High-frequency behavior of the longitudinal impedance for a cavity of general shape

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An ultrarelativistic beam bunch traveling along the axis of an azimuthally symmetric cavity of general shape connected to a beam pipe generates wake fields. An integral equation is derived for the longitudinal electric component of these wake fields at the beam-pipe radius. The kernel for this equation is approximated at high frequency for several different cavity shapes and the resulting integral equation is solved for the electric field at the beam-pipe radius. The longitudinal coupling impedance is obtained and shown to vary inversely as the square root of the frequency, in agreement with recent results obtained by others using different approximation techniques. The results are also shown to depend only on the axial extent of the cavity at the beam-pipe radius, but otherwise are independent of cavity shape for four different cavity geometries. In addition, the results are shown to be in agreement with the analytic properties of the coupling impedance as a function of complex frequency required by causality for an ultrarelativistic beam.

I. INTRODUCTION

The behavior of the coupling impedance for an obstacle in a beam pipe at high frequency has been of considerable importance because of recent interest in the transport and acceleration of short bunches.[1] The slow falloff of the real part at high frequency was obtained for a pillbox by Lawson[2] by means of a diffraction model, and confirmed in an approximate calculation by Dôme.[3] Heifets and Kheifets[4] use an iteration method to derive Dôme's result for the real part; namely, a falloff as $\omega^{-1/2}$. In this paper, we start with the integral equation for the field, and obtain an approximate kernel valid for high frequency for an obstacle of general shape. We then obtain a solution for the average field for a pillbox, leading to a confirmation of the results of Dôme[3] for both the real and imaginary parts of the impedance.

The analysis is then applied to three different triangular cavities, where we obtain exactly the same high-frequency behavior, as might be expected from the diffraction model.

II. ANALYSIS FOR A CAVITY OF GENERAL SHAPE

Gluckstern and Zotter[5] have derived an integral equation for the electric field at the pipe radius. Since we shall use this integral equation as a starting point for several of our analyses, we duplicate the derivation here.

Let us consider a beam pipe of radius a which enters and leaves an azimuthally symmetric cavity of general shape, as shown in Fig. 1. The longitudinal impedance can be obtained by field matching at $r=a$ in the usual manner. Specifically we identify the source fields in the ultrarelativistic limit as

$$E_z^{(s)}=0,\quad Z_0H_\phi^{(s)}=-\frac{Z_0I_0}{2\pi r}e^{-jkz}=-E_r^{(s)} \tag{2.1}$$

to which we add the pipe fields to obtain

$$E_z(r,z)=\int_{-\infty}^{\infty}dq\;A(q)e^{-jqz}\frac{J_0(Kr)}{J_0(Ka)}, \tag{2.2}$$

$$Z_0H_\phi(r,z)=-\frac{Z_0I_0}{2\pi r}e^{-jkz}-jk\int_{-\infty}^{\infty}dq\;A(q)e^{-jqz}\frac{J_0'(Kr)}{KJ_0(Ka)}. \tag{2.3}$$

Here $k=\omega/c$, the suppressed time dependence is $\exp(j\omega t)$, $Z_0=120\pi\ \Omega$ is the impedance of free space, and I_0 is the arbitrary driving current. We have defined

$$K=\sqrt{k^2-q^2} \tag{2.4}$$

and take the contour in the q plane as shown in Fig. 2 relative to the zeros of $J_0(Ka)$ so that we have only outgoing waves for the pipe fields as $z\to\pm\infty$. If we define $E_z(a,z)\equiv f(z)$ at the pipe radius, we have

$$f(z)=\int_{-\infty}^{\infty}dq\;A(q)e^{-jqz},\qquad A(q)=\frac{1}{2\pi}\int_0^g dz\,f(z)e^{jqz}, \tag{2.5}$$

where $f(z)$ vanishes outside the interval $0<z<q$.

The field in the cavity region outside the pipe region will be expanded into an orthonormal set of cavity

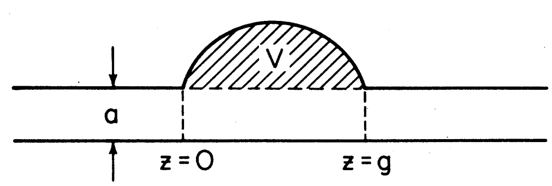

FIG. 1. Azimuthally symmetric cavity with beam pipes.

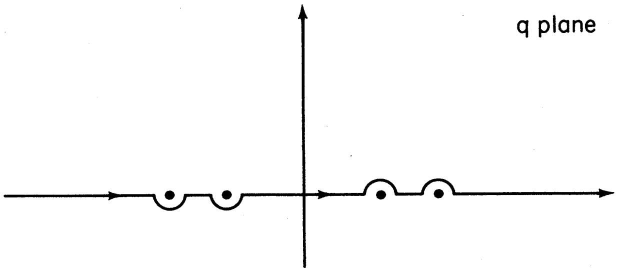

FIG. 2. Contour in the complex q plane.

modes[6] in the region V which satisfies the normal metallic boundary conditions on the outer wall of the cavity as well as at the pipe radius, $r=a$. These modes satisfy

$$\nabla\times\mathbf{E}_l=k_l\mathbf{H}_l,\quad \nabla\times\mathbf{H}_l=k_l\mathbf{E}_l\ , \tag{2.6}$$

and

$$\int_V \mathbf{E}_l\cdot\mathbf{E}_m\,dv=\int_V \mathbf{H}_l\cdot\mathbf{H}_m\,dv=\delta_{lm}\ . \tag{2.7}$$

Maxwell's equations for the actual fields are

$$Z_0\nabla\times\mathbf{H}=jk\mathbf{E},\quad \nabla\times\mathbf{E}=-jkZ_0\mathbf{H}\ . \tag{2.8}$$

If we (scalar) multiply the first by $\mathbf{E}_l$, the second by $\mathbf{H}_l$, integrate over V, integrate by parts, and use the known boundary conditions on $\mathbf{E}$, $\mathbf{H}$, $\mathbf{E}_l$, and $\mathbf{H}_l$, we find

$$k_lV_l=-jkZ_0I_l-2\pi a\int_0^g dz\,f(z)h_l(z) \tag{2.9}$$

and

$$k_lZ_0I_l=jkV_l\ . \tag{2.10}$$

Here

$$V_l=\int_V \mathbf{E}\cdot\mathbf{E}_l dv,\quad \mathbf{E}=\sum V_l\mathbf{E}_l\ , \tag{2.11}$$

$$I_l=\int \mathbf{H}\cdot\mathbf{H}_l dv,\quad \mathbf{H}=\sum I_l\mathbf{H}_l\ , \tag{2.12}$$

and $h_l\equiv H_\phi^{(l)}(a,z)$ is the normalized azimuthal magnetic field at the pipe radius.

Solving for I_l, we find

$$Z_0I_l=\frac{2\pi ajk}{k^2-k_l^2}\int_0^g dz\,f(z)h_l(z)\ , \tag{2.13}$$

which enables us to write the actual magnetic field at $r=a$, in terms of $f(z)$, as

$$Z_0H_\phi(a,z)=2\pi ajk\int_0^g dz'f(z')\sum_l\frac{h_l(z)h_l(z')}{k^2-k_l^2}\ . \tag{2.14}$$

Equation (2.14) is a special case of the general expression for the fields in the interior of a cavity as a function of the fields on the boundary. The simple form of Eq. (2.14) is a direct result of our using a Green's function for which $\partial G/\partial n=0$ on the boundary of the cavity, as explained in Appendix A.

The source and pipe fields at $r=a$ are given by Eqs. (2.3) and (2.5) as

$$Z_0H_\phi(a,z)=-\frac{jka}{2\pi}\int_0^g dz'f(z')\int_{-\infty}^{\infty}dq\,e^{-jq(z-z')}\hat{\mathcal{J}}(q)-\frac{Z_0I_0}{2\pi a}e^{-jkz}\ , \tag{2.15}$$

where

$$\hat{\mathcal{J}}(q)\equiv\frac{J_0'(Ka)}{KaJ_0(Ka)}\ . \tag{2.16}$$

Equating Eqs. (2.14) and (2.15) then leads to the integral equation for $f(z)$:

$$\int_0^g dz'f(z')[K_p(z'-z)+K_c(z,z')]=j\frac{Z_0I_0}{ka^2}e^{-jkz}\ , \tag{2.17}$$

where the kernel in the pipe is

$$K_p(u)=\int_{-\infty}^{\infty}dq\,e^{-jqu}\hat{\mathcal{J}}(q) \tag{2.18}$$

and where the kernel in the cavity is

$$K_c(z,z')=4\pi^2\sum_l\frac{h_l(z)h_l(z')}{k^2-k_l^2}\ . \tag{2.19}$$

The sum in Eq. (2.19) is over all azimuthally symmetric modes in the annular cavity volume V.

We can obtain a more explicit form for $K_p(u)$ in Eq. (2.18) by using the identity

$$\hat{\mathcal{J}}(q)\equiv\frac{J_0'(Ka)}{KaJ_0(Ka)}=2\sum_{s=1}^{\infty}\frac{1}{K^2a^2-j_s^2}=-2\sum_{s=1}^{\infty}\frac{1}{q^2a^2-b_s^2}\ , \tag{2.20}$$

where j_s are the zeros of $J_0(x)$ and where

$$b_s=(k^2a^2-j_s^2)^{1/2},\quad \beta_s=(j_s^2-k^2a^2)^{1/2}\ . \tag{2.21}$$

For positive u, the contour in Eq. (2.18) can be closed in the lower half-plane, enclosing the poles at $qa=b_s$ and $qa=-j\beta_s$. For negative u, the contour can be closed in the upper half-plane, enclosing the poles at $qa=-b_s$ and $qa=j\beta_s$. The result for $K_p(u)$ is then

$$K_p(u)=\frac{2\pi j}{a}\sum_{s=1}^{\infty}\frac{e^{-jb_s|u|/a}}{b_s}\ , \tag{2.22}$$

where b_s is to be replaced by $-j\beta_s$ when $j_s>ka$.

The longitudinal impedance of the cavity can be written as

$$\frac{Z(k)}{Z_0}=\frac{1}{Z_0I_0}\int_{-\infty}^{\infty}dz\,e^{jkz}E_z(0,z)=\frac{2\pi}{Z_0I_0}A(k)=\frac{1}{Z_0I_0}\int_0^g dz\,f(z)e^{jkz}\ . \tag{2.23}$$

If we renormalize $f(z)$ to include the factor Z_0I_0/ka^2, we find

$$\int_0^g dz'F(z')[K_p(|z'-z|)+K_c(z,z')]=je^{-jkz}\ , \tag{2.24}$$

with

$$\frac{Z(k)}{Z_0}=\frac{1}{ka^2}\int_0^g dz\, F(z)e^{jkz} . \tag{2.25}$$

One can absorb the factor $\exp(jkz)$ into $F(z)$ to obtain

$$\int_0^g dz' G(z')[\hat{K}_p(z'-z)+\hat{K}_c(z',z)]=\frac{2\pi j}{a} \tag{2.26}$$

and

$$\frac{Z(k)}{Z_0}=\frac{1}{2\pi ka}\int_0^g dz\, G(z) , \tag{2.27}$$

where

$$F(z)=\frac{a}{2\pi}G(z)e^{-jkz} , \tag{2.28}$$

$$\hat{K}_p(z'-z)=e^{-jk(z'-z)}K_p(|z'-z|) , \tag{2.29}$$

$$\hat{K}_c(z',z)=e^{-jk(z'-z)}K_c(z',z) . \tag{2.30}$$

The solution of Eq. (2.26) for $G(z')$ can then be used to find the longitudinal impedance.

III. EVALUATION OF THE KERNELS FOR HIGH FREQUENCY

Equation (2.27) suggests that rapidly varying components of $G(z)$ will not contribute significantly to the impedance. As a consequence, at high frequency our effort is directed toward obtaining the nonoscillatory part of $\hat{K}_p$ and $\hat{K}_c$ in Eqs. (2.29) and (2.30).

For the pipe kernel, we have, from Eq. (2.22), with $u=z'-z$,

$$\hat{K}_p(u)=\frac{2\pi j}{a}\sum_{s=1}^{\infty}\frac{e^{-jku-jb_s|u|/a}}{b_s} . \tag{3.1}$$

Since $b_s^2=k^2a^2-j_s^2$, we expect Eq. (3.1) to show rapid oscillations unless we have $u<0$ *and* $j_s \ll ka$. Our smooth approximation to $\hat{K}_p(u)$ is therefore obtained by setting $b_s=ka$ in the denominator and

$$\frac{b_s}{a}\simeq k-\frac{j_s^2}{2ka^2} \tag{3.2}$$

in the exponent, leading to

$$\langle \hat{K}_p(u)\rangle_k \simeq \begin{cases} 0, & u>0 , \\ (2\pi j/ka^2)\sum_{s=1}^{\infty}\exp(j|u|j_s^2/2ka^2) , & u<0 , \end{cases} \tag{3.3}$$

where $\langle\ \rangle_k$ implies a local average over k. The important contributions in the sum are those for which

$$j_s \sim (ka^2/|u|)^{1/2} \gg 1 . \tag{3.4}$$

As a consequence the sum over s can be approximated by an integral, leading to

$$\langle \hat{K}_p(z'-z)\rangle_k \simeq \begin{cases} 0, & z'>z , \\ \dfrac{j-1}{a}\left[\dfrac{\pi}{k(z-z')}\right]^{1/2}, & z'<z . \end{cases} \tag{3.5}$$

In order to evaluate the cavity kernel, we must choose a cavity geometry. For a pillbox of outer radius b, the magnetic field is

$$H_\phi^{(l)}(r,z)=AP_1(\sigma_m r)\cos(n\pi z/g) , \tag{3.6}$$

where

$$\sigma_m^2+(n\pi/g)^2=k_l^2 , \tag{3.7}$$

$$P_0(\sigma_m r)=Y_0(\sigma_m r)J_0(\sigma_m b)-J_0(\sigma_m r)Y_0(\sigma_m b) , \tag{3.8}$$

$$P_0'(\sigma_m r)=-P_1(\sigma_m r)=Y_0'(\sigma_m r)J_0(\sigma_m b)-J_0'(\sigma_m r)Y_0(\sigma_m b) . \tag{3.9}$$

The normalization of $H_\phi^{(l)}(r,z)$ requires that

$$\pi A^2 g(1+\delta_{n0})\int_a^b r\,dr\,P_1^2(\sigma_m r)=1 . \tag{3.10}$$

Since the boundary conditions for the cavity modes require $P_0(\sigma_m a)=0$, we find

$$A^2=\frac{2}{\pi g(1+\delta_{n0})}\frac{1}{b^2P_1^2(\sigma_m b)-a^2P_1^2(\sigma_m a)} . \tag{3.11}$$

We therefore obtain, for $K_c(z,z')$,

$$K_c(z,z') =\frac{8\pi}{g}\sum_{n=0}^{\infty}\frac{\cos n\pi\frac{z}{g}\cos n\pi\frac{z'}{g}}{1+\delta_{n0}} \times\sum_{m=0}^{\infty}\frac{P_1^2(\sigma_m a)[b^2P_1^2(\sigma_m b)-a^2P_1^2(\sigma_m a)]^{-1}}{k^2-\left[\dfrac{n\pi}{g}\right]^2-\sigma_m^2} . \tag{3.12}$$

As we shall show shortly, the dominant region for the sum over m is that for large m in which case the asymptotic form of the Bessel functions can be used to obtain

$$P_0^2(\sigma_m a)\simeq\frac{4}{\pi^2\sigma_m^2 ab}\sin^2\sigma_m(b-a) , \tag{3.13}$$

$$P_1^2(\sigma_m a)\simeq\frac{4}{\pi^2\sigma_m^2 ab}\cos^2\sigma_m(b-a) , \tag{3.14}$$

$$P_1^2(\sigma_m b)=\frac{4}{\pi^2\sigma_m^2 b^2} . \tag{3.15}$$

In this approximation, the wave number σ_m is given approximately by

$$\sigma_m \simeq m\pi/(b-a) \tag{3.16}$$

and

$$\frac{P_1^2(\sigma_m a)}{b^2P_1^2(\sigma_m b)-a^2P_1^2(\sigma_m a)}\simeq\frac{1}{a(b-a)} . \tag{3.17}$$

We therefore write

$$K_c(z,z')\equiv K_c^+(z,z')+K_c^-(z,z') , \tag{3.18}$$

where

$$K_c^{\pm}(z,z') \simeq \frac{4\pi}{ag(b-a)} \times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\cos n\pi\left[\frac{z'\pm z}{g}\right]}{(1+\delta_{n0})\left[k^2-\left[\frac{n\pi}{g}\right]^2-\left[\frac{m\pi}{b-a}\right]^2\right]} . \quad (3.19)$$

We now wish to extract the nonoscillatory part of $\hat{K}_c(z,z')$, defined by

$$\hat{K}_c(z,z')=e^{-jk(z'-z)}[\hat{K}_c^{+}(z,z')+\hat{K}_c^{-}(z,z')] . \quad (3.20)$$

In Appendix B we show that the main contributions to the sums occur for

$$\left|n-\frac{kg}{\pi}\right| \sim \frac{g}{|z'-z|} \geq 1, \quad m \sim \frac{b-a}{\pi}\left[\frac{k}{|z'-z|}\right]^{1/2} \gg 1 \quad (3.21)$$

and that the nonoscillatory part can be obtained by converting Eq. (3.19) to integrals over the continuous variables m and n. The result for the nonoscillatory part is

$$\langle \hat{K}_c(z,z')\rangle_k \simeq \begin{cases} 0, & z'>z ,\\ \frac{j-1}{a}\left[\frac{\pi}{k(z-z')}\right]^{1/2}, & z'<z , \end{cases} \quad (3.22)$$

identical to the result for $\langle \hat{K}_p(z'-z)\rangle_k$ in Eq. (3.5). Here $\langle\ \rangle_k$ once again implies a local average over k.

IV. SOLUTION TO THE INTEGRAL EQUATION FOR LARGE k

The approximate values of $\langle \hat{K}_p(z'-z)\rangle_k$ in Eq. (3.5) and $\langle \hat{K}_c(z,z')\rangle_k$ in Eq. (3.22) lead to the integral equation for $G(z')$:

$$\int_0^g \frac{dz'G(z')}{\sqrt{z-z'}} \simeq \frac{(1-j)\sqrt{\pi k}}{2} . \quad (4.1)$$

The solution to Eq. (4.1) is easily obtained by taking Laplace transforms, and is

$$G(z') \simeq \frac{1-j}{2\pi}\frac{\sqrt{\pi k}}{\sqrt{z'}} . \quad (4.2)$$

One then obtains the longitudinal impedance from Eq. (2.27):

$$\frac{Z(k)}{Z_0} \simeq \frac{1-j}{2\pi a}\frac{\sqrt{g}}{\sqrt{k\pi}} . \quad (4.3)$$

The high-frequency impedance in Eq. (4.3) agrees exactly with the result obtained by Dôme[3] using a different approximation scheme. Heifets and Kheifets[4] also obtain the same result for the real part of $Z(k)$.

V. TRIANGULAR CAVITY

The cavity kernel at high frequency was obtained in Sec. III for a pillbox cavity. In the process we found that for high frequency, the cavity modes were those corresponding to a two-dimensional Cartesian problem where the Bessel functions were replaced by sines and cosines. This is a simple consequence of the fact that $k(b-a) \gg 1$. In addition, the sums over modes could be replaced by integrals over the mode numbers. It is therefore possible to explore other cavity geometries in which two-dimensional solutions of the wave equation can be written down explicitly.

One such geometry is shown in Fig. 3 for a cavity with a 45°, 45°, 90° right-triangular cross section, where the (right-handed) coordinate system is taken to be (z,s,ζ). The orthonormal eigenmodes for the magnetic field are easily seen to be

$$H_{\zeta}^{m,n}(z,s)=\frac{2\left[\cos\frac{m\pi z}{g}\cos\frac{n\pi s}{g}+\cos\frac{n\pi z}{g}\cos\frac{m\pi s}{g}\right]}{g\sqrt{2\pi a}\,(1+\delta_{mn})^{1/2}(1+\delta_{n0})^{1/2}} , \quad (5.1)$$

where $\partial H/\partial n$ vanishes along all cavity boundaries. (The length of the cavity in the ζ direction is taken to be $2\pi a$.) The magnetic field at the pipe radius ($s=0$) is therefore

$$h_l(z)=\frac{1}{g}\left[\frac{2}{\pi a}\right]^{1/2}(1+\delta_{m0})^{-1/2}(1+\delta_{n0})^{-1/2} \times\left[\cos\frac{m\pi z}{g}+\cos\frac{n\pi z}{g}\right] . \quad (5.2)$$

From Eq. (2.5), we find for the cavity kernel

$$K_c \simeq \frac{4\pi}{ag^2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\left[\cos\frac{m\pi z'}{g}+\cos\frac{n\pi z'}{g}\right]\left[\cos\frac{m\pi z}{g}+\cos\frac{n\pi z}{g}\right]}{[k^2-(\pi/g)^2(m^2+n^2)](1+\delta_{m0})(1+\delta_{n0})} , \quad (5.3)$$

where we take into account the double counting of modes for $m\to n$, $n\to m$ by dividing by two. As before, the only terms which survive, after averaging over rapid oscillations at high frequency, are those involving the combination $z'-z$, so that we obtain

$$K_c(z,z') \simeq \frac{4\pi}{ag^2}\sum_{n=0}^{\infty}\frac{\cos\frac{n\pi(z-z')}{g}}{1+\delta_{n0}}\sum_{m=0}^{\infty}\frac{(1+\delta_{m0})^{-1}}{k^2-(\pi/g)^2(m^2+n^2)} . \quad (5.4)$$

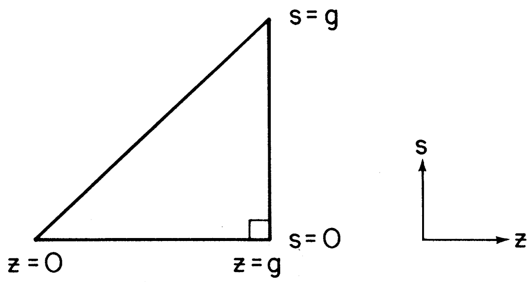

FIG. 3. Right-triangular cavity on a side.

As shown in Appendix B, the main contributions to the nonoscillatory part of

$$\hat{K}_c(z,z')=e^{-jk(z'-z)}K_c(z,z') \tag{5.5}$$

occur for high frequency near

$$\left| n-\frac{kg}{\pi} \right| \sim \frac{g}{|z'-z|} \geq 1, \quad \frac{m\pi}{g} \sim \left[\frac{k}{|z'-z|} \right]^{1/2} \gg 1 \ , \tag{5.6}$$

leading to

$$\langle \hat{K}_c(z,z') \rangle_k \simeq \begin{cases} 0, & z'>z \ , \\ \dfrac{j-1}{a} \left[\dfrac{\pi}{k(z-z')} \right]^{1/2}, & z'<z \ , \end{cases} \tag{5.7}$$

identical to what we found for the pillbox.

A similar calculation leads to the same result for the triangular cavities shown in Figs. 4(a) and 4(b). This leads to the conclusion that the high-frequency behavior is independent of the shape of the cavity, as might have been expected from a diffraction model in the short-wavelength limit.

The similarity of the impedance for the cavities in Figs. 3 and 4(a) turns out to be a general property of the impedances and wake functions for asymmetric cavities which are reflections of one another. This was noted by Gluckstern and Zotter[5] who used the symmetry properties of $K_p(|z'-z|)$ and $K_c(z',z)$ to prove this general result for all frequencies.

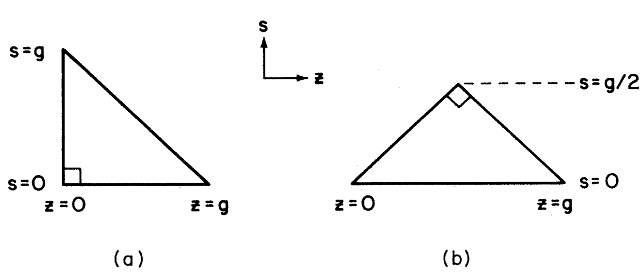

FIG. 4. (a) Reversed right-triangular cavity on a side. (b) Right-triangular cavity on its hypotenuse.

VI. CAUSALITY

It is of interest to test the implications of causality on the result in Eq. (4.3). The definition of $Z(k)$ in terms of the wake function for an ultrarelativistic point charge can be used to demonstrate that $Z(k)$ is analytic in the *lower* half complex k plane, with

$$Z(-k)=Z^*(k), \quad R(-k)=R(k), \quad X(-k)=-X(k) \ . \tag{6.1}$$

Use of the Hilbert transform pair allows us to write the following integral relations for $R(k)$ and $X(k)$:

$$X(k)=\frac{2k}{\pi}\mathrm{P}\int_0^\infty \frac{dk'R(k')}{k'^2-k^2} \ , \tag{6.2}$$

$$R(k)=-\frac{2}{\pi}\mathrm{P}\int_0^\infty \frac{dk'k'X(k')}{k'^2-k^2} \ . \tag{6.3}$$

If we assume a high-frequency dependence for $R(k)$ of the form

$$R(k)\simeq\frac{A}{\sqrt{k}} \ ,$$

we can obtain the high-frequency behavior of $X(k)$ from Eq. (6.2) as

$$\begin{aligned} X(k) &\simeq \frac{A}{\pi}\mathrm{P}\int_0^\infty \frac{dk'}{\sqrt{k'}} \left[\frac{1}{k'-k} - \frac{1}{k'+k} \right] \\ &\simeq \frac{2A}{\pi}\mathrm{P}\int_0^\infty du \left[\frac{1}{u^2-k} - \frac{1}{u^2+k} \right] \\ &\simeq \frac{A}{\pi\sqrt{k}} \left[\ln\left| \frac{u-\sqrt{k}}{u+\sqrt{k}} \right| - 2\arctan\frac{u}{\sqrt{k}} \right]_0^\infty \ . \end{aligned} \tag{6.4}$$

After taking the limits at $u=\infty$ and 0, we find

$$X(k)\simeq -\frac{A}{\sqrt{k}} \ , \tag{6.5}$$

implying the form

$$Z(k)=(1-j)\frac{A}{\sqrt{k}} \ , \tag{6.6}$$

totally consistent with Eq. (4.5).

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APPENDIX A

We shall apply Green's theorem

$$\int dv[\phi(\nabla^2+k^2)\psi-\psi(\nabla^2+k^2)\phi] = \int dS'[\phi(n'\cdot\nabla\psi)-\psi(n'\cdot\nabla\phi)] \quad \text{(A1)}$$

to the rectangular magnetic field components, with

$$\psi(\mathbf{x}')=G(\mathbf{x},\mathbf{x}'), \quad \phi(\mathbf{x}')=H_i(\mathbf{x}') , \quad \text{(A2)}$$

and

$$(\nabla_{x'}^2+k^2)G(\mathbf{x},\mathbf{x}')=\delta(\mathbf{x}'-\mathbf{x}) . \quad \text{(A3)}$$

The resulting expression for the magnetic field in the interior of a source-free cavity can be shown to be

$$\mathbf{H}(\mathbf{x})=\frac{jk}{Z_0}\int dS' G(\mathbf{x},\mathbf{x}')[\mathbf{n}'\times\mathbf{E}(\mathbf{x}')] + \int dS'\mathbf{H}(\mathbf{x}')\frac{\partial G(\mathbf{x},\mathbf{x}')}{\partial n'} . \quad \text{(A4)}$$

In our application, we construct the Green's function $G(\mathbf{x},\mathbf{x}')$ as

$$G(\mathbf{x},\mathbf{x}')=\sum_l \frac{\mathbf{H}_l(\mathbf{x})\cdot\mathbf{H}_l(\mathbf{x}')}{k^2-k_l^2} , \quad \text{(A5)}$$

where $\mathbf{H}_l(\mathbf{x})$ are the orthonormal magnetic-field modes of wave number k_l, defined in Eqs. (2.6) and (2.7). Clearly

$$(\nabla^2+k^2)\sum_l \frac{\mathbf{H}_l(\mathbf{x})\cdot\mathbf{H}_l(\mathbf{x}')}{k^2-k_l^2} = \sum_l \mathbf{H}_l(\mathbf{x})\cdot\mathbf{H}_l(\mathbf{x}') = \delta(\mathbf{x}'-\mathbf{x}) , \quad \text{(A6)}$$

as required by Eq. (A3). Moreover, the functions $\mathbf{H}_l(\mathbf{x})$ are required to satisfy the "metallic" boundary condition

$$\frac{\partial \mathbf{H}_l^{(t)}(\mathbf{x})}{\partial n}=0 \quad \text{and} \quad H_l^{(n)}(x)=0 , \quad \text{(A7)}$$

where t and n denote the tangential and normal (to the cavity boundary) directions. Our Green's function in Eq. (A5) therefore satisfies the boundary condition

$$\frac{\partial G(\mathbf{x},\mathbf{x}')}{\partial n'}=0 \quad \text{(A8)}$$

and Eq. (A4) reduces to

$$\mathbf{H}(\mathbf{x})=\frac{jk}{Z_0}\int dS' G(\mathbf{x},\mathbf{x}')[\mathbf{n}'\times\mathbf{E}(\mathbf{x}')] . \quad \text{(A9)}$$

In our application, Eq. (A9) reduces to

$$Z_0 H_\phi(a,z)=2\pi a jk \int_0^g dz' f(z')\sum_l \frac{h_l(z)h_l(z')}{k^2-k_l^2} , \quad \text{(A10)}$$

as in Eq. (2.14), with $h_l(z)$ being the ϕ component of $\mathbf{H}_l(\mathbf{x})$ at the beam-pipe radius.

APPENDIX B

Our task is to find the nonoscillatory part of

$$\hat{K}_c^{\pm}(z,z') \simeq \frac{4\pi}{ag(b-a)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{e^{-jk(z'-z)}\cos n\pi\left[\dfrac{z'\pm z}{g}\right]}{(1+\delta_{n0})\left[k^2-\left[\dfrac{n\pi}{g}\right]-\left[\dfrac{m\pi}{b-a}\right]^2\right]} . \quad \text{(B1)}$$

It is clear that the only surviving term will be $\hat{K}_c^-(z,z')$, where the oscillatory part of the cosine term will cancel that of $\exp[-jk(z'-z)]$ in the vicinity

$$n \sim \frac{kg}{\pi} . \quad \text{(B2)}$$

Specifically we find, in the region defined by Eq. (B2),

$$\langle \hat{K}_c^+(z,z')\rangle_k \simeq 0 \quad \text{(B3)}$$

and

$$\langle \hat{K}_c(z,z')\rangle_k \simeq \frac{2\pi}{ag(b-a)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\exp[j(n-kg/\pi)\theta]}{(1+\delta_{n0})\left[\dfrac{2k\pi}{g}\left[\dfrac{kg}{\pi}-n\right]-\left[\dfrac{m\pi}{b-a}\right]^2\right]} , \quad \text{(B4)}$$

where $\theta\equiv\pi(z'-z)/g$. Equation (B4) clearly indicates that the important contributions come from

$$\left|n-\frac{kg}{\pi}\right| \sim \frac{g}{|z'-z|} \geq 1 \quad \text{(B5)}$$

and consequently from

$$\frac{m\pi}{b-a} \sim \left[\frac{k}{|z'-z|}\right]^{1/2} \gg 1 . \quad \text{(B6)}$$

The sum over m can therefore by approximated by an integral, leading to

$$\langle \hat{K}_c(z,z')\rangle_k \simeq \frac{\sqrt{\pi}}{a\sqrt{2kg}} \sum_{n=0}^{\infty} \frac{e^{j(n-kg/\pi)\theta}}{1+\delta_{n0}} \times \begin{cases} \dfrac{j}{\sqrt{kg/\pi-n}}, & n<\dfrac{kg}{\pi}, \\ -\dfrac{1}{\sqrt{n-kg/\pi}}, & n>\dfrac{kg}{\pi}, \end{cases} \tag{B7}$$

where the sign following the curly bracket for $n > kg/\pi$ is determined by giving n a small positive imaginary part to guarantee eventual decay of the cavity modes in time.

The sum over n in Eq. (B7) can be extended to $n=-\infty$ and the term involving δ_{n0} can be dropped without affecting the result. If one writes

$$\frac{kg}{\pi} = N + \Delta , \tag{B8}$$

where N is the nearest integer to kg/π and where $-\frac{1}{2} \le \Delta < \frac{1}{2}$, Eq. (B7) becomes

$$\langle K_c(z,z')\rangle_k \simeq -\left[\frac{\pi}{2kg}\right]^{1/2} \sum_{p=-\infty}^{\infty} e^{j(p-\Delta)\theta} \times \begin{cases} \dfrac{j}{(\Delta-p)^{1/2}}, & p<\Delta , \\ -\dfrac{1}{(p-\Delta)^{1/2}}, & p>\Delta . \end{cases} \tag{B9}$$

If one averages over Δ (corresponding to a "local" average over k) the sum over p becomes an integral over the continuous variable w defined by $p=\Delta\pm w$, with the result

$$\langle \hat{K}_c(z,z')\rangle_k \simeq \frac{1}{a}\left[\frac{\pi}{2kg}\right]^{1/2} \int_0^{\infty} \frac{dw}{\sqrt{w}} (je^{-jw\theta} - e^{jw\theta}) . \tag{B10}$$

Using

$$\int_0^{\infty} \frac{dw}{\sqrt{w}} e^{jwx} = \left[\frac{1}{|x|}\right]^{1/2} (1\pm j) \tag{B11}$$

depending on whether $x \gtrless 0$, one finds

$$\langle \hat{K}_c(z,z')\rangle_k \simeq \frac{\sqrt{\pi}}{2a\sqrt{kg|z'-z|}}[j(1\mp j)-(1\pm j)] \tag{B12}$$

or

$$\langle K_c(z,z')\rangle_k \simeq \begin{cases} 0, & z'>z , \\ \dfrac{(j-1)\sqrt{\pi}}{a\sqrt{k(z-z')}}, & z'<z , \end{cases} \tag{B13}$$

identical to the result in Eq. (3.5). We have therefore shown that, for large k, the nonoscillatory part of $\hat{K}_c(z,z')$ is correctly given by treating the sums over m and n as integrals.

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