Longitudinal impedance of a periodic structure at high frequency

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An ultrarelativistic beam bunch traveling along the axis of an azimuthally symmetric cavity of general shape connected to a beam pipe generates wake fields. We extend an earlier derivation of the integral equation for the axial electric field at the beam-pipe radius to the case of a periodic structure. This equation is then solved in the high-frequency limit, and we show that the reciprocal of the impedance per cell has a particularly simple form, closely related to the admittance for a single cavity. We also show that the imaginary part of the impedance per cell varies as ω^{-1} and the real part of the impedance per cell varies as $\omega^{-3/2}$. Finally, the results are shown to satisfy the requirements of causality for an ultrarelativistic beam.

I. INTRODUCTION

We have recently derived[1,2] an integral equation for the electric field at the pipe radius in the presence of an azimuthally symmetric cavity of arbitrary shape in a beam pipe of circular cross section, and shown that the high-frequency limit of the coupling impedance is approximately independent of the cavity shape. In this paper, the derivation is extended to several cavities, as well as to a periodic structure. We then explore the high-frequency limit of the impedance per cell and show that the admittance per cell has a particularly simple form. The real part of the impedance is shown to vary as $\omega^{-3/2}$, in agreement with the behavior suggested by Heifets and Kheifets,[3] and the imaginary part varies as ω^{-1}.

II. ANALYSIS

The starting point for the analysis is the integral equation obtained for the electric field in the obstacle at the pipe radius.[1,2] Specifically, we have

$$\int_0^g dz' F(z')[K_p(|z'-z|)+K_c(z',z)] = je^{-jkz} \tag{2.1}$$

and

$$\frac{Z(k)}{Z_0} = \frac{1}{ka^2}\int_0^g dz\, F(z) e^{jkz} . \tag{2.2}$$

Here $kc/2\pi$ is the frequency, a is the pipe radius, $Z_0 = 120\pi\ \Omega$ is the impedance of free space, and the azimuthally symmetric cavity, of general shape in the r,z plane, extends axially from $z=0$ to $z=g$ at the pipe radius $r=a$. Apart from a constant, $F(z)$ is the axial electric field for $r=a$ and $0<z<g$. The component of the kernel from the "pipe field" is

$$K_p(|u|) = \frac{2\pi j}{a}\sum_{s=1}^{\infty}\frac{e^{-jb_s|u|/a}}{b_s} , \tag{2.3}$$

where

$$b_s = (k^2a^2 - j_s^2)^{1/2}, \quad \beta_s = (j_s^2 - k^2a^2)^{1/2} . \tag{2.4}$$

Here j_s is the sth zero of the Bessel function $J_0(x)$ and b_s is to be replaced by $-j\beta_s$ when $j_s > ka$. The component of the kernel from the "cavity fields" is

$$K_c(z',z) = 4\pi^2 \sum_l \frac{h_l(z)h_l(z')}{k^2-k_l^2} , \tag{2.5}$$

where the orthonormal (azimuthally symmetric) modes of the cavity (with an imaginary metal wall at $r=a$) are defined by

$$\nabla\times\mathbf{e}_l = k_l\mathbf{h}_l, \quad \nabla\times\mathbf{h}_l = k_l\mathbf{e}_l ,$$
$$\int \mathbf{e}_l\cdot\mathbf{e}_m\, dv = \int \mathbf{h}_l\cdot\mathbf{h}_m\, dv = \delta_{lm} , \tag{2.6}$$

and where

$$h_l(z) \equiv [h_l(a,z)]_\phi$$

is the azimuthal component of the normalized magnetic field at $r=a$.

III. ANALYSIS FOR SEVERAL CAVITIES

A parallel analysis for several cavities connected by a common beam pipe leads to an equation very similar to Eq. (2.1). Specifically, one obtains

$$\sum_m \int_m dz'_m F(z'_m)[K_p(|z'_m - z_n|) + \delta_{mn}K_c(z'_m, z_n)] = je^{-jkz_n} , \tag{3.1}$$

where z'_m and z_n denote the variables z' and z within cavity m and n, and $\int_m dz'_m$ is over cavity m. The coupling between different cavities now occurs through the pipe kernel while the cavity kernels are diagonal.

For a periodic structure consisting of identical cavities whose centers are a distance L apart, we can write

$$z'_m = mL + t', \quad 0 \le t' \le g ,$$
$$z_n = nL + t, \quad 0 \le t \le g . \tag{3.2}$$

The solution to Eq. (3.1) clearly requires

$$F(z'_m)=e^{-jkz'_m}F(t') , \tag{3.3}$$

where $F(t')$ is the same in each cavity. Equation (3.1) then becomes

$$\int_0^g dt' F(t')\hat{K}(t',t) + \int_0^g dt' F(t') \sum_{m\neq n} \hat{K}_p[(m-n)L+t'-t]=j , \tag{3.4}$$

where, from Eq. (2.3),

$$\hat{K}_p(u)=\frac{2\pi j e^{-jku}}{a}\sum_{s=1}^{\infty}\frac{e^{-jb_s|u|/a}}{b_s} . \tag{3.5}$$

The one-cell kernel $\hat{K}(t',t)$ is that obtained earlier for a single cavity,

$$\hat{K}(t',t)=\hat{K}_p(t'-t)+4\pi^2 e^{-jk(t'-t)}\sum_l \frac{h_l(t')h_l(t)}{k^2-k_l^2} , \tag{3.6}$$

and the impedance per cell $Z_L(k)$, from Eq. (2.2), is

$$\frac{Z_L(k)}{Z_0}=\frac{1}{ka^2}\int_0^g dt\, F(t) . \tag{3.7}$$

The sum over m in Eq. (3.4) is independent of n and can be evaluated by introducing a convergence factor $\exp(-\epsilon|m|)$, leading to

$$\begin{aligned} K_s(v) &\equiv \sum_{m\neq n}\hat{K}_p[(m-n)L+t'-t] \\ &= \frac{2\pi j}{a}\sum_{m\neq 0} e^{-j\theta(m+v)}\sum_{s=1}^{\infty}\frac{e^{-j\phi_s|m+v|}}{b_s} \\ &= \frac{2\pi j}{a}\sum_{s=1}^{\infty}\frac{e^{-j(\theta v+\phi_s)}}{b_s} \\ &\quad\times\left[\frac{e^{-j\phi_s v}}{e^{j\theta}-e^{-j\phi_s}}+\frac{e^{j\phi_s v}}{e^{-j\theta}-e^{-j\phi_s}}\right] , \end{aligned} \tag{3.8}$$

where

$$v=(t'-t)/L,\quad \theta=kL,\quad \phi_s=b_sL/a . \tag{3.9}$$

IV. HIGH-FREQUENCY LIMIT

We have shown in some detail[1] that, for large k, the sum over s in Eq. (2.3) or (3.8) can be converted to an integral over the continuous variable s. If we make the substitution $x\equiv b_s/ka=\phi_s/\theta$, the sum over s can be written as

$$\begin{aligned} \sum_{s=1}^{\infty}\frac{G}{b_s} &\to \frac{1}{\pi}\int_0^{\infty}\frac{G\,dj_s}{b_s}=-\frac{1}{\pi}\int_{ka}^{-j\infty}\frac{G\,db_s}{(k^2a^2-b_s^2)^{1/2}} \\ &= \frac{j}{\pi}\int_1^{-j\infty}\frac{G\,dx}{\sqrt{x^2-1}} , \end{aligned} \tag{4.1}$$

where we have chosen the sign so that $j_s=(k^2a^2-b_s^2)^{1/2}=jka(x^2-1)^{1/2}$ approaches $+\infty$ for large positive s. We then have for the coupling kernel

$$K_s(v)\simeq -\frac{2}{a}\int_1^{-j\infty}\frac{dx}{\sqrt{x^2-1}}e^{-j\theta(v+x)} \times\left[\frac{e^{-j\theta(vx+1)}}{1-e^{-j\theta(x+1)}}+\frac{e^{j\theta(vx+1)}}{1-e^{j\theta(1-x)}}\right] . \tag{4.2}$$

Apart from the apparent divergence at $x=1$ in the second term in the large parentheses the integral is convergent, particularly at large x because $|v|<1$, and $\mathrm{Im}(x)<0$. If we expand the denominators within the large parentheses in powers of $\exp[-j\theta(x\pm1)]$, we have for the individual terms

$$e^{-j\theta(1+x)(1+v+n)} \text{ and } e^{+j\theta(1-x)(1-v+n)} ,$$

with $n\geq 0$. When we now consider the average over the oscillations for large $\theta=kL$, it appears that all terms average to zero in the first term and all terms average to zero in the second term, except near $x=1$. We will therefore return to Eq. (3.8) and obtain the average over frequency by evaluating the second term in the large parentheses near $x=1$.

The region $x\simeq 1$ is equivalent to $j_s \ll ka$, where we can write

$$\phi_s=\frac{L}{a}(k^2a^2-j_s^2)^{1/2}\simeq kL-\frac{j_s^2L}{2ka^2}=\theta-\frac{j_s^2L}{2ka^2} . \tag{4.3}$$

If we now put $b_s=ka$, $\phi_s=kL$, and

$$e^{-j\theta}-e^{-j\phi_s}=-je^{-j\theta}\frac{j_s^2L}{2ka^2} \tag{4.4}$$

in the second term in the large parentheses in Eq. (3.8), we find the approximate result for the smoothed coupling kernel

$$K_s(v)\simeq -\frac{2\pi}{ka^2}\sum_{s=1}^{\infty}\frac{2ka^2}{j_s^2L}=-\frac{\pi}{L} , \tag{4.5}$$

where we have used the identity $\sum_{s=1}^{\infty} j_s^{-2}=\frac{1}{4}$.

The one-cell kernel has been evaluated[1] for high frequency and is

$$\hat{K}(t',t)\simeq\begin{cases}0, & t'>t,\\ \dfrac{2(j-1)\sqrt{\pi}}{a\sqrt{k(t-t')}}, & t'<t.\end{cases} \tag{4.6}$$

We therefore have, from Eqs. (3.4), (3.8), and (4.5),

$$\frac{1+j}{\sqrt{\pi k}}\int_0^t\frac{dt'F(t')}{\sqrt{t-t'}}+\frac{ja}{2L}\int_0^g dt'F(t')\simeq\frac{a}{2\pi} . \tag{4.7}$$

The solution to Eq. (4.7) can be shown to be

$$F(t')=\frac{B}{\sqrt{t'}},\quad \int_0^g F(t)dt=2B\sqrt{g} , \tag{4.8}$$

where

$$\frac{1+j}{\sqrt{\pi k}}B\pi+\frac{ja}{2L}B2\sqrt{g}=\frac{a}{2\pi} . \tag{4.9}$$

We then obtain, for the admittance (per cell) from Eq. (3.7),

$$Z_0 Y_L(k) \simeq \frac{ka^2}{2B\sqrt{g}} = 2\pi ka \left[\frac{1+j}{2} \left[\frac{\pi}{kg} \right]^{1/2} + \frac{ja}{2L} \right] . \tag{4.10}$$

We have therefore expressed the admittance as a simple sum of the one-cell admittance and a modification due to the periodicity which is proportional to ka. The single-cell impedance is the same as that obtained by Dôme,[4] and by Heifets and Kheifets,[5] and corresponds to a $k^{-1/2}$ behavior at large frequency. But for the periodic structure the behavior for large k is strongly influenced by the added term in Eq. (4.10), which is much larger than the single-cell admittance. The impedance per cell can then be written, for large k, as

$$\frac{Z_L(k)}{Z_0} \simeq \frac{-jL}{\pi ka^2} \left[1 + \frac{1+j}{j} \frac{L}{a} \left[\frac{\pi}{kg} \right]^{1/2} \right]^{-1}$$
$$\simeq \frac{-jL}{\pi ka^2} + \frac{(1+j)L^2}{\pi ka^3} \left[\frac{\pi}{kg} \right]^{1/2} , \tag{4.11}$$

consisting of an imaginary part which decreases as k^{-1} and a smaller real part which decreases as $k^{-3/2}$. This result for the real part was obtained by Heifets and Kheifets,[3] although we find a different coefficient.

It is also possible to make an estimate of the transition region at which the frequency behavior changes from $k^{-3/2}$ to $k^{-1/2}$ for a large but finite number of cells, M. The contribution for finite M comes from the second term in Eq. (3.8), where we now include the oscillatory factor

$$1 - e^{jM(\phi - \phi_s)}$$

in performing the sum over m. For the value of $\theta - \phi_s$ given in Eq. (4.3), the neglect of the oscillating term corresponds to the assumption

$$M|\theta - \phi_s| = \frac{MLj_s^2}{2ka^2} \gg 1 \tag{4.12}$$

for the periodic result. Since the important values of j_s are of order 1, Eq. (4.12) corresponds to the transition frequency

$$k_t a \sim ML/a , \tag{4.13}$$

in agreement with that given by Heifets and Kheifets.[3]

V. CAUSALITY

In the analysis of the high-frequency behavior of a single obstacle,[1] we used the required analytic properties of the coupling impedance in the complex k plane to demonstrate that the $k^{-1/2}$ behavior implied an imaginary part equal to the negative of the real part. Let us explore the implications of causality on the types of behavior suggested by the result in Eq. (4.11).

It is straightforward to show that causality implies that $Z(k)$ is analytic in the lower half-plane, satisfying

$$Z(-k) = Z^*(k), \quad R(-k) = R(k), \quad X(-k) = -X(k) , \tag{5.1}$$

and the equations for the transform pairs

$$X(k) = \frac{2k}{\pi} \mathrm{P} \int_0^\infty \frac{dk' R(k')}{k'^2 - k^2} \tag{5.2}$$

and

$$R(k) = -\frac{2}{\pi} \mathrm{P} \int_0^\infty \frac{dk' k' X(k')}{k'^2 - k^2} . \tag{5.3}$$

If we consider that portion of the reactance which is of the form

$$X(k) \simeq \frac{B}{k^{3/2}} , \tag{5.4}$$

the high-frequency part of $R(k)$ in Eq. (5.3) is dominated by the behavior near the singularity at $k' = k$. Specifically, we have

$$R(k) \simeq -\frac{B}{\pi k} \mathrm{P} \int_0^\infty \frac{dk'}{\sqrt{k'}} \left[\frac{1}{k'-k} - \frac{1}{k'+k} \right]$$
$$\simeq -\frac{2B}{\pi k} \mathrm{P} \int_0^\infty du \left[\frac{1}{u^2 - k} - \frac{1}{u^2 + k} \right]$$
$$\simeq -\frac{B}{\pi k^{3/2}} \left[\ln \left| \frac{u - \sqrt{k}}{u + \sqrt{k}} \right| - 2 \arctan \frac{u}{\sqrt{k}} \right]_0^\infty . \tag{5.5}$$

Evaluating Eq. (5.5) at $u = \infty$ and 0 leads to

$$R(k) \simeq \frac{B}{k^{3/2}} . \tag{5.6}$$

At the same time Eq. (5.2) implies a reactance whose high-frequency dependence is given by

$$X(k) \simeq -\frac{2}{\pi k} \int_0^\infty dk' R(k') \tag{5.7}$$

since Eq. (5.6) guarantees the convergence of the integral in Eq. (5.7) as $k' \to \infty$. We therefore conclude that the high-frequency behavior of the impedance is of the form

$$Z(k) \simeq -\frac{jC}{k} + \frac{(1+j)B}{k^{3/2}} , \tag{5.8}$$

in agreement with the form in Eq. (4.11). In addition, we have demonstrated the existence of the sum rule

$$\int_0^\infty dk\, R(k) = \frac{Z_0 L}{2a^2} \tag{5.9}$$

for a periodic structure, from which one can estimate the value of k below which $R(k)$ must depart from $k^{-3/2}$ behavior.

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