{\theta}$	0	$\{\tau^r\}$	0
$\frac{\dot{u}}{\theta}$	$(\gamma-1)\{C_+\tau^3, C_-\tau^q\}$	$(\gamma-1)\{\tau^q\}$	0
$\frac{E}{\theta^2}$	$\{C_+\tau^2, C_-\tau^{-p}\}$	$\{\tau^{-p}\}$	$\{C_+\tau^2, C_-\tau^{-p}\}$
$\frac{H}{\theta^2}$	0	$\{\tau^q\}$	$\{C_+\tau^3, C_-\tau^q\}$
$\frac{R^*}{\theta^2}$	$\{C_+\tau^2, C_-\tau^r\}$, $\gamma\neq 1$ $\{C_+\tau^2, 0\}$, $\gamma=1$	0	0
$\frac{S^*}{\theta^2}$	$As\frac{R^*}{\theta^2}$	0	$\{C_+\tau^2, C_-\tau^r\}$
$\frac{C^*}{\theta^2}$	0	0	$\{C_+\tau^3, C_-\tau^s\}$

The time dependence in the vector perturbation case, except that of σ/θ, holds in general, that is, independently of the assumption that $\tau \ll 1$. The time dependence given for σ/θ *does* depend on the assumption $\tau \ll 1$ [see Eq. (5.8)]. Note that the constant of proportionality factors omitted in the vector perturbation case in Table II are dimensionless and that they do, in general, contain the separation constant k. This follows since, as noted in the previous section, the time dependence of V_c is independent of k. We discuss the results contained in Table II for the scalar, vector, and tensor perturbations separately.

1. *Scalar perturbations*

The time dependence of ϵ_m consists of two modes, one $(C_+\tau^2)$ which increases into the future, and one (C_-t^{-r}) which decreases into the future (except in the case of stiff matter when it is a constant mode). These two modes are usually referred to as the (relatively) *increasing* (C_+) and (relatively) *decreasing* (C_-) *modes*, respectively (Liang[14] and Bardeen[2]). We note, however, that the density contrast $\delta\mu$, will not, in general, contain two such modes since the behavior of the independent modes is gauge dependent. For example, in the synchronous gauge when $p=\mu/3$ the corresponding modes in $\delta\mu$ are $\{t, t^{1/2}\}$, both of which increase into the future.

We see from Table II that all of the nonzero quantities, except $\dot{u}/\theta$, contain the *same* increasing mode. However, the behavior of the C_- mode in R^*/θ^2, S^*/θ^2, and $\dot{u}/\theta$ is dependent on the equation of state. Indeed for $\gamma > \frac{10}{9}$ the C_- mode in R^*/θ^2, S^*/θ^2 is an *increasing* mode, and is a constant mode when $\gamma = \frac{10}{9}$. The behavior of the C_- mode in $\dot{u}/\theta$ is the same about $\gamma = \frac{4}{3}$. The C_- mode in all other nonzero quantities coincides with that of ϵ_m.

The result that $\epsilon_m \approx \sigma/\theta$ for $\tau \ll 1$ was first discovered by Liang (in fact Liang[14] considered the density contrast $\delta\mu$, but used a gauge in which $\delta\mu$ and ϵ_m coincide) and has been discussed, for example, by Bardeen.[2] We see from Table II that E/θ^2 also has the same time dependence as ϵ_m and σ/θ. [That E/θ^2 and ϵ_m have the same time dependence actually follows directly from Eq. (5.3).]

2. *Vector perturbations*

As in the scalar case, whether or not the vector contribution to a quantity increases or decreases into the future depends on the equation of state. It is remarkable that when $\tau \ll 1$ the exponent in the power of τ of the vector perturbation contribution to any quantity coincides with the exponent of τ of the C_- mode of the scalar perturbation contribution to that quantity. The two "new" quantities that arise from the inclusion of vector perturbations are $\omega_\alpha{}^\beta$ and $H_\alpha{}^\beta$. We see from Table II that the time dependence of H/θ^2 coincides with the vector contribution (and C_--mode scalar contribution) to $\dot{u}/\theta$ (provided $\gamma \neq 1$). As far as the vorticity is concerned, the time dependence of ω/θ coincides with the C_--mode scalar perturbation contribution to the spatial curvature (providing $\gamma \neq 1$). It is worth noting that the vorticity scalar itself has time dependence given by

$$\omega \approx \eta^{2(3\gamma-5)/(3\gamma-2)} ,$$

which decreases into the future for $\gamma < \frac{5}{3}$, is constant if $\gamma = \frac{5}{3}$, and increases into the future if $\gamma > \frac{5}{3}$. [This time dependence has been derived (using different arguments) by, for example, Barrow and Tipler[17]].

3. *Tensor perturbations*

Once more any nonzero tensor mode coincides with the corresponding nonzero scalar and vector modes. In particular the shear tensor has an increasing (C_+) and decreasing (C_-) mode. The Cotton-York tensor is the only quantity that contains purely tensor perturbations. We see that the C_+-mode contribution to C^*/θ^3 is an increasing mode, and that the C_--mode contribution is *also* an increasing mode for $\gamma \neq 1$. When $\gamma = 1$ the C_--mode contribution to C^*/θ^3 is a constant mode. The Cotton-York tensor itself contains two decreasing modes for all γ: namely,

$$\{ C_+ \tau^{-6/(3\gamma-2)}, C_- \tau^{3(\gamma-4)/(3\gamma-2)} \} .$$

Finally we note that the results of Table II can be expressed in terms of comoving proper time t in the background spacetime using

$$\eta \propto t^{(3\gamma-2)/3\gamma} .$$

VIII. CONCLUSION

In this paper we have used Bardeen's gauge-invariant formalism to derive a cosmologically oriented characterization of the different perturbation modes that can arise within the linearized theory and have also analyzed the relationship between various geometric and physical quantities at the linear level. In conclusion we will indicate how the results derived in Secs. III–VII can be useful in the interpretation of exact solutions of the EFE's.

Consider the $k=0$ class II Szekeres[18] solutions of the EFE's with *irrotational* dust as source. These solutions have been analyzed in detail by Goode and Wainwright.[19] The line element can be written in the form (Goode and Wainwright[19])

$$ds^2 = -dt^2 + t^{4/3}(dx^2 + dy^2 + H^2 dz^2) , \tag{8.1}$$

where

$$H = A - \tfrac{9}{10}\beta_+ t^{2/3} - \tfrac{8}{9}\beta_- t^{-1} \tag{8.2}$$

and

$$A = 1 - \tfrac{1}{2}\beta_+[(x-\alpha)^2 + (y-\delta)^2] . \tag{8.3}$$

In (8.2) and (8.3) $\beta_\pm$, α, and δ are sufficiently smooth functions of z. Relative to these coordinates the fluid four-velocity and energy density are, respectively,

$$u = \frac{\partial}{\partial t}, \quad \mu = \frac{4A}{3H} t^{-2} ,$$

and the Einstein–de Sitter model arises when $\beta_\pm \equiv 0$. Consider a time interval $0 < t_1 \le t \le t_2$ and suppose that $\beta_\pm$ are restricted by

$$\beta_\pm \le 0, \quad 0 \le |\beta_+| t_2{}^{2/3} \le \epsilon, \quad 0 \le |\beta_-| t_1{}^{-1} \le \epsilon ,$$

where t_1, t_2, and ϵ are positive constants. It has been shown (Goode and Wainwright[19]) that, when $\epsilon \ll 1$, the above solution can be considered as being, in a well-defined sense, a perturbation of the Einstein–de Sitter model. In the linear approximation (defined by $\epsilon \ll 1$) it is easily shown (using the formulas given in Goode and Wainwright[19]) that the time dependence of all of the quantities in the above solution coincides with that given in the first column of Table II of the previous section. Further, in all of the Szekeres solutions, the slices orthogonal to the fluid flow are conformally flat (Berger, Eardley, and Olson,[10] and Szafron and Collins[20]). Thus in these solutions we have $\omega_a{}^b = C^*{}_a{}^b = 0$, and so the gauge-invariant characterization given in Theorem 5.6 suggests that the above solution should be interpreted as a scalar perturbation of the Einstein–de Sitter model. A similar interpretation is also possible for the remaining Szekeres models. This interpretation is certainly consistent with the results of Bonnor[21] on the nonradiative property of the Szekeres solutions.

It is reassuring that the results from the linearized theory are reflected in the linearization of exact solutions. It is worth mentioning, however, that the above solution does not satisfy the basic assumption of the linearized theory, namely, that $\tilde{g}_{ab} = g_{ab} + h_{ab}$ with $|h_{ab}| \ll 1$, relative to the present coordinates. Indeed, from (8.1)–(8.3), we see that, even in the linear approximation mentioned above, the perturbed and background metrics are diverging as $x^2 + y^2 \rightarrow +\infty$.

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