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Detection of the gravitomagnetic field using an orbiting superconducting gravity gradiometer. Theoretical principles

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The angular momentum of the Earth produces gravitomagnetic components of the Riemann curvature tensor, which are of the order of 10^{-10} of the Newtonian tidal terms arising from the mass of the Earth. These components could be detected in principle by sensitive superconducting gravity gradiometers currently under development. We lay out the theoretical principles of such an experiment by using the parametrized post-Newtonian formalism to derive the locally measured Riemann tensor in an orbiting proper reference frame, in a class of metric theories of gravity that includes general relativity. A gradiometer assembly consisting of three gradiometers with axes at mutually right angles measures three diagonal components of a 3×3 "tidal tensor," related to the Riemann tensor. We find that, by choosing a particular assembly orientation relative to the orbit and taking a sum and difference of two of the three gradiometer outputs, one can isolate the gravitomagnetic relativistic effect from the large Newtonian background.

I. INTRODUCTION AND SUMMARY

According to general relativity, moving matter produces a gravitational field that is similar in many ways to the magnetic field produced by moving charges. On one level, this phenomenon is simply a consequence of the fact that general relativity is compatible with Lorentz invariance (at least locally), just as are Maxwell's equations. On a more concrete level, this phenomenon can be seen directly in the weak-field, slow-motion limit of general relativity, where the field equations can be written approximately in the form (see, for example, Ref. 1)

$$\nabla\cdot\mathbf{E}_g \approx -4\pi\rho,\quad \nabla\times\mathbf{E}_g \approx -\partial\mathbf{H}_g/\partial t\ ,$$
$$\nabla\cdot\mathbf{H}_g \approx 0,\quad \nabla\times\mathbf{H}_g \approx -16\pi\rho\mathbf{v}+\partial\mathbf{E}_g/\partial t\ , \tag{1.1}$$

where ρ is the matter density and $\mathbf{v}$ the velocity, $c=G=1$, and where

$$\mathbf{E}_g = -\nabla\phi-\partial\mathbf{A}/\partial t,\quad \mathbf{H}_g=\nabla\times\mathbf{A}\ . \tag{1.2}$$

The "potentials" ϕ and $\mathbf{A}$ are related to the spacetime metric $g_{\mu\nu}$ by

$$\phi\approx-\tfrac{1}{2}(g_{00}+1),\quad A_i\approx g_{0i}\ . \tag{1.3}$$

The field $\mathbf{E}_g$ is sometimes called the "gravitoelectric" field, and $\mathbf{H}_g$ is called the "gravitomagnetic" field. The equations of motion can be written in a form that approximately parallels the Lorentz force equation, although there are additional terms. Some of these terms come from the spatial part of the metric g_{ij}, which has no counterpart in electromagnetism. It should be kept in mind that this is only an approximation to general relativity, and is valid only in a specific coordinate system.

Because of the similarity between these equations and those of electromagnetism, one can use "lines of force," "right-hand rules," and so on, to determine qualitatively the fields of various source configurations and the forces these fields produce on matter, just as in electromagnetism.

The gravitoelectric field corresponds at lowest order to the Newtonian gravitational acceleration, and is well tested experimentally (fifth and sixth forces notwithstanding), as are many of the so-called "post-Newtonian" corrections predicted by general relativity.[2]

For a rotating body, such as the Earth, the gravitomagnetic field looks just like the magnetic dipole field of a rotating charged sphere. If one places a spinning object, such as a gyroscope, in this field it will precess, just as will a current loop in a magnetic field. This phenomenon is usually called the "dragging of inertial frames," since one usually defines a local, "nonrotating," inertial frame in relativity to be a freely falling frame whose spatial axes are attached to gyroscopes. It is also called the "Lense-Thirring effect," after the two scientists who first studied its consequence for celestial mechanics in 1918.[3] The importance of this effect in astrophysics has been emphasized by some who claim that the forces associated with the gravitomagnetic field around rotating supermassive black holes may play a role in the formation of jets in quasars and other systems.[4,1(b)]

Alternative metric theories of gravitation also predict the existence of the gravitomagnetic field, albeit possibly with a different numerical size. Although several solar-system tests have tightly constrained the range of viable numerical coefficients for the gravitomagnetic field, these tests involved verification of Lorentz invariance in gravitational interactions, through the absence of "preferred-frame effects."[5] To date, the gravitomagnetic field itself has not been measured or detected.

One effort to attempt to detect the gravitomagnetic field has been continuing since the early 1960s: the Stanford Gyroscope Experiment, or in NASA terminology, Gravity Probe B (GP-B). (Gravity Probe A was a 1976 rocket gravitational red-shift experiment.) In this experiment, a set of four superconducting-niobium-coated spherical quartz gyroscopes will be flown in a low Earth polar orbit, and the precession of the gyroscopes relative to the distant stars will be measured. The predicted effect is about 42 milliarcseconds per year, and the accuracy goal of the experiment is about 2 milliarcseconds per year. Current plans call for a shuttle-launched experiment around 1994. For a recent overview of this experiment, see Ref. 6.

Other proposals have been made in recent years to look for this effect by alternative means. One proposal is to study the precession of the plane of oscillation of a Foucault pendulum at the south pole;[7] the pendulum must be placed at the south pole to reduce to a controllable level the natural rotation of the pendulum plane caused by the rotation of the direction of the vertical. Another is to measure the relative precession of the line of nodes of a pair of satellites with supplementary inclination angles;[8] the inclinations must be supplementary in order to cancel the dominant nodal precession caused by the Earth's Newtonian gravitational multipole moments. None of these proposals has come to fruition yet.

The recent development of extremely precise, superconducting gravity gradiometers[9] offers a potentially promising means of detecting the gravitomagnetic field of the Earth. A gravity gradiometer measures the local gradient of the gravitational force, which corresponds to the tidal gravitational force. In relativistic language, the tidal force is proportional to the Riemann curvature tensor of spacetime. For a gradiometer at rest, the tidal force comes from the Newtonian potential and its post-Newtonian corrections, in other words, from the gravitoelectric field (there are also contributions from the spatial part of the metric). However, for a gradiometer in orbit, there is an additional contribution to the tidal force from the product of the orbital velocity and the Riemann tensor component produced by the gravitomagnetic field (in the electromagnetic analogy, these forces would be gradients of the magnetic Lorentz force). The existence of this effect was first pointed out and estimated by Braginsky and Polnarev,[10] and was later elaborated theoretically by Mashhoon and Theiss.[11] For a low Earth orbit, this force is approximately 10^{-10} of the Newtonian tidal term. Superconducting gravity gradiometers are now under development that can measure the tidal force to 10^{-12} of the Newtonian value. This makes the detection of the gravitomagnetic field of the Earth possible, at least in principle.

One of the immediate questions is whether one can detect the small gravitomagnetic term buried beneath the enormous (10^{10} times larger) Newtonian signal. Our initial results show that this is possible in principle. The gradiometers under development actually consist of an array of three gradiometers, whose axes are at mutually right angles, and as a result, they measure three orthogonal components of the tidal gravitational force. We have found that, if the gradiometers are oriented in such a way that two of the axes are at angles of 45° relative to the orbital plane, then, when the *difference* between the outputs of the two arms is taken, the Newtonian and post-Newtonian gravitoelectric terms *cancel exactly,* leaving the gravitomagnetic term isolated.

Specifically, we consider a three-axis gradiometer whose axes are described by mutually orthogonal unit basis vectors $\hat{\mathbf{p}}$, $\hat{\mathbf{q}}$, and $\hat{\mathbf{r}}$. The basis is parallel transported, in other words, the axes are assumed to be tied to gyroscopes. A test particle located a distance l from the origin along the $\hat{\mathbf{p}}$ axis, constrained to move in the $\hat{\mathbf{p}}$ direction, experiences an acceleration in the $\hat{\mathbf{p}}$ direction given by

$$\ddot{\xi}_{\hat{p}} = -lK_{\hat{p}\hat{p}} , \tag{1.4}$$

where $K_{\hat{p}\hat{p}}$ is the $\hat{\mathbf{p}}\,\hat{\mathbf{p}}$-diagonal component of the "tidal matrix"

$$K_{\hat{i}\hat{j}} \equiv R_{\hat{0}\hat{i}\hat{0}\hat{j}} , \tag{1.5}$$

where $R_{\hat{0}\hat{i}\hat{0}\hat{j}}$ are components of the Riemann curvature tensor on the basis $\hat{\mathbf{e}}_{\hat{j}}$, with the "time" basis four-vector $e_{\hat{0}}$, given by the four-velocity u of the frame. Equation (1.4) comes from the equation of geodesic deviation.

We calculate the components $R_{\hat{0}\hat{i}\hat{0}\hat{j}}$ using the parametrized post-Newtonian (PPN) formalism,[12] which treats the post-Newtonian limit of a broad class of metric theories of gravity, including general relativity, in terms of a set of dimensionless parameters whose values vary from theory to theory. We assume that the Earth is spherical, with asymptotically measured mass M and angular momentum J. At the moment of the observation, the orbit is described by the instantaneous Keplerian orbit elements: eccentricity $e=0$, semimajor axis a, inclination relative to the Earth's equator i, and angle of nodes $\Omega=0$. The orientation of the three-axis gradiometer is as follows (see Fig. 1): the $\hat{\mathbf{r}}$ axis lies in the orbital plane, an angle $\pi/2-\psi_0$ from the nodal direction; the $\hat{\mathbf{p}}$

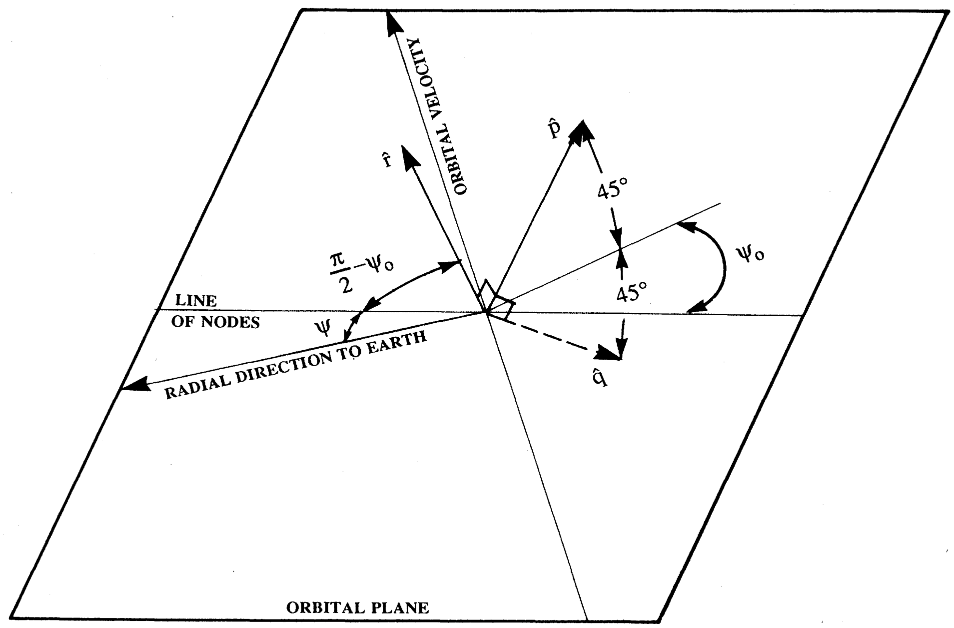


FIG. 1. Orientation of a three-axis gradiometer relative to the orbital plane. The $\hat{\mathbf{r}}$ axis is in the orbital plane, at an angle $\pi/2-\Psi_0$ from the line of nodes of the orbit. The orthogonal axes $\hat{\mathbf{p}}$ and $\hat{\mathbf{q}}$ are, respectively, 45° above and below the plane. The orbital phase Ψ determines the angle between the radial direction and the line of nodes.

and $\hat{\mathbf{q}}$ axes lie, respectively, 45° above and below the orbital plane. The angle ψ_0 is a constant, representing the arbitrary initial orientation of the gradiometer about an axis normal to the orbit, determined when the gradiometer is at the line of nodes. Under these assumptions, the components $K_{\hat{i}\hat{j}}$ can be written as the sum of three tidal matrices, a Newtonian term $K^N_{\hat{i}\hat{j}}$ a post-Newtonian gravitoelectric term $K^E_{\hat{i}\hat{j}}$ and the post-Newtonian gravitomagnetic term $K^M_{\hat{i}\hat{j}}$. In the chosen orientation, it turns out that the $\hat{\mathbf{p}}\hat{\mathbf{p}}$ and $\hat{\mathbf{q}}\hat{\mathbf{q}}$ components of the Newtonian and gravitoelectric tidal matrices are equal, in other words,

$$K^N_{\hat{p}\hat{p}}=K^N_{\hat{q}\hat{q}}=(M/4a^3)(1-3\cos 2\bar{\psi}) , \tag{1.6a}$$

$$K^E_{\hat{p}\hat{p}}=K^E_{\hat{q}\hat{q}}=(M^2/2a^4)(A+C+3B\cos 2\bar{\psi}) , \tag{1.6b}$$

while, for the gravitomagnetic tidal matrix, these components are not equal. In fact,

$$K^M_{\hat{p}\hat{p}}=-\frac{3\Delta J v}{a^4}\{\tfrac{1}{2}\cos i(1-\cos 2\bar{\psi}) -\sin i[2\sin(2\bar{\psi}+\psi_0)+\sin\psi_0]\} , \tag{1.7a}$$

$$K^M_{\hat{q}\hat{q}}=-\frac{3\Delta J v}{a^4}\{\tfrac{1}{2}\cos i(1-\cos 2\bar{\psi}) +\sin i[2\sin(2\bar{\psi}+\psi_0)+\sin\psi_0]\} , \tag{1.7b}$$

where ψ is the orbital phase angle at the moment of observation, and $\bar{\psi}\equiv\psi-\psi_0$. The coefficients A, B, C, and Δ are given by

$$A\equiv\tfrac{1}{2}(4\beta+\gamma-5) , \tag{1.8a}$$

$$B\equiv\tfrac{1}{6}(8\beta+7\gamma-9) , \tag{1.8b}$$

$$C\equiv-2\beta-\gamma+3 , \tag{1.8c}$$

$$\Delta\equiv\tfrac{1}{8}(4\gamma+4+\alpha_1) , \tag{1.8d}$$

where γ, β, and α_1 are PPN parameters. In general relativity ($\gamma=\beta=1$, $\alpha_1=0$), these coefficients take the values $A=C=0$ and $B=\Delta=1$. To the necessary order, $v=(M/a)^{1/2}$. The dependence on angles of these components can be understood as follows: because of the assumed spherical symmetry of the Earth, the Newtonian and gravitoelectric terms depend only on the orientation of the $\hat{\mathbf{r}}$ axis relative to the radial direction, which according to Fig. 1 depends only on $\bar{\psi}$. However, the gravitomagnetic term depends as well on the angular momentum vector of the Earth, whose orientation relative to the orbit is determined by the inclination i and the line of nodes; thus the angle ψ_0 of the $\hat{\mathbf{r}}$ axis relative to the nodal line appears, in addition to $\bar{\psi}$.

Now, in a gravity gradiometer, the test mass located along the $\hat{\mathbf{p}}$ direction is constrained to move in that direction, and compensating forces are applied in that direction to prevent or control the actual motion of the mass. These forces then represent the measured signal. Thus the three gradiometer signals are proportional to the three diagonal components of $K_{\hat{i}\hat{j}}$. Thus, by taking sums and differences of the signals along gradiometers $\hat{\mathbf{p}}$ and $\hat{\mathbf{q}}$, we can separate the gravitomagnetic term from the dominant Newtonian term: namely,

$$K^+\equiv\tfrac{1}{2}(K_{\hat{p}\hat{p}}+K_{\hat{q}\hat{q}}) =\frac{1}{4}\frac{M}{a^3}(1-3\cos 2\bar{\psi})+\frac{1}{2}\frac{M^2}{a^4}(A+C+3B\cos 2\bar{\psi}) -\frac{3}{2}\frac{\Delta J v}{a^4}\cos i(1-\cos 2\bar{\psi}) , \tag{1.9a}$$

$$K^{-}\equiv\tfrac{1}{2}(K_{\hat{p}\hat{p}}-K_{\hat{q}\hat{q}})=\frac{3\Delta Jv}{a^4}\sin i[2\sin(2\bar{\psi}+\psi_0)+\sin\psi_0]\ . \tag{1.9b}$$

We also consider a gradiometer which rotates as it orbits the Earth, with angular velocity $\omega_0=d\psi/dt$ about the normal to the orbital plane. In this case, the $\hat{\mathbf{r}}$ axis maintains a fixed orientation relative to the radial direction. We call this an "Earth-pointing" orientation. Because of the rotation relative to a parallel transported frame, centrifugal forces will be present, so that the acceleration along the $\hat{\mathbf{p}}$ direction is now given by

$$\ddot{\xi}_{\hat{p}}=-l\tilde{K}_{\hat{p}\hat{p}}=-l[K_{\hat{p}\hat{p}}+(\omega_0\cdot\hat{\mathbf{p}})^2-\omega_0^2]\ . \tag{1.10}$$

In this case, the sum and difference signals for $\tilde{K}$ are

$$\begin{aligned}\tilde{K}^{+}&\equiv\tfrac{1}{2}(\tilde{K}_{\hat{p}\hat{p}}+\tilde{K}_{\hat{q}\hat{q}})\\&=\frac{1}{4}\frac{M}{a^3}(1-3\cos2\psi_0)\\&\quad-\tfrac{1}{2}\omega_0^2+\frac{1}{2}\frac{M^2}{a^4}(A+C+3B\cos2\psi_0)\\&\quad-\frac{3}{2}\frac{\Delta Jv}{a^4}\cos i(1-\cos2\psi_0)\ ,\end{aligned} \tag{1.11a}$$

$$\begin{aligned}\tilde{K}^{-}&\equiv\tfrac{1}{2}(\tilde{K}_{\hat{p}\hat{p}}-\tilde{K}_{\hat{q}\hat{q}})\\&=\frac{3\Delta Jv}{a^4}\sin i[2\sin\bar{\psi}+\sin(\bar{\psi}+2\psi_0)]\ ,\end{aligned} \tag{1.11b}$$

showing again the separation of the gravitomagnetic term from the Newtonian gravitational and centrifugal terms.

Although this separation is possible in principle, the important issue is whether it is achievable in practice. Among the sources of error that must be considered are pointing errors that would alter the 45° orientation and make the cancellation imperfect; these errors can come from a variety of sources, including inadequate pointing accuracy in the spacecraft, precession of the basis directions, and changes in the orbital elements. Misalignment of the gradiometer axes (failure of the gradiometers to be at exactly mutual right angles) also will affect the cancellation of the Newtonian and gravitoelectric terms in the difference signals. Other sources of error include tidal forces due to the Moon, the Earth's multipole moments, the Sun, etc., and small rotations of the gradiometers relative to the nonrotating or Earth-pointing frames that would introduce spurious inertial forces. Detailed analyses of such error sources will be left to future publications.

The structure of this paper is as follows: In Sec. II we use the parametrized post-Newtonian formalism to obtain the Riemann tensor as measured by a freely falling observer in a proper reference frame in orbit around the Earth. Section III applies this formalism to measurements made by an ideal three-axis gradiometer; that is, one with perfectly aligned axes and perfect pointing accuracy. We consider both an inertially guided or parallel-transported gradiometer assembly (one whose axes are fixed by gyroscopes) and an "Earth-pointing" gradiometer that is made to rotate at the orbital angular velocity in such a way that it maintains a fixed orientation relative to the radial direction. In both cases we demonstrate the cancellation of Newtonian and gravitoelectric effects with the proper 45° orientation. Section IV presents concluding remarks.

We use units in which $G=c=1$; commas denote partial differentiation, while semicolons denote covariant differentiations; greek indices run over the values 0, 1, 2, and 3, while latin indices run over the values 1, 2, and 3; other conventions are those of Misner, Thorne, and Wheeler.[13]

II. POST-NEWTONIAN RIEMANN TENSOR IN A PROPER REFERENCE FRAME

A. Proper reference frame of a freely falling observer

Consider an observer who moves through spacetime on a geodesic (free-fall) with four-velocity u. Orthogonal to the observer's four-velocity is a set of orthonormal basis vectors $e_{\hat{i}}$ with the property that, on the observer's world line,

$$u\cdot e_{\hat{i}}=0\ . \tag{2.1}$$

With the definition $u\equiv e_{\hat{0}}$, the tetrad of basis vectors $e_{\hat{\alpha}}$ satisfies

$$e_{\hat{\alpha}}\cdot e_{\hat{\beta}}=\eta_{\alpha\beta}=\mathrm{diag}(-1,1,1,1)\ . \tag{2.2}$$

Ordinarily, one demands that the tetrad be parallel-transported through spacetime as the observer moves along the trajectory. However, we will admit the possibility that the spatial basis vectors rotate relative to parallel-transported basis vectors. In this case, the basis vectors satisfy the equation

$$\nabla_u e_{\hat{\alpha}}=\overleftrightarrow{\Phi}\cdot e_{\hat{\alpha}}\ , \tag{2.3}$$

where

$$\Phi^{\mu\nu}\equiv u_\alpha\omega_\beta\epsilon^{\alpha\beta\mu\nu}\ . \tag{2.4}$$

The quantity $\epsilon^{\alpha\beta\mu\nu}$ is the completely antisymmetric Levi-Civita symbol, and the four-vector ω is orthogonal to the four-velocity ($\omega\cdot u=0$). Its purely spatial components in the proper reference frame represent the angular velocity of the basis relative to a parallel-transported basis.

Associated with this basis is a coordinate system: $x^{\hat{0}}\equiv\tau$ is proper time along the observer's trajectory as measured by atomic clocks, and $x^{\hat{j}}$ are proper distances along the three orthogonal spatial directions as measured by rods. The equations of motion of a particle at location $x^{\hat{j}}$, with coordinate velocity $v^{\hat{j}}\equiv dx^{\hat{j}}/dx^{\hat{0}}$, are given by[14]

$$d^2x^{\hat{j}}/dx^{\hat{0}2}+[\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{x})]^{\hat{j}}+(\dot{\boldsymbol{\omega}}\times\mathbf{x})^{\hat{j}}+2(\boldsymbol{\omega}\times\mathbf{v})^{\hat{j}}+x^{\hat{k}}R_{\hat{0}\hat{j}\hat{0}\hat{k}}+O(xvR)=F^{\hat{j}}/m\ , \tag{2.5}$$

where we represent spatial vectors in the proper frame using bold face, and where $R_{\hat{0}\hat{j}\hat{0}\hat{k}}$ are the tetrad components of the Riemann curvature tensor, and $F^{\hat{j}}$ represents any nongravitational forces that might constrain the motion of the particle. An overdot denotes a derivative with respect to $x^{\hat{0}}$.

It will be useful to adopt a matrix representation for the spatial vectors and tensors in the proper frame. We represent the spatial basis vectors $\hat{\mathbf{e}}_{\hat{j}}$ by

$$\hat{\mathbf{e}}_{\hat{1}}=\begin{bmatrix}1\\0\\0\end{bmatrix},\quad \hat{\mathbf{e}}_{\hat{2}}=\begin{bmatrix}0\\1\\0\end{bmatrix},\quad \hat{\mathbf{e}}_{\hat{3}}=\begin{bmatrix}0\\0\\1\end{bmatrix}, \tag{2.6}$$

and a spatial vector as

$$\mathbf{x}=x^{\hat{j}}\hat{\mathbf{e}}_{\hat{j}}=\begin{bmatrix}x^{\hat{1}}\\x^{\hat{2}}\\x^{\hat{3}}\end{bmatrix}. \tag{2.7}$$

Under a change of basis to $\hat{\mathbf{e}}_{\hat{j}'}$, we have

$$\hat{\mathbf{e}}_{\hat{j}'}=\Lambda_{\hat{j}'}{}^{\hat{k}}\hat{\mathbf{e}}_{\hat{k}}. \tag{2.8}$$

(The transformation need not be orthogonal, as we will see in our analysis of alignment errors in future publications.) The matrix representation of $\Lambda_{\hat{j}'}{}^{\hat{k}}$ is

$$\Lambda_{\hat{j}'}{}^{\hat{k}}=\begin{bmatrix}\Lambda_1{}^1 & \Lambda_1{}^2 & \Lambda_1{}^3\\ \Lambda_2{}^1 & \Lambda_2{}^2 & \Lambda_2{}^3\\ \Lambda_3{}^1 & \Lambda_3{}^2 & \Lambda_3{}^3\end{bmatrix}. \tag{2.9}$$

Then the vector x transforms according to matrix multiplication

$$x=\Lambda^T x', \tag{2.10a}$$

$$x'=(\Lambda^T)^{-1}x, \tag{2.10b}$$

where T denotes the transpose. If we now define the angular velocity matrix ω to be the matrix with components

$$\omega_{\hat{i}\hat{j}}=-\epsilon_{\hat{i}\hat{j}\hat{k}}\omega_{\hat{k}}=\begin{bmatrix}0 & -\omega_3 & \omega_2\\ \omega_3 & 0 & -\omega_1\\ -\omega_2 & \omega_1 & 0\end{bmatrix}, \tag{2.11}$$

and the "tidal matrix" to be

$$K_{\hat{i}\hat{j}}\equiv R_{\hat{0}\hat{i}\hat{0}\hat{j}}, \tag{2.12}$$

then the equation of motion (2.5) becomes the matrix equation

$$\ddot{x}+\omega^2 x+\dot{\omega}x+2\omega v+Kx=F/m. \tag{2.13}$$

In a gravity gradiometer, compensating forces are applied to the proof masses to keep them at a fixed location in the proper frame; in other words, $\ddot{x}=v=0$. The output of the gradiometer is the force required to achieve this; it depends on, in addition to the tidal tensor itself, the trajectory of the gradiometer and the orientation of the spatial basis vectors (and on the evolution of that orientation with time).

B. Parametrized post-Newtonian metric

To analyze the ingredients of the tidal matrix, we will work in the post-Newtonian limit of gravitational theory. Because ultimately any measurement of the gravitomagnetic field is a test of general relativity against alternative theories of gravity, we shall work in the parametrized post-Newtonian (PPN) formalism.[12] We shall restrict attention to so-called "semiconservative" theories of gravity, those which possess conservation laws for total momentum (this class includes all theories based on an action principle); we also exclude theories with "Whitehead effects." Of the ten PPN parameters, only four are relevant: γ, β, α_1, and α_2. In general relativity, these have the values $(\gamma,\beta,\alpha_1,\alpha_2)=(1,1,0,0)$. We shall also ignore "preferred-frame" effects that result from the possible nonvanishing of α_1 and α_2; this amounts to dropping terms in the PPN metric that depend on the velocity of the observer $\mathbf{w}$ relative to the rest frame of the Universe. Such effects can be treated separately from the present analysis.

The PPN coordinate system is a quasi-Cartesian coordinate system (t,x^j), where t is time measured at asymptotically flat spatial infinity, and where the spatial coordinate directions are tied to the distant stars. In this coordinate system, and with the simplifications listed above, the PPN metric takes the form

$$g_{00}=-1+2U-2\beta U^2+(2\gamma+2)\Phi_1+2(3\gamma-2\beta+1)\Phi_2+6\gamma\Phi_4, \tag{2.14a}$$

$$g_{0i}=-\tfrac{1}{2}(4\gamma+3+\alpha_1-\alpha_2)V_i-\tfrac{1}{2}(1+\alpha_2)W_i, \tag{2.14b}$$

$$g_{ij}=(1+2\gamma U)\delta_{ij}, \tag{2.14c}$$

where

$$\begin{aligned} U&=\int\frac{\rho'}{|\mathbf{x}-\mathbf{x}'|}d^3x', \quad \Phi_1=\int\frac{\rho' v'^2}{|\mathbf{x}-\mathbf{x}'|}d^3x',\\ \Phi_2&=\int\frac{\rho' U'}{|\mathbf{x}-\mathbf{x}'|}d^3x', \quad \Phi_4=\int\frac{p'}{|\mathbf{x}-\mathbf{x}'|}d^3x',\\ V_i&=\int\frac{\rho' v_i'}{|\mathbf{x}-\mathbf{x}'|}d^3x',\\ W_i&=\int\frac{\rho'[\mathbf{v}'\cdot(\mathbf{x}-\mathbf{x}')](x-x')_i}{|\mathbf{x}-\mathbf{x}'|^3}d^3x'. \end{aligned} \tag{2.15}$$

Here, ρ, p, and v_j are the mass-energy density, pressure, and velocity of matter in the source.

We now idealize the source to be a stationary, rotating, nearly spherical body, whose center of mass is at rest. Although the Earth has significant multipole moments that will affect the gradiometer response as well as its orbital motion, to the necessary accuracy they can be treated separately as purely Newtonian effects. We will not consider them in this paper. Thus the potentials in $g_{\mu\nu}$ can be expanded in inverse powers of $r=|\mathbf{x}|$. By choosing the origin of the PPN coordinate system appropriately, we can eliminate the "dipole" (r^{-2}) term from the expansion of the potentials in g_{00}. The result is

$$g_{00}=-1+2U^*-2\beta U^{*2}, \tag{2.16a}$$

$$g_{0i}=-\tfrac{1}{2}(4\gamma+4+\alpha_1)V_i, \tag{2.16b}$$

$$g_{ij}=(1+2\gamma U^*)\delta_{ij}, \tag{2.16c}$$

where

$$U^*=M/r \ , \tag{2.17a}$$

$$V_i=-\tfrac{1}{2}\epsilon_{ijk}x^jJ^k/r^3=-\tfrac{1}{2}(\hat{\mathbf{n}}\times\mathbf{J})_i/r^2 \ , \tag{2.17b}$$

where $\hat{\mathbf{n}}=\mathbf{x}/r$, and where M and J^i are the asymptotically measured mass and angular momentum of the body, given to post-Newtonian order, by

$$M=\int \rho\, d^3x[1+(\gamma+1)v^2+(3\gamma-2\beta+1)U+3\gamma p/\rho] \ , \tag{2.18a}$$

$$\mathbf{J}=\int \rho(\mathbf{x}\times\mathbf{v})d^3x \ . \tag{2.18b}$$

It is useful to note that M is equivalent to the "active" gravitational mass in semiconservative theories of gravity (see Ref. 12, pp. 148–151).

C. Tetrad components of the Riemann tensor

At a given instant, we choose the orthonormal tetrad $e_{\hat{\alpha}}$ in such a way that the spatial vectors are related to the asymptotic PPN coordinate spatial vectors by a boost to the instantaneous velocity of the frame, together with a scale change to maintain orthonormality with respect to the PPN metric. To the required post-Newtonian order, the resulting tetrad vectors are given by[15]

$$e^{\mu}_{\hat{0}}=u^{\mu} \ , \tag{2.19a}$$

$$e^{0}_{\hat{j}}=v_j+O(3) \ , \tag{2.19b}$$

$$e^{k}_{\hat{j}}=(1-\gamma U^*)\delta^k_j+\tfrac{1}{2}v_jv_k+O(4) \ , \tag{2.19c}$$

where the order symbol O denotes post-Newtonian orders, with $v\approx O(1)$, $v^2\approx U^*\approx O(2)$, and so on. We can now relate the "electric" tetrad components of the Riemann tensor $R_{\hat{0}\hat{j}\hat{0}\hat{k}}$ to the PPN coordinate components $R_{\mu\nu\alpha\beta}$, with the result

$$\begin{aligned}R_{\hat{0}\hat{j}\hat{0}\hat{k}}=&4e^{[0}_{\hat{0}}e^{m]}_{\hat{j}}e^{[0}_{\hat{0}}e^{n]}_{\hat{k}}R_{0m0n}+4e^{[0}_{\hat{0}}e^{m]}_{(\hat{j}}e^{p}_{\hat{k})}e^{n}_{\hat{0}}R_{0mnp}\\&+e^{m}_{\hat{0}}e^{n}_{\hat{j}}e^{p}_{\hat{0}}e^{q}_{\hat{k}}R_{mnpq} \ ,\end{aligned} \tag{2.20}$$

where parentheses around indices denote symmetrization, and square brackets denote antisymmetrization. Noting that the PPN components have the relative orders

$$\begin{aligned}&R_{0m0n}\sim O(2)+O(4), \quad R_{0mnp}\sim O(3) \ ,\\&R_{mnpq}\sim O(2)+O(4) \ ,\end{aligned} \tag{2.21}$$

and that the components of the four-velocity are given by

$$u^0=1+U^*+\tfrac{1}{2}v^2, \quad u^j=v^j(1+U^*+\tfrac{1}{2}v^2) \ , \tag{2.22}$$

we obtain, to the necessary order,

$$\begin{aligned}R_{\hat{0}\hat{j}\hat{0}\hat{k}}=&[1+2(1-\gamma)U^*+v^2]R_{0j0k}-v^mv_{(j}R_{k)0m0}\\&-2v^nR_{0(jk)n}+v^mv^nR_{mjnk} \ .\end{aligned} \tag{2.23}$$

The components $R_{\mu\nu\alpha\beta}$ are given by

$$R_{\mu\nu\alpha\beta}=g_{\mu\lambda}(\Gamma^{\lambda}{}_{\nu\beta,\alpha}-\Gamma^{\lambda}{}_{\nu\alpha,\beta}+\Gamma^{\lambda}{}_{\sigma\alpha}\Gamma^{\sigma}{}_{\nu\beta}-\Gamma^{\lambda}{}_{\sigma\beta}\Gamma^{\sigma}{}_{\nu\alpha}) \ , \tag{2.24}$$

where $\Gamma^{\alpha}{}_{\beta\gamma}$ are the Christoffel symbols, given to the needed order by

$$\begin{aligned}&\Gamma^0{}_{0i}=-U^*_{,i}+(\beta-1)(U^{*2})_{,i}, \quad \Gamma^0{}_{00}=0 \ ,\\&\Gamma^i{}_{00}=-U^*_{,i}+(\beta+\gamma)(U^{*2})_{,i} \ ,\\&\Gamma^i{}_{jk}=\gamma(\delta_{ij}U^*_{,k}+\delta_{ik}U^*_{,j}-\delta_{jk}U^*_{,i}) \ ,\\&\Gamma^0{}_{ij}=\tfrac{1}{2}(4\gamma+4+\alpha_1)V_{(i,j)} \ ,\\&\Gamma^i{}_{0j}=-\tfrac{1}{2}(4\gamma+4+\alpha_1)V_{[i,j]} \ .\end{aligned} \tag{2.25}$$

Substituting Eqs. (2.25) into (2.24) and thence into (2.23), we obtain, to post-Newtonian order,

$$\begin{aligned}R_{\hat{0}\hat{j}\hat{0}\hat{k}}=&-[1-2(\beta+\gamma-1)U^*+(\gamma+1)v^2]U^*{}_{,jk}\\&+(2\beta+2\gamma-1)U^*{}_{,j}U^*{}_{,k}-\gamma\delta_{jk}|\nabla U^*|^2\\&+(2\gamma+1)v^mv_{(j}U^*{}_{,k)m}-\gamma\delta_{jk}v^mv^nU^*{}_{,mn}\\&+\tfrac{1}{2}(4\gamma+4+\alpha_1)(v^nV_{n,jk}-v^nV_{(j,k)n}) \ .\end{aligned} \tag{2.26}$$

It is useful to note that the spatial trace of $R_{\hat{0}\hat{j}\hat{0}\hat{k}}$ is the Ricci tensor $R_{\hat{0}\hat{0}}$, given from Eq. (2.26) by

$$\begin{aligned}R_{\hat{0}\hat{0}}=&-[1-2(\beta+\gamma-1)U^*+(\gamma+1)v^2]\nabla^2U^*\\&+(2\beta-\gamma-1)|\nabla U^*|^2\\&+(1-\gamma)v^mv^nU^*{}_{,mn}+\tfrac{1}{2}(4\gamma+4+\alpha_1)v^n\nabla^2V_n \ .\end{aligned} \tag{2.27}$$

In vacuum ($\nabla^2U^*=\nabla^2V_j=0$) and for general relativity ($\beta=\gamma=1$), $R_{\hat{0}\hat{0}}$ vanishes, as it must, according to Einstein's equations. Note that, in other theories of gravity, it does not necessarily vanish in vacuum.

Substituting the forms of U^* and V_j from Eqs. (2.17) into Eq. (2.26), we obtain

$$R_{\hat{0}\hat{j}\hat{0}\hat{k}}=R^{N}_{\hat{0}\hat{j}\hat{0}\hat{k}}+R^{E}_{\hat{0}\hat{j}\hat{0}\hat{k}}+R^{M}_{\hat{0}\hat{j}\hat{0}\hat{k}} \ , \tag{2.28}$$

where the Newtonian (N), gravitoelectric post-Newtonian (E), and gravitomagnetic (M) terms are given by

$$R^{N}_{\hat{0}\hat{j}\hat{0}\hat{k}}=(M/r^3)(\delta_{jk}-3n_jn_k) \ , \tag{2.29a}$$

$$\begin{aligned}R^{E}_{\hat{0}\hat{j}\hat{0}\hat{k}}=&-(M^2/r^4)[(2\beta+3\gamma-2)\delta_{jk}-(8\beta+8\gamma-7)n_jn_k]\\&+(M/r^3)\{(2\gamma+1)[v^2\delta_{jk}-v_jv_k+3(\mathbf{v}\cdot\hat{\mathbf{n}})n_{(j}v_{k)}]\\&-3\gamma(\mathbf{v}\cdot\hat{\mathbf{n}})^2\delta_{jk}-3(\gamma+1)n_jn_kv^2\} \ ,\end{aligned} \tag{2.29b}$$

$$\begin{aligned}R^{M}_{\hat{0}\hat{j}\hat{0}\hat{k}}=&-3(\gamma+1+\tfrac{1}{4}\alpha_1)r^{-4}[3(\mathbf{v}\times\mathbf{J})_{(j}n_{k)}+(\hat{\mathbf{n}}\times\mathbf{J})_{(j}v_{k)}\\&+\hat{\mathbf{n}}\cdot(\mathbf{v}\times\mathbf{J})(\delta_{jk}-5n_jn_k)\\&-5(\hat{\mathbf{n}}\cdot\mathbf{v})(\hat{\mathbf{n}}\times\mathbf{J})_{(j}n_{k)}] \ .\end{aligned} \tag{2.29c}$$

This form is valid for a general orbit outside a stationary rotating body, with tetrad basis vectors given instantaneously by Eq. (2.19).

D. PPN equations of motion

The orbit of the gradiometer is determined by a solution of the PPN equations of motion. Using the geodesic equation and the metric, Eqs. (2.16), it is straightforward to derive the equation of motion

$$\begin{aligned}\frac{d^2\mathbf{x}}{dt^2}=&-\frac{M\mathbf{x}}{r^3}[1-2(\gamma+\beta)M/r+\gamma v^2]\\&+2(\gamma+1)\frac{M}{r^3}\mathbf{v}(\mathbf{x}\cdot\mathbf{v})\\&-\tfrac{1}{2}(4\gamma+4+\alpha_1)\frac{1}{r^3}[\mathbf{v}\times\mathbf{J}-\tfrac{3}{2}(\hat{\mathbf{n}}\times\mathbf{J})(\hat{\mathbf{n}}\cdot\mathbf{v})\\&\qquad+\tfrac{3}{2}\hat{\mathbf{n}}\mathbf{v}\cdot(\hat{\mathbf{n}}\times\mathbf{J})]\ . \qquad (2.30)\end{aligned}$$

We now restrict attention to a circular orbit, with $\ddot{r}=0$ and $\dot{r}=\hat{\mathbf{n}}\cdot\mathbf{v}=0$. With this restriction, the preceding equations take the form

$$v^2=\frac{M}{r}[1-2(\gamma+\beta)M/r+\gamma v^2]-\tfrac{1}{4}(4\gamma+4+\alpha_1)\frac{1}{r^3}\mathbf{J}\cdot\mathbf{h}\ , \qquad (2.31a)$$

$$\frac{d\mathbf{h}}{dt}=-\tfrac{1}{4}(4\gamma+4+\alpha_1)\frac{1}{r^2}\mathbf{v}(\hat{\mathbf{n}}\cdot\mathbf{J})\ , \qquad (2.31b)$$

where $\mathbf{h}\equiv\mathbf{x}\times\mathbf{v}$ is the orbital angular momentum per unit mass. Note from Eq. (2.31b) that the magnitude $|\mathbf{h}|$ is constant, the vector $\mathbf{h}$ varies by an effect of first order in J, and hence that, through first order in J, a circular orbit is compatible with Eq. (2.31a).

We now specify the circular orbit using instantaneous osculating orbit elements:[16] eccentricity $e=0$, semimajor axis a, inclination relative to the equator i, and angle of nodes Ω. Because of post-Newtonian perturbations, these orbital elements, in particular i and Ω, will not be strictly constant, but instead they will vary with time. Notice that, in order to evaluate the post-Newtonian terms in the Riemann tensor, only the Newtonian orbit (with fixed elements) is needed, while the full post-Newtonian orbit is needed to evaluate the Newtonian term. However, since the Newtonian term depends only on a and $\hat{\mathbf{n}}$, it is sufficient to write $\hat{\mathbf{n}}$ in terms of the instantaneous osculating orbit elements evaluated at the time of observation of the tidal tensor. The effects of possible variations of these orbit elements due to post-Newtonian and other orbit corrections will be the subject of a separate publication.

With these definitions, the unit vector $\hat{\mathbf{n}}$ of the instantaneous orbital position is given in terms of the original PPN basis $\hat{\mathbf{e}}_i$ by

$$\hat{\mathbf{n}}\equiv\hat{\mathbf{e}}_x\cos\Psi+(\hat{\mathbf{e}}_y\cos i+\hat{\mathbf{e}}_z\sin i)\sin\Psi\ , \qquad (2.32)$$

where Ψ is the orbital phase. We have chosen the z axis to be parallel to the angular momentum vector $\mathbf{J}$ of the Earth, and the x axis to be parallel to the line of nodes (so that $\Omega=0$ at the instant in question). The coordinate velocity unit vector $\hat{\mathbf{v}}$ and the orbital angular momentum unit vector $\hat{\mathbf{h}}\equiv\mathbf{h}/|\mathbf{h}|$ can be determined to Newtonian order, since they appear only in post-Newtonian terms; they are given by

$$\begin{aligned}\hat{\mathbf{v}}&=-\hat{\mathbf{e}}_x\sin\Psi+(\hat{\mathbf{e}}_y\cos i+\hat{\mathbf{e}}_z\sin i)\cos\Psi\ ,\\\hat{\mathbf{h}}&=-\hat{\mathbf{e}}_y\sin i+\hat{\mathbf{e}}_z\cos i\ .\end{aligned} \qquad (2.33)$$

In the post-Newtonian terms, we can use the Newtonian relations for a circular orbit: $v^2=M/a$ and $\mathbf{v}\cdot\hat{\mathbf{n}}=0$.

E. Tidal matrix for a circular orbit

Substituting Eqs. (2.32) and (2.33) into Eqs. (2.29), and writing in matrix form, we obtain the tidal matrices

$$K^N=\frac{M}{a^3}\begin{bmatrix}-\frac{1}{2}-\frac{3}{2}\cos 2\Psi & -\frac{3}{2}\cos i\,\sin 2\Psi & -\frac{3}{2}\sin i\,\sin 2\Psi\\ -\frac{3}{2}\cos i\,\sin 2\Psi & 1-\frac{3}{2}\cos^2 i(1-\cos 2\Psi) & -\frac{3}{2}\sin i\,\cos i(1-\cos 2\Psi)\\ -\frac{3}{2}\sin i\,\sin 2\Psi & -\frac{3}{2}\sin i\,\cos i(1-\cos 2\Psi) & 1-\frac{3}{2}\sin^2 i(1-\cos 2\Psi)\end{bmatrix}, \qquad (2.34a)$$

$$K^E=\frac{M^2}{a^4}\begin{bmatrix}A+3B\cos 2\Psi & 3B\cos i\,\sin 2\Psi & 3B\sin i\,\sin 2\Psi\\ 3B\cos i\,\sin 2\Psi & A\cos^2 i+C\sin^2 i-3B\cos^2 i\cos 2\Psi & \sin i\,\cos i(A-C-3B\cos 2\Psi)\\ 3B\sin i\,\sin 2\Psi & \sin i\,\cos i(A-C-3B\cos 2\Psi) & A\sin^2 i+C\cos^2 i-3B\sin^2 i\cos 2\Psi\end{bmatrix}, \qquad (2.34b)$$

$$K^M=\frac{6\Delta Jv}{a^4}\begin{bmatrix}\cos i\cos^2\Psi & -(1-\frac{3}{2}\cos^2 i)\sin 2\Psi & \frac{3}{2}\sin i\,\cos i\,\sin 2\Psi\\ -(1-\frac{3}{2}\cos^2 i)\sin 2\Psi & \cos i(1-5\sin^2 i)\sin^2\Psi & -\sin i(\frac{3}{2}-2\cos^2\Psi-5\cos^2 i\sin^2\Psi)\\ \frac{3}{2}\sin i\,\cos i\,\sin 2\Psi & -\sin i(\frac{3}{2}-2\cos^2\Psi-5\cos^2 i\sin^2\Psi) & -\cos i(1-5\sin^2 i\sin^2\Psi)\end{bmatrix}, \qquad (2.34c)$$

where the coefficients A, B, C, and Δ are given by Eqs. (1.8).

It is useful to reexpress the tidal matrices in a tetrad basis that is more tailored to the orbit: a basis $\hat{\mathbf{e}}_{i'}$ in which the $\hat{\mathbf{x}}'$ and $\hat{\mathbf{z}}'$ vectors lie in the orbital plane, while the $\hat{\mathbf{y}}'$ vector is orthogonal to the orbital plane. The $\hat{\mathbf{x}}'$ vector makes an angle Ψ_0 with the $\hat{\mathbf{x}}$ direction (line of nodes). These vectors are related to the original orthonormal basis by a rotation about the $\hat{\mathbf{x}}$ axis by the angle $i-\pi/2$, and a rotation about the $\hat{\mathbf{y}}'$ axis by an angle $-\Psi_0$ (Fig. 2):

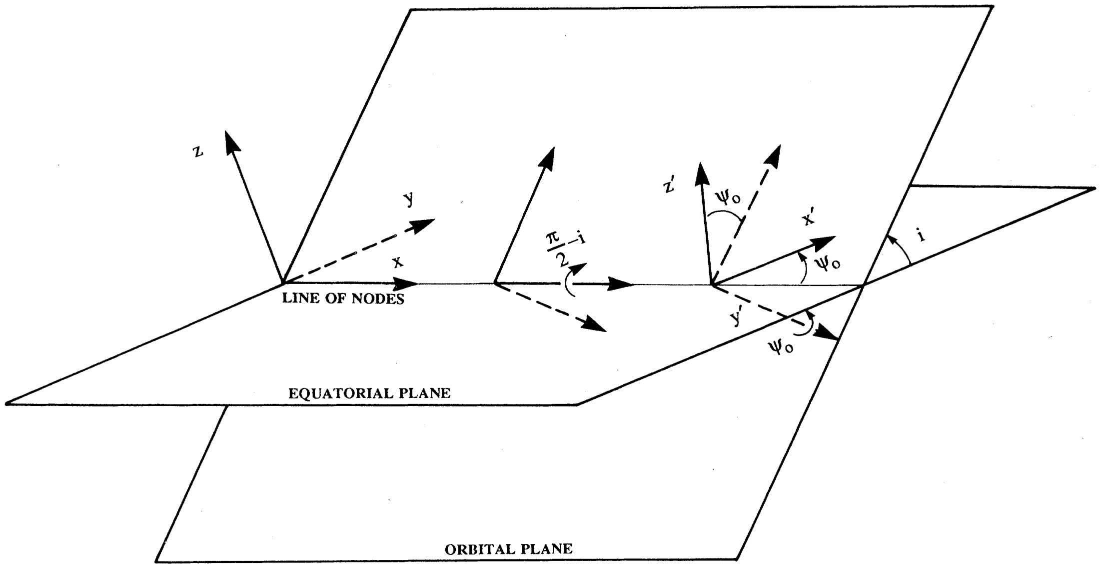


FIG. 2. Relationship between the $\hat{\mathbf{x}}\hat{\mathbf{y}}\hat{\mathbf{z}}$ basis and the $\hat{\mathbf{x}}'\hat{\mathbf{y}}'\hat{\mathbf{z}}'$ basis. The original basis is first rotated about the line of nodes by $i-\pi/2$ to bring the z axis down to the orbital plane, then rotated about the new y' axis by the arbitrary angle $-\Psi_0$.

$$\hat{\mathbf{e}}_{\hat{x}'}=\hat{\mathbf{e}}_{\hat{x}}\cos\Psi_0+\sin\Psi_0(\hat{\mathbf{e}}_{\hat{y}}\cos i+\hat{\mathbf{e}}_{\hat{z}}\sin i),\quad \hat{\mathbf{e}}_{\hat{y}'}=\hat{\mathbf{e}}_{\hat{y}}\sin i-\hat{\mathbf{e}}_{\hat{z}}\cos i,\quad \hat{\mathbf{e}}_{\hat{z}'}=-\hat{\mathbf{e}}_{\hat{x}}\sin\Psi_0+\cos\Psi_0(\hat{\mathbf{e}}_{\hat{y}}\cos i+\hat{\mathbf{e}}_{\hat{z}}\sin i)\ . \tag{2.35}$$

Matrices then transform according to $A_{\rm NEW}=R^{-1}A_{\rm OLD}R$, where

$$R=\begin{bmatrix}\cos\Psi_0 & 0 & -\sin\Psi_0\\ \sin\Psi_0\cos i & \sin i & \cos\Psi_0\cos i\\ \sin\Psi_0\sin i & -\cos i & \cos\Psi_0\sin i\end{bmatrix}\ . \tag{2.36}$$

In this basis, the tidal matrices simplify to

$$K^N=\frac{M}{a^3}\begin{bmatrix}-\frac{1}{2}(1+3\cos2\overline{\Psi}) & 0 & -\frac{3}{2}\sin2\overline{\Psi}\\ 0 & 1 & 0\\ -\frac{3}{2}\sin2\overline{\Psi} & 0 & -\frac{1}{2}(1-3\cos2\overline{\Psi})\end{bmatrix}, \tag{2.37a}$$

$$K^E=\frac{M^2}{a^4}\begin{bmatrix}A+3B\cos2\overline{\Psi} & 0 & 3B\sin2\overline{\Psi}\\ 0 & C & 0\\ 3B\sin2\overline{\Psi} & 0 & A-3B\cos2\overline{\Psi}\end{bmatrix}, \tag{2.37b}$$

$$K^M=\frac{6\Delta Jv}{a^4}\begin{bmatrix}\frac{1}{2}\cos i(1+\cos2\overline{\Psi}) & -\frac{1}{2}\sin i[2\sin i(2\overline{\Psi}+\Psi_0)+\sin\Psi_0] & \frac{1}{2}\cos i\sin2\overline{\Psi}\\ -\frac{1}{2}\sin i[2\sin(2\overline{\Psi}+\Psi_0)+\sin\Psi_0] & -\cos i & \frac{1}{2}\sin i[2\cos(2\overline{\Psi}+\Psi_0)-\cos\Psi_0]\\ \frac{1}{2}\cos i\sin2\overline{\Psi} & \frac{1}{2}\sin i[2\cos(2\overline{\Psi}+\Psi_0)-\cos\Psi_0] & \frac{1}{2}\cos i(1-\cos2\overline{\Psi})\end{bmatrix}, \tag{2.37c}$$

where $\overline{\Psi}\equiv\Psi-\Psi_0$.

III. DETECTION WITH AN IDEAL GRAVITY GRADIOMETER

A. Measurements made by an ideal gradiometer

In an ideal gravity gradiometer, suitable nongravitational forces are applied to confine the motion of a proof mass to a specified linear direction, and to read out its motion. Thus for a given direction, described by a unit vector $\hat{\mathbf{p}}$, both the displacement and velocity are confined to the direction $\hat{\mathbf{p}}$, and thus the Coriolis term $[(\boldsymbol{\omega}\times\mathbf{v})$ in Eq. (2.5) or ωv in Eq. (2.12)] and the angular acceleration term $[(\dot{\boldsymbol{\omega}}\times\mathbf{x})$ or $\dot{\omega}x]$ are canceled by the forces provided by the massive, rigid channel along which the proof mass moves. The force directed along the direction $\hat{\mathbf{p}}$ can be characterized by a restoring force proportional to the displacement from equilibrium along $\hat{\mathbf{p}}$, a damping force proportional to the velocity along $\hat{\mathbf{p}}$, and a feedback force. If we write, for the vector locating the proof mass in the $\hat{\mathbf{p}}$ gradiometer,

$$\mathbf{x}_{(\hat{\mathbf{p}})}=\hat{\mathbf{p}}(l+\xi_{\hat{\mathbf{p}}}) , \tag{3.1}$$

where l is the equilibrium displacement of the proof mass from the origin of the proper frame, and $\xi_{\hat{\mathbf{p}}}$ is the scalar displacement along the $\hat{\mathbf{p}}$ direction, then the equation of motion (2.5) referred to the $\hat{\mathbf{p}}$ direction can be written as

$$d^2\xi_{\hat{\mathbf{p}}}/d\tau^2+(\omega_G/Q)d\xi_{\hat{\mathbf{p}}}/d\tau+\omega_G^2\xi_{\hat{\mathbf{p}}}=-l\tilde{K}_{\hat{\mathbf{p}}\hat{\mathbf{p}}}+F_{\hat{\mathbf{p}}}/m , \tag{3.2}$$

where ω_G is the resonant frequency of the gradiometer, Q is the quality factor (Q/ω_G is the damping time), $F_{\hat{\mathbf{p}}}$ is the feedback force, and

$$\tilde{K}_{\hat{\mathbf{p}}\hat{\mathbf{p}}}=(\boldsymbol{\omega}\cdot\hat{\mathbf{p}})^2-\omega^2+\hat{\mathbf{p}}^{\hat{\jmath}}\hat{\mathbf{p}}^{\hat{k}}R_{\hat{0}\hat{\jmath}\hat{0}\hat{k}} , \tag{3.3}$$

or in matrix notation

$$\tilde{K}_{\hat{\mathbf{p}}\hat{\mathbf{p}}}=\omega^2_{\hat{\mathbf{p}}\hat{\mathbf{p}}}+K_{\hat{\mathbf{p}}\hat{\mathbf{p}}} . \tag{3.4}$$

Thus a "three-axis gradiometer," consisting of three such gradiometers oriented along three perfectly orthogonal directions $\hat{\mathbf{p}}$, $\hat{\mathbf{q}}$, and $\hat{\mathbf{r}}$, measures the three "diagonal" components of $\tilde{K}$ in the $\hat{\mathbf{p}}\hat{\mathbf{q}}\hat{\mathbf{r}}$ basis.

The central problem in detecting the gravitomagnetic effect is the measurement or elimination of the dominant Newtonian tidal forces, represented in the tidal matrix K^N, Eq. (2.37a). Notice that the $\hat{\mathbf{x}}'\hat{\mathbf{y}}'$ and $\hat{\mathbf{y}}'\hat{\mathbf{z}}'$ components of K^N vanish (as, fortuitously, do those of K^E). Therefore, we need a device that is sensitive to only those components. A pair of gradiometers whose sensitive axes are in the $\hat{\mathbf{x}}'\hat{\mathbf{y}}'$ plane, but rotated each by 45° about the $\hat{\mathbf{z}}'$ axis, measures the combinations of components $(K_{\hat{x}'\hat{x}'}+2K_{\hat{x}'\hat{y}'}+K_{\hat{y}'\hat{y}'})/2$ and $(K_{\hat{x}'\hat{x}'}-2K_{\hat{x}'\hat{y}'}+K_{\hat{y}'\hat{y}'})/2$. The difference between the two gradiometer outputs is sensitive only to $K_{\hat{x}'\hat{y}'}$, and thus measures *only* the gravitomagnetic contribution K^M. A similar cancellation occurs if the assembly is rotated by 45° about the $\hat{\mathbf{x}}'$ axis.

In order to study this cancellation in detail, and in particular to analyze potential sources of error, it is most convenient to work in a tetrad basis that has already been rotated by the requisite 45°, and to orient the three-axis gradiometer along the axes of this new basis. We shall consider both an inertially guided (parallel transported) system, and a system forced to rotate in such a manner that one axis of the gradiometer maintains a fixed orientation relative to the radius vector from Earth. We will focus on the 45° rotation about the $\hat{\mathbf{z}}'$ axis. The other possibility mentioned above can be seen to be physically equivalent to changing the orientation angle Ψ_0 to $\Psi_0+\pi/2$, and then rotating by 45° about the $\hat{\mathbf{z}}'$ axis.

B. Inertially guided system

Consider a proper frame whose basis is parallel transported, so that the angular velocity $\boldsymbol{\omega}$ is zero. At the moment of observation, this proper frame is chosen to coincide with the orthonormal tetrad obtained from the PPN coordinate frame, Eqs. (2.19). At subsequent times, this frame may differ from the tetrad of Eqs. (2.19) appropriate to the subsequent velocity by certain secular terms that arise from the equations of parallel transport.[11] A complete post-Newtonian treatment which takes these secular terms into account will be deferred to a future publication. Except for these secular terms, the matrix $\tilde{K}$ is then the same as the tidal matrix K of Eqs. (2.37). We now choose the gradiometer axes $\hat{\mathbf{p}}$, $\hat{\mathbf{q}}$, and $\hat{\mathbf{r}}$ to correspond to a rotation of the $\hat{\mathbf{e}}_{\hat{i}'}$ basis by $-45°$ about the $\hat{\mathbf{z}}'$ direction. Thus the $\hat{\mathbf{r}}$ gradiometer axis lies in the orbital plane parallel to the $\hat{\mathbf{z}}'$ axis, while the $\hat{\mathbf{p}}$ and $\hat{\mathbf{q}}$ gradiometer axes lie 45°, respectively, above and below the orbital plane (see Fig. 3). Thus

$$\hat{\mathbf{p}}=(\hat{\mathbf{e}}_{\hat{x}'}-\hat{\mathbf{e}}_{\hat{y}'})/\sqrt{2},\quad \hat{\mathbf{q}}=(\hat{\mathbf{e}}_{\hat{x}'}+\hat{\mathbf{e}}_{\hat{y}'})/\sqrt{2},\quad \hat{\mathbf{r}}=\hat{\mathbf{e}}_{\hat{z}'} . \tag{3.5}$$

The rotation matrix is given by

$$R=\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} . \tag{3.6}$$

In the new basis the tidal matrices become

$$K^N=\frac{M}{a^3}\begin{bmatrix} \frac{1}{4}(1-3\cos 2\overline{\Psi}) & -\frac{3}{4}(1+\cos 2\overline{\Psi}) & -\frac{3}{2\sqrt{2}}\sin 2\overline{\Psi} \\ -\frac{3}{4}(1+\cos 2\overline{\Psi}) & \frac{1}{4}(1-3\cos 2\overline{\Psi}) & -\frac{3}{2\sqrt{2}}\sin 2\overline{\Psi} \\ -\frac{3}{2\sqrt{2}}\sin 2\overline{\Psi} & -\frac{3}{2\sqrt{2}}\sin 2\overline{\Psi} & -\frac{1}{2}(1-3\cos 2\overline{\Psi}) \end{bmatrix} , \tag{3.7a}$$

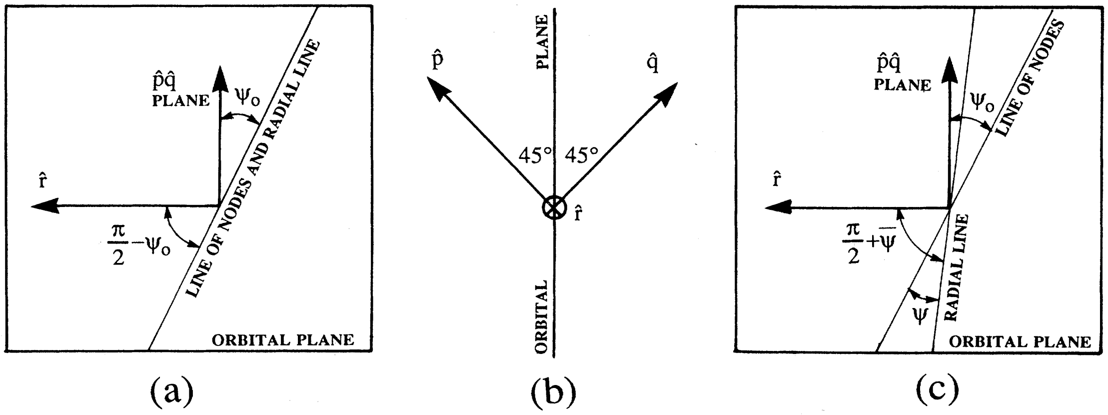


FIG. 3. The $\hat{\mathbf{p}}\,\hat{\mathbf{q}}\,\hat{\mathbf{r}}$ basis for the inertially guided system. (a) Top view at orbital phase $\Psi=0$, when the radial line coincides with the line of nodes. The $\hat{\mathbf{r}}$ axis is $\pi/2-\Psi_0$ from the line of nodes. (b) Side view, showing the $\hat{\mathbf{p}}$ and $\hat{\mathbf{q}}$ axes 45° above and below the orbital plane. (c) Top view at orbital phase Ψ. The angle between the $\hat{\mathbf{r}}$ axis and the line of nodes is unchanged because of inertial guidance, but the angle between it and the radial direction has changed to $\pi/2+\bar{\Psi}$, because of the motion of the satellite.

$$K^E=\frac{M^2}{a^4}\begin{pmatrix} \frac{1}{2}(A+C+3B\cos 2\bar{\Psi}) & \frac{1}{2}(A-C+3B\cos 2\bar{\Psi}) & \frac{3}{\sqrt{2}}B\sin 2\bar{\Psi} \\ \frac{1}{2}(A-C+3B\cos 2\bar{\Psi}) & \frac{1}{2}(A+C+3B\cos 2\bar{\Psi}) & \frac{3}{\sqrt{2}}B\sin 2\bar{\Psi} \\ \frac{3}{\sqrt{2}}B\sin 2\bar{\Psi} & \frac{3}{\sqrt{2}}B\sin 2\bar{\Psi} & A-3B\cos 2\bar{\Psi} \end{pmatrix}, \tag{3.7b}$$

$$K^M=\frac{6\Delta J v}{a^4}\begin{pmatrix} \begin{gathered}-\frac{1}{4}\cos i(1-\cos 2\bar{\Psi})\\ +\frac{1}{2}\sin i[2\sin(2\bar{\Psi}+\Psi_0)\\ +\sin\Psi_0]\end{gathered} & \frac{1}{4}\cos i(3+\cos 2\bar{\Psi}) & \begin{gathered}\frac{1}{2\sqrt{2}}\cos i\sin 2\bar{\Psi}\\ -\frac{1}{2\sqrt{2}}\sin i[2\cos(2\bar{\Psi}+\Psi_0)\\ -\cos\Psi_0]\end{gathered} \\ \frac{1}{4}\cos i(3+\cos 2\bar{\Psi}) & \begin{gathered}-\frac{1}{4}\cos i(1-\cos 2\bar{\Psi})\\ -\frac{1}{2}\sin i[2\sin(2\bar{\Psi}+\Psi_0)\\ +\sin\Psi_0]\end{gathered} & \begin{gathered}\frac{1}{2\sqrt{2}}\cos i\sin 2\bar{\Psi}\\ +\frac{1}{2\sqrt{2}}\sin i[2\cos(2\bar{\Psi}+\Psi_0)\\ -\cos\Psi_0]\end{gathered} \\ \begin{gathered}\frac{1}{2\sqrt{2}}\cos i\sin 2\bar{\Psi}\\ -\frac{1}{2\sqrt{2}}\sin i[2\cos(2\bar{\Psi}+\Psi_0)\\ -\cos\Psi_0]\end{gathered} & \begin{gathered}\frac{1}{2\sqrt{2}}\cos i\sin 2\bar{\Psi}\\ +\frac{1}{2\sqrt{2}}\sin i[2\cos(2\bar{\Psi}+\Psi_0)\\ -\cos\Psi_0]\end{gathered} & \frac{1}{2}\cos i(1-\cos 2\bar{\Psi}) \end{pmatrix}. \tag{3.7c}$$

Notice that, in K^N and K^E, the $\hat{\mathbf{p}}\hat{\mathbf{p}}$ and $\hat{\mathbf{q}}\hat{\mathbf{q}}$ components, corresponding to the gradiometers oriented at 45° relative to the orbital plane, are, respectively, identical, while in K^M they are not. Thus, by monitoring the sum and difference of the outputs of the gradiometers in the $\hat{\mathbf{p}}$ and $\hat{\mathbf{q}}$ directions, one can partially separate the gravitomagnetic effect from the Newtonian and post-Newtonian gravitoelectric effects.

Defining the sum K^+ and difference K^- outputs appropriately, we find

$$\begin{aligned} K^+ &\equiv \tfrac{1}{2}(K_{\hat{p}\hat{p}}+K_{\hat{q}\hat{q}}) \\ &=\frac{1}{4}\frac{M}{a^3}(1-3\cos 2\bar{\Psi})+\frac{1}{2}\frac{M^2}{a^4}(A+C+3B\cos 2\bar{\Psi}) \\ &\quad -\frac{3}{2}\frac{\Delta J v}{a^4}\cos i(1-\cos 2\bar{\Psi})\,, \end{aligned} \tag{3.8a}$$

$$K^{-}\equiv\tfrac{1}{2}(K_{\hat{p}\hat{p}}-K_{\hat{q}\hat{q}})=\frac{3\Delta J v}{a^4}\sin i[2\sin(2\bar{\Psi}+\Psi_0)+\sin\Psi_0]\ . \tag{3.8b}$$

Notice that the cancellation of the Newtonian and post-Newtonian contributions in K^- is independent of the angle Ψ_0. For a polar orbit ($i=\pi/2$), the gravitomagnetic term drops out of K^+, and we obtain a complete separation.

C. Earth-pointing system

As an alternative, we consider an orientation in which the gradiometer assembly rotates as it orbits the Earth in such a way that it maintains a fixed orientation relative to the radial direction. To obtain such an orientation, we first rotate the basis $\hat{\mathbf{e}}_{\hat{i}'}$ about the $\hat{\mathbf{y}}'$ direction at the uniform angular velocity $\omega_0=d\Psi/dt$, then rotate the basis about the new $\hat{\mathbf{z}}$ direction by 45°. Because the proper frame is rotating relative to a parallel transported frame, we must now include the centrifugal acceleration matrix corresponding to ω_0, given in the $\hat{\mathbf{e}}_{\hat{i}}'$ basis by

$$\omega=\omega_0\begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ -1 & 0 & 0\end{bmatrix}\ . \tag{3.9}$$

First, the rotation with angular velocity ω_0 yields the matrices

$$\omega^2=-\omega_0^2\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}, \tag{3.10a}$$

$$K^N=\frac{M}{a^3}\begin{bmatrix}-\frac{1}{2}(1+3\cos2\Psi_0) & 0 & \frac{3}{2}\sin2\Psi_0\\ 0 & 1 & 0\\ \frac{3}{2}\sin2\Psi_0 & 0 & -\frac{1}{2}(1-3\cos2\Psi_0)\end{bmatrix}, \tag{3.10b}$$

$$K^E=\frac{M^2}{a^4}\begin{bmatrix}A+3B\cos2\Psi_0 & 0 & -3B\sin2\Psi_0\\ 0 & C & 0\\ -3B\sin2\Psi_0 & 0 & A-3B\cos2\Psi_0\end{bmatrix}, \tag{3.10c}$$

$$K^M=\frac{6\Delta J v}{a^4}\begin{bmatrix}\frac{1}{2}\cos i(1+\cos2\Psi_0) & -\frac{1}{2}\sin i[2\sin\bar{\Psi}+\sin(\bar{\Psi}+2\Psi_0)] & -\frac{1}{2}\cos i\sin2\Psi_0\\ -\frac{1}{2}\sin i[2\sin\bar{\Psi}+\sin(\bar{\Psi}+2\Psi_0)] & -\cos i & \frac{1}{2}\sin i[2\cos\bar{\Psi}-\cos(\bar{\Psi}+2\Psi_0)]\\ -\frac{1}{2}\cos i\sin2\Psi_0 & \frac{1}{2}\sin i[2\cos\bar{\Psi}-\cos(\bar{\Psi}+2\Psi_0)] & \frac{1}{2}\cos i(1-\cos2\Psi_0)\end{bmatrix}. \tag{3.10d}$$

In this case, because of the centrifugal accelerations, the measured $\tilde{K}$ is equal to $K+\omega^2$. Rotating by 45° to obtain the Earth-pointing $\hat{\mathbf{p}}\,\hat{\mathbf{q}}\,\hat{\mathbf{r}}$ basis, we find that the relevant matrices take the form

$$\omega=\frac{\omega_0}{\sqrt{2}}\begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 1\\ -1 & -1 & 0\end{bmatrix}, \tag{3.11a}$$

$$\omega^2=-\omega_0^2\begin{bmatrix}\frac{1}{2} & \frac{1}{2} & 0\\ \frac{1}{2} & \frac{1}{2} & 0\\ 0 & 0 & 1\end{bmatrix}, \tag{3.11b}$$

$$K^N=\frac{M}{a^3}\begin{bmatrix}\frac{1}{4}(1-3\cos2\Psi_0) & -\frac{3}{4}(1+\cos2\Psi_0) & \frac{3}{2\sqrt{2}}\sin2\Psi_0\\ -\frac{3}{4}(1+\cos2\Psi_0) & \frac{1}{4}(1-3\cos2\Psi_0) & \frac{3}{2\sqrt{2}}\sin2\Psi_0\\ \frac{3}{2\sqrt{2}}\sin2\Psi_0 & \frac{3}{2\sqrt{2}}\sin2\Psi_0 & -\frac{1}{2}(1-3\cos2\Psi_0)\end{bmatrix}, \tag{3.11c}$$

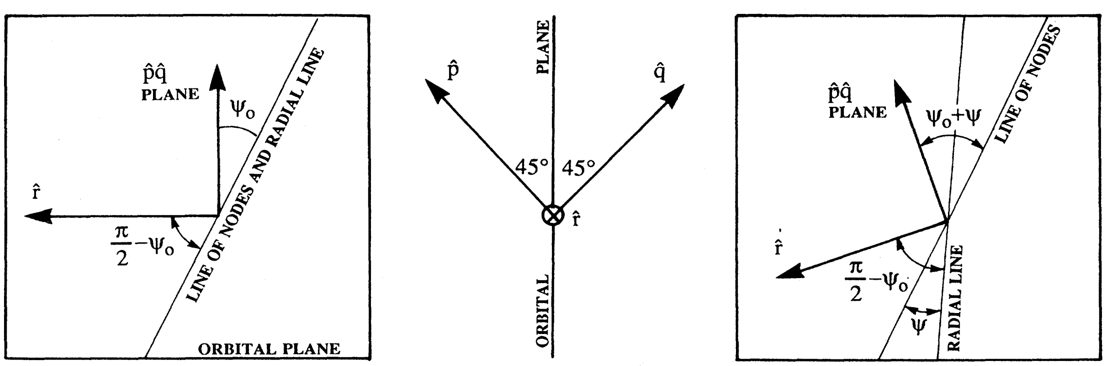
$$K^E=\frac{M^2}{a^4}\begin{bmatrix} \frac{1}{2}(A+C+3B\cos 2\Psi_0) & \frac{1}{2}(A-C+3B\cos 2\Psi_0) & -\frac{3}{\sqrt{2}}B\sin 2\Psi_0 \\ \frac{1}{2}(A-C+3B\cos 2\Psi_0) & \frac{1}{2}(A+C+3B\cos 2\Psi_0) & -\frac{3}{\sqrt{2}}B\sin 2\Psi_0 \\ -\frac{3}{\sqrt{2}}B\sin 2\Psi_0 & -\frac{3}{\sqrt{2}}B\sin 2\Psi_0 & A-3B\cos 2\Psi_0 \end{bmatrix}, \tag{3.11d}$$

$$K^M=\frac{6\Delta J v}{a^4}\begin{bmatrix} \begin{array}{l} -\frac{1}{4}\cos i(1-\cos 2\Psi_0) \\ +\frac{1}{2}\sin i[2\sin\bar{\Psi} \\ \quad +\sin(\bar{\Psi}+2\Psi_0)] \end{array} & \frac{1}{4}\cos i(3+\cos 2\Psi_0) & \begin{array}{l} -\frac{1}{2\sqrt{2}}\cos i\sin 2\Psi_0 \\ -\frac{1}{2\sqrt{2}}\sin i[2\cos\bar{\Psi} \\ \quad -\cos(\bar{\Psi}+2\Psi_0)] \end{array} \\ \frac{1}{4}\cos i(3+\cos 2\Psi_0) & \begin{array}{l} -\frac{1}{4}\cos i(1-\cos 2\Psi_0) \\ -\frac{1}{2}\sin i[2\sin\bar{\Psi} \\ \quad +\sin(\bar{\Psi}+2\Psi_0)] \end{array} & \begin{array}{l} -\frac{1}{2\sqrt{2}}\cos i\sin 2\Psi_0 \\ +\frac{1}{2\sqrt{2}}\sin i[2\cos\bar{\Psi} \\ \quad -\cos(\bar{\Psi}+2\Psi_0)] \end{array} \\ \begin{array}{l} -\frac{1}{2\sqrt{2}}\cos i\sin 2\Psi_0 \\ -\frac{1}{2\sqrt{2}}\sin i[2\cos\bar{\Psi} \\ \quad -\cos(\bar{\Psi}+2\Psi_0)] \end{array} & \begin{array}{l} -\frac{1}{2\sqrt{2}}\cos i\sin 2\Psi_0 \\ +\frac{1}{2\sqrt{2}}\sin i[2\cos\bar{\Psi} \\ \quad -\cos(\bar{\Psi}+2\Psi_0)] \end{array} & \frac{1}{2}\cos i(1-\cos 2\Psi_0) \end{bmatrix} \tag{3.11e}$$

In the Earth-pointing orientation, the angle between the $\hat{\mathbf{r}}$ axis and the radial direction is fixed to be its initial value $\pi/2-\Psi_0$, while the angle between the $\hat{\mathbf{r}}$ axis and the nodal line varies, and is given by $\pi/2-\Psi_0-\Psi$ (Fig. 4). As a consequence, the tidal matrices here can be obtained from those of the inertially guided system by the replacements $\bar{\Psi}\rightarrow-\Psi_0$ and $\Psi_0\rightarrow\Psi+\Psi_0=\bar{\Psi}+2\Psi_0$. Again, the $\hat{p}\hat{p}$ and $\hat{q}\hat{q}$ components of K^N, K^E, and of ω^2 are equal, independently of Ψ_0, while those of K^M are not.

Thus the sum and difference signals for $\tilde{K}$ yield

$$\begin{aligned} \tilde{K}^+ &\equiv \tfrac{1}{2}(\tilde{K}_{\hat{p}\hat{p}}+\tilde{K}_{\hat{q}\hat{q}}) \\ &= \frac{1}{4}\frac{M}{a^3}(1-3\cos 2\Psi_0)-\tfrac{1}{2}\omega_0^2 \\ &\quad +\frac{1}{2}\frac{M^2}{a^4}(A+C+3B\cos 2\Psi_0) \\ &\quad -\frac{3}{2}\frac{\Delta J v}{a^4}\cos i(1-\cos 2\Psi_0)\ , \end{aligned} \tag{3.12a}$$



(a) (b) (c)

FIG. 4. The $\hat{\mathbf{p}}\hat{\mathbf{q}}\hat{\mathbf{r}}$ basis for the Earth-pointing system. (a) and (b) same as Fig. 3. (c) Top view at orbital phase Ψ. The angle between the $\hat{\mathbf{r}}$ axis and the radial line is unchanged because the system is rotated to maintain a fixed orientation relative to the Earth. The angle between the $\hat{\mathbf{r}}$ axis and the line of nodes is now $\pi/2-(\Psi_0+\Psi)$.

$$\tilde{K}^{-} \equiv \tfrac{1}{2}(\tilde{K}_{\hat{p}\hat{p}} - \tilde{K}_{\hat{q}\hat{q}}) = \frac{3\Delta J v}{a^4} \sin i [2\sin\overline{\Psi} + \sin(\overline{\Psi} + 2\Psi_0)] \;. \tag{3.12b}$$

The cancellation of the Newtonian, centrifugal, and post-Newtonian contributions in $\tilde{K}^{-}$ is again independent of the angle Ψ_0. Again, for a polar orbit, the gravitomagnetic term drops out of the "sum" signal, giving a complete separation.

Particular choices of the angle Ψ_0 are worth discussion. Notice that the three diagonal components of K^N vanish simultaneously when

$$2\Psi_0 = \arccos\tfrac{1}{3} \;. \tag{3.13}$$

With this choice of Ψ_0, the three gradiometer axes are in an "umbrella orientation;"[9] i.e., each axis makes an identical angle $\arctan\sqrt{2}$ with respect to the radial direction. Therefore the background Newtonian gravity gradients vanish for all three axes, although the comparable centrifugal accelerations remain. On the other hand, the Newtonian gradients plus the centrifugal accelerations vanish in $\tilde{K}_{\hat{p}\hat{p}}$ and $\tilde{K}_{\hat{q}\hat{q}}$ simultaneously when

$$2\Psi_0 = \arccos(-\tfrac{1}{3}) \;, \tag{3.14}$$

where we use the fact that $\omega_0^2 = M/a^3$ for a circular orbit. Since the only nonvanishing terms in this orientation are the relativistic corrections, the requirement to match the readout or "scale" factors in the $\hat{p}$ and $\hat{q}$ gradiometers to achieve the proper cancellation of the Newtonian signal may be alleviated.

IV. CONCLUDING REMARKS

We have demonstrated that the gravitomagnetic terms in the locally measured Riemann tensor or gravity gradients can be separated in principle from the Newtonian and gravitoelectric post-Newtonian background in either an inertial or an Earth-pointing orientation using a three-axis gradiometer oriented in such a way that it detects appropriate components of the tidal matrix. In order to make estimates of sizes of effects and accuracy requirements for an experiment designed to carry out such a measurement, it is useful to define the dimensionless parameters ϵ and μ by

$$\epsilon \equiv M/a, \quad \mu \equiv J/\omega_0 a^3 \tag{4.1}$$

and to use the Newtonian approximation

$$\omega_0^2 \simeq M/a^3 \;. \tag{4.2}$$

Then, for an orbit of radius a around the Earth (radius R),

$$\omega_0^2 = 1.54\times10^{-6}(R/a)^3\,\text{sec}^{-2} \;,$$
$$\omega_0/2\pi = 1.98\times10^{-4}(R/a)^{3/2}\,\text{sec}^{1} \;, \tag{4.3}$$
$$\epsilon = 6.97\times10^{-10}(R/a), \quad \mu = 1.36\times10^{-11}(R/a)^{3/2} \;.$$

To be specific, we shall assume a 650-km-altitude polar circular orbit, as in GP-B. The relevant parameters are then $\omega_0^2 = 1.15\times10^{-6}$ sec^{-2} ($\omega_0/2\pi = 1.7\times10^{-4}$ Hz), $\epsilon = 6.3\times10^{-10}$, and $\mu = 1.2\times10^{-11}$.

Equations (3.8b) and (3.12b) show that the amplitude of the gravitomagnetic signal in the difference output is $9\Delta\mu\omega_0^2$. For general relativity, at a 650-km altitude, this becomes 1.3×10^{-16} sec^{-2}. In order to resolve this signal with a signal-to-noise ratio of 100 in a year ($t = \pi\times10^7$ sec), the noise level of each gradiometer at the signal frequency of 3.4×10^{-4} Hz (inertial orientation) or 1.7×10^{-4} Hz (Earth-pointing) must be 10^{-14} sec^{-2} Hz$^{-1/2}$, or $10^{-5}E$ Hz$^{-1/2}$, where E denotes one Eötvös unit, 10^{-9} sec^{-2}. This sensitivity is within the capability of the superconducting gravity gradiometers under development.[9]

In practice, however, alignment errors in the gradiometer caused by relative misalignment of the three sensitive axes and by pointing errors of the spacecraft will mix undesirable matrix components with the gravitomagnetic terms. In order to resolve the gravitomagnetic field with a signal-to-noise ratio of 100, the appropriate components of the pointing errors must be reduced to 10^{-12} rad upon averaging for one year. This implies a pointing stability requirement of 10^{-3} arcsec Hz$^{-1/2}$ at the signal frequency which is comparable to that of GP-B (Ref. 11). Another important characteristic of the gradiometer is the match and stability of the scale factors that relate the experimentally measured output in each gradiometer to the corresponding component of K and $\tilde{K}$, since the outputs of two gradiometers are differenced to reject the large Newtonian gradient. These and other sources of error will be the subject of future papers in this series.

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