# Exact Langevin equation in a cosmological setting

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Recently, various authors have used Langevin equations to model physical processes in the early Universe. Much of this work is based on a flat-space effective potential, neglecting potential complications associated with a curved spacetime. This paper considers a simple statistical-mechanical model of time-dependent, bilinearly coupled oscillators, which leads to an exact Langevin equation that mocks a field theory in a time-dependent background. This equation exhibits a nonlocal time-dependent change in the effective potential which vanishes identically when the oscillators are time independent.

## I. INTRODUCTION

Langevin equations and their associated Fokker-Planck realizations have been used for many years in statistical physics to model various dynamical processes which include dissipative effects. In the past several years, these equations have also made their appearance in quantum field theory. One obvious example is stochastic quantization.[1] Another more recent application has been to problems involving quantum field theory in the early Universe, impacting, e.g., on phase transitions or Starobinskii's paradigm of stochastic inflation.[2] Closely related to this are phenomenological models which it is believed could mimic the "dynamical relaxation" of a quantum system in a cosmological setting.[3] And finally, there is a recent suggestion that simple statistical models of a coupled "subsystem" and "bath" could shed light on the "quantum-to-classical transition" in the early Universe.[4]

Much of the aforementioned work in a cosmological context is based on flat-space field theory, the effects of an expanding universe being allowed for only in a phenomenological way. However, it has been argued that flat-space effective potentials, upon which these analyses are based, could in fact miss important curved-space contributions.[5]

To assess the validity of such criticisms, it is very useful to have some model, however simplistic, in which an exact Langevin equation can be derived without any phenomenological assumptions. One such model, which permits the derivation of an exact Langevin equation in a time-independent setting, has been considered by several authors[6,7] in the statistical-mechanics literature. In its simplest form, this model considers a test particle immersed in a "bath" of field oscillators to which it is coupled via a bilinear mode-mode coupling.

The purpose of this paper is to generalize Zwanzig's[7] treatment of this model by allowing for explicit time dependences in the oscillator frequencies and mode-mode couplings. With suitable choices for these time dependences, this should enable one to model in a reasonable way the modes of some field in an expanding universe. It will be seen that this time dependence does in fact lead to a new, and potentially significant, effect, namely, a finite shifting of the effective potential, which would not have been incorporated into the most naive generalizations of an ordinary Langevin equation.

Section II of this paper considers this generalized model for generic time-dependent couplings and frequencies. Section III then discusses possible applications of the model in a cosmological setting.

## II. A SIMPLE MODEL

The model to be analyzed here is a variant of one that has been studied extensively in the past, in which one considers a system of interest, the "subsystem," interacting with a "bath" of oscillators. Previous studies have focused on such problems as the evolution of the dynamical variables of the subsystem,[6] and on the derivation of nonlinear Langevin equations. It is this latter aspect, considered several years ago by Zwanzig,[7] which is of primary interest here. Specifically, what Zwanzig studied was the evolution of a test particle in an arbitrary potential, immersed in a sea of oscillators to which it is connected via a time-independent, bilinear mode-mode coupling. His principal result was the derivation of an exact nonlocal Langevin equation for the test particle which, in certain cases, could be well approximated by a Markovian equation. The object here is to generalize this simple model to a situation in which all the potentials and mode couplings are permitted to manifest a more or less arbitrary time dependence.

Consider a system characterized by the Hamiltonian

$$H=\tfrac{1}{2}v^2+U(x,t)+\sum_k \tfrac{1}{2}p_k^2+\sum_k \tfrac{1}{2}\omega_k^2(q_k-\gamma_k x/\omega_k^2)^2 \,, \tag{1}$$

where $\omega_k$ and $\gamma_k$ are arbitrary smooth real functions of time $t$, and the potential $U(x,t)$ is also allowed an explicit time dependence. This leads immediately to equations of motion

$$\dot{x}=v\ ,$$
$$\dot{v}=-\partial U/\partial x+\sum_k \gamma_k(q_k-\gamma_k x/\omega_k^2)\ ,$$
$$\dot{q}_k=p_k\ ,$$

and (2)

$$\dot{p}_k=-\omega_k^2 q_k+\gamma_k x\ .$$

The object here is to solve formally for the evolution of each $q(t)$ in terms of $x$ and $v$ at past times $s<t$, and, by inserting that formal solution back into the equation for $\dot{v}(t)$, to obtain a nonlocal evolution equation.

Let $S_k(t)$ and $C_k(t)$ denote two linearly independent solutions to the homogeneous equation

$$\frac{d^2\psi_k(t)}{dt^2}+\omega_k^2(t)\psi_k(t)=0\ , \tag{3}$$

which, without loss of generality, may be assumed to satisfy initial conditions $C_k(0)=\dot{S}_k(0)=1$ and $S_k(0)=\dot{C}_k(0)=0$. If the $\omega_k$'s were time independent, these would simply be sines and cosines. And, presuming that $\omega_k\neq 0$, but that $|\dot{\omega}_k|$ is not too large, they will still be oscillatory. In terms of $S_k$ and $C_k$, one then verifies exactly that

$$q_k(t)=q_k(0)C_k(t)+p_k(0)S_k(t)+\int_0^t ds\,\gamma_k(s)x(s)[S_k(t)C_k(s)-S_k(s)C_k(t)]\ . \tag{4}$$

By substituting (4) into the equation for $\dot{v}(t)$, one obtains the desired Langevin equation.

The integral in (4) contributes to the "memory term" in the equation for $\dot{v}(t)$, and, as such, it is inconvenient that the integrand vanishes in the coincidence limit $s\to t$. This difficulty can be avoided by replacing $S_k(s)$ and $C_k(s)$ by $-\ddot{S}_k/\omega_k^2$ and $-\ddot{C}_k/\omega_k^2$ and then integrating by parts. By grouping terms together suggestively, one then sees that

$$\dot{v}(t)=-\frac{\partial U(x(t),t)}{\partial x}-\int_0^t ds[K(t,s)v(s)-M(t,s)x(s)]+F_s(t)\ , \tag{5}$$

where the "stochastic force"

$$F_s(t)=\sum_k \gamma_k(t)[Q_k(0)C_k(t)+p_k(0)S_k(t)] \tag{6}$$

involves only initial conditions at time $t=0$. Here the quantity

$$Q_k(0)\equiv q_k(0)-\gamma_k(0)x(0)/\omega_k^2(0)\ . \tag{7}$$

In terms of the Wronskian of solutions

$$W_k(s,t)\equiv\dot{S}_k(s)C_k(t)-\dot{C}_k(s)S_k(t)\ , \tag{8}$$

and the function

$$A_k(s,t)=\frac{\gamma_k(s)\gamma_k(t)}{\omega_k^2(s)} \tag{9}$$

and its time derivative

$$B_k(s,t)=\frac{\partial}{\partial s}A_k(s,t)\ , \tag{10}$$

the memory functions then assume the forms

$$K(s,t)=\sum_k W_k(s,t)A_k(s,t) \tag{11}$$

and

$$M(s,t)=-\sum_k W_k(s,t)B_k(s,t)\ . \tag{12}$$

Note that, whereas $M(t,s)$ vanishes when $\dot{\gamma}_k=\dot{\omega}_k=0$, $K(t,s)$ does not. Indeed, in this limit (5) reduces to the Zwanzig[7] form

$$\dot{v}(t)=-\frac{\partial U(x(t),t)}{\partial x}-\int_0^t ds\,K(t-s)v(s)+F_s(t)\ , \tag{13}$$

where

$$K(t-s)=\sum_k(\gamma_k^2/\omega_k^2)\cos\omega_k(t-s) \tag{14}$$

and

$$F_s(t)=\sum_k\gamma_k\left[Q_k(0)\cos\omega_k t+p_k(0)\frac{\sin\omega_k t}{\omega_k}\right]\ . \tag{15}$$

By contrasting Eqs. (5) and (13), one sees that time-dependent frequencies and mode-mode couplings induce a systematic but nonlocal $x$-dependent force which, in a field-theoretic setting, could be interpreted as a time-dependent effective mass (although this "mass" need not be real).

Consider, by way of illustration, an ensemble of initial conditions for which

$$\langle Q_k(0)\rangle=\langle p_k(0)\rangle=0\ , \tag{16}$$

so that $\langle F_s(t)\rangle\equiv 0$, and suppose further that the second moments of $p_k$ and $Q_k$ are initially thermal, so that

$$\langle p_j(0)p_k(0)\rangle=\omega_j(0)\omega_k(0)\langle Q_j(0)Q_k(0)\rangle=k_B T\delta_{jk} \tag{17}$$

and

$$\langle Q_j(0)p_k(0)\rangle\equiv 0\ . \tag{18}$$

It then follows exactly that

$$\langle F_s(t)F_s(s)\rangle=\sum_k\gamma_k(t)\gamma_k(s)k_BT\left[\frac{C_k(t)C_k(s)}{\omega_k^2(0)}+S_k(t)S_k(s)\right]\ . \tag{19}$$

In the limit that $\gamma_k$ and $\omega_k$ are independent of time, one sees that the autocorrelation function (19) is in fact a function only of $|t-s|$, and indeed, one concludes that

$$\langle F_s(t)F_s(s)\rangle = k_B T K(t-s) \ . \tag{20}$$

This implies that the model system under consideration here would in fact satisfy a fluctuation-dissipation theorem, the autocorrelation function (20) providing a "diffusion" that precisely balances the "dynamical friction" in (12) (Ref. 8). The existence of such a fluctuation-dissipation theorem, and even the fact that (20) is a function only of $|t-s|$, is, however, a special consequence of time-translation invariance, so that, in general, when $\dot{\gamma}_k$ and/or $\dot{\omega}_k \neq 0$ and the Hamiltonian is not time independent, no simple analogue of (20) will obtain.

The exact equation (5) can simplify significantly in the limit that $S_k(s)$ and $C_k(s)$, and hence $W(s,t)$, is a rapidly oscillating function of $s$. Specifically, if the $S_k$'s and $C_k$'s oscillate rapidly and the distribution of frequencies is "generic," one might anticipate that $K(t,s)$ and $M(t,s)$ will be non-negligible only when $|t-s|$ is small. It then seems natural to *assume* that

$$K(t,s) \simeq K(t)\delta(t-s)$$

and (21)

$$M(t,s) \simeq M(t)\delta(t-s) \ ,$$

in which case one is led to a "local" equation of the form

$$\dot{v}(t) = -\frac{\partial U(x(t),t)}{\partial x} + M(t)x(t) - K(t)v(t) + F_s(t) \ . \tag{22}$$

The further assumption that $\langle F_s(t)F_s(s)\rangle$ may also be treated as nearly delta correlated in time, i.e., proportional to $\delta(t-s)$, then implies a truly Markovian description. (Strictly speaking, the validity of such a Markov approximation must of course be checked explicitly for any given choice of frequencies and couplings.) Equation (22) is an approximate Langevin equation involving a new effective potential $U_{\text{eff}} = U - \frac{1}{2}M(t)x^2$ and a time-dependent viscosity term $K(t)v$. Note that, whereas $K$ is intrinsically positive, the correction to the effective potential can be either positive or negative, depending on the detailed time dependence of the $\dot{\gamma}_k$'s and $\dot{\omega}_k$'s.

It is clear that the initial thermal distribution assumed above for the "bath" of oscillators may not be trivially justifiable for a quantum field in the early Universe. However, one might argue plausibly, in the spirit, e.g., of Vilenkin and Ford,[9] that particles created very early on will thermalize quickly, and that this model will be reasonable after this thermalization.[10] Note also that, although this model is classical, its quantum generalization is completely straightforward: all that one need do is reinterpret the dynamical variables as Heisenberg operators. In this case, the formal dynamics are essentially unchanged although, in the low-temperature limit, "quantum fluctuations" analogous to those described by Ford, Kac, and Mazur[6] can lead to qualitatively different effects.

## III. DISCUSSION

The model considered in this paper, albeit simplisitc, appears robust with respect to the solutions obtained in the senses (1) that no explicit form was assumed for the potential $U$ and (2) that one did not need to solve explicitly for the modes of the field.

The key point simply is that time dependences in the frequencies and couplings will lead generically to potentially significant effects which could alter predictions based on a naive "jazzing-up" of a time-independent model. Thus, e.g., if the potential $U$ is fine tuned to yield a "slow roll" or some other desired features, this fine-tuning could be lost completely in the effective $U_{\text{eff}} = U - \frac{1}{2}Mx^2$ (recall, e.g., that the fine-tuned effective potential in inflationary scenarios must be assumed to be *very* flat). This suggests, but certainly does not prove, that the study of phase transitions and the like in the framework of quantum field theory in a time-dependent background may, as has been argued before,[5] require more than the most obvious extensions of earlier work in flat spacetimes.

In this regard, it is useful to comment on the issue of how some simple analogue of this model might be realized in a realistic cosmological setting. The obvious point is that, for a homogeneous and isotropic cosmology, such as that assumed for the simplest power-law or exponential inflaton, the "natural" modes of a free field evolve independently. There is no mode-mode coupling at all.

Aside from allowing simply for nonlinearities in the field equation, which will not lead to the sort of bilinear mode-mode couplings assumed in Sec. II, there are at least two mechanisms by which the desired couplings could arise. (1) Following Vilenkin and Ford[9] or Brandenberger,[3] one can consider a pair of bilinearly interacting fields, a "bath field" $\Phi$ and a "subsystem field" $\psi$. With de Sitter space as the background spacetime, this may be interpreted as modeling the interaction of an "inflaton field" with some background radiation. (2) Alternatively, if the Universe is not static up to an overall scale factor (as it is for a homogeneous and isotropic universe), perturbations therefrom will induce effective mode-mode couplings of this general form.[11]

One may note that this sort of Langevin equation can also be derived quite generally,[12] for more or less arbitrary initial conditions, by considering a free field in an inflating universe and, following Starobinskii,[2] by introducing a time-dependent splitting of the field into short- and long-wavelength components. Such a setting is, however, very different from that considered here, the effective interactions arising in that case as a consequence of the fact that what one means by "system" or "bath" is in fact time dependent.

Note also that, although this has not been done here, it is straightforward to use the Markov limit (22) of the Langevin equation (5) to formulate a Fokker-Planck description. This would seem a worthwhile exercise if the model were more constrained, i.e., if $U$ were chosen for some specific inflationary model and realistic values for other parameters such as the $\gamma_k$'s were known.

And note finally that the principal limitation of the toy model considered here, namely, the linearity of the oscillator equations of motion and the simple bilinearity of the couplings between the "subsystem" and the "bath," can, at least in principle, be relaxed. This may be of fundamental importance since the model Hamiltonian (1) implies, perhaps unrealistically for a quantum field in the early Universe, that the "bath field" is essentially free and linear. It has been assumed (cf. Ref. 2) that this should be approximately true for the shorter-wavelength modes even for a nonlinear field theory (e.g., a $\lambda\Phi^4$ theory) but that this assumption can be justified is not completely obvious.

In any event, what really underlay the possibility of formulating the desired Langevin equation here was the implementation of a formal splitting of the system into "subsystem" plus "bath," each of which satisfies a closed "subdynamics," whereby nonlocal evolution equations for $\{x(t),v(t)\}$ involve $\{q_k(s),p_k(s)\}$ for $s<t$ only through the propagation of an initial condition $\{q_k(0),p_k(0)\}$ reflected in the stochastic force. For the simple linear model treated here, this was well nigh trivial; but, even in more complicated nonlinear settings, this splitting should be implementable quite generally through the introduction of appropriate projection operators.[13] The net result thereof will be a more complicated nonlocal Langevin equation, which one can again hope to analyze approximately.

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