Hadron scattering in the large-N_c limit as a problem in linear algebra

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We study meson-baryon scattering in the nonrelativistic quark model, to leading order in $1/N_c$. The problem is elegantly expressed as a matrix equation $\mathbf{T}=\Pi\cdot\mathbf{T}$, where $\mathbf{T}$ is the scattering matrix and Π is a projection operator expressible as an $18j$ symbol of the second kind. The transformation that diagonalizes Π is the crossing relation between s-channel and t-channel partial-wave amplitudes. The solutions to the equation are those scattering amplitudes which satisfy the "$I_t=J_t$ rule" and the "proportionality rule," two large-N_c selection rules which have been derived using Skyrmion and one-boson-exchange techniques as well. The projection-equation approach is also extended to operators other than the the meson-baryon $\mathbf{T}$ matrix.

I. OVERVIEW

In this paper we study two-flavor meson-baryon scattering to leading order in the $1/N_c$ expansion. Our tool is the old nonrelativistic quark model, generalized to N_c colors, although (as we shall make clear in Sec. II) the discussion could equally well be framed in terms of the Skyrme model.

The partial-wave meson-baryon $\mathbf{T}$ matrix will be shown to obey a matrix equation of the form

$$\mathbf{T}=\Pi(L,L')\cdot\mathbf{T}\ , \tag{1}$$

with L and L' the initial and final orbital angular momenta, respectively. This turns out to be a highly nontrivial constraint on the scattering. For each L and L', $\Pi(L,L')$ is a numerical matrix—in fact, a projection operator—formed from the various isospin and angular momentum invariants in the problem; its construction will lead us to an excursion into the forgotten realm of the higher j symbols. The solutions to Eq. (1), the eigenvectors of $\Pi(L,L')$ with eigenvalue unity, are the scattering amplitudes that obey two large-N_c selection rules that were recently derived using both Skyrmion[1] and one-boson-exchange[2] techniques: the "$I_t=J_t$ rule" and the "proportionality rule" described later in this introductory section.

Before delving into specifics we briefly review some prominent developments in the large-N_c approach to strong-interaction physics. Hopefully, this will place our present findings in the proper perspective. Over the past 15 years, the $1/N_c$ expansion has provided much qualitative insight into the properties of hadrons.[3–5] Although originally formulated by 't Hooft as a means of organizing the quark-gluon diagrams of QCD (Ref. 3), it has been gainfully applied to approximation schemes as diverse as the nonrelativistic quark model,[6] the Skyrme model,[7] and one-boson exchange.[2] Admittedly, we are still far from having a complete description of the large-N_c world—such a description would entail specifying the full hadron spectrum and scattering matrix, an unimaginable task. Nevertheless, we can already point to substantial progress on both fronts.

Large-N_c spectroscopy. Perhaps the most interesting feature of the large-N_c hadron spectrum is the existence of infinite towers of states (quite aside from Regge trajectories). In the meson sector, asymptotic freedom requires the presence of an infinite number of resonances in each J^{PC} channel. Both the masses and the mass splittings scale as N_c^0 and the resonances are narrow, with widths $\sim N_c^{-1}$, the dominant decay modes being resonant two-body final states. Exotic four-quark states are suppressed.

The role played by baryons in the $1/N_c$ expansion was clarified by Witten,[5] who showed that, while their masses scale as N_c, as one would naively expect for an object with N_c valence quarks, their sizes and shapes have a smooth, N_c-independent limit. These attributes, Witten noted, are those of a monopole or soliton, an analogy that proved extremely fruitful, culminating in the resurgence of the Skyrme model.[8,7]

Among the nonstrange baryons in larger N_c, one finds a tower of positive-parity states with equal spin and isospin:

$$I=J=\frac{1}{2},\frac{3}{2},\ldots,\frac{N_c}{2}\qquad\text{(fermionic baryons, } N_c \text{ odd)} \tag{2a}$$

or

$$I=J=0,1,2,\ldots,\frac{N_c}{2}\qquad\text{(bosonic baryons, } N_c \text{ even)}\ . \tag{2b}$$

In our world, with $N_c=3$, this tower comprises only the nucleon and the Δ. Viewed from the perspective of the nonrelativistic quark model, these are the ground-state baryons formed from N_c quarks in S-wave orbitals bound by a confining potential. Alternatively, in the Skyrme model, they are the quantized rotational exitations of the "hedgehog" soliton[7]

$$U_0 = \exp[iF(r)\hat{\mathbf{r}}\cdot\tau] . \quad (3)$$

From either point of view, these states are expected to be degenerate and stable in the large-N_c limit, with mass splittings[7] of order N_c^{-1}, and decay widths[9] of order N_c^{-3} in the chiral limit.

Additionally, for any N_c, there is a rich array of resonances, mostly with $I \neq J$, in which the vibrational and rotational modes of the quarks are excited. These states can be formed from the $I=J$ baryons discussed above through pion scattering, for example, and will have mass splittings and decay widths characteristically of order N_c^0. As Mattis and Karliner demonstrated in their leading-order large-N_c Skyrme model analysis,[10] the baryon spectrum in large N_c is surprisingly predictive in our world, in which $N_c=3$. Almost all the observed N^* and Δ^* resonances could be accounted for in their scheme, to within 7% accuracy of the measured masses, after a two-parameter fit of the constants in the Skyrme Lagrangian.

Large-N_c scattering matrix. Some interesting things can also be said about the scattering matrix in large N_c. Meson scattering is especially simple: mesons are stable and noninteracting to leading order in $1/N_c$. In contrast, baryon scattering, viewed from either the quark-gluon[5] or the soliton picture,[11] is thought to be extremely complicated. The calculation of the scattering matrix can be expressed as a saddle-point problem, which requires the solution of nonlinear classical equations. Not surprisingly, most of the attention has therefore focused on the intermediate case of meson-baryon collisions. The fact that mesons and baryons have different mass scales in large N_c is a boon: the baryons can be treated as nonrecoiling objects, essentially external potentials, and the scattering of the mesons is governed by the *linear* equations for small perturbations about the baryon background.[5]

When the baryons involved in the collisions are members of the $I=J$ tower (2), one can go much further by exploiting a peculiar symmetry called "K spin," which is defined as the vector sum of isospin and angular momentum:

$$\mathbf{K}=\mathbf{I}+\mathbf{J} .$$

Of course, there can be no preferred alignment of spatial and isospin axes; one ought really to define a family of K-spin operators $\mathbf{K}_A$ in which the relative orientation is parametrized by an arbitrary SU(2) matrix A:

$$\mathbf{K}_A=\mathbf{I}+R_A\cdot\mathbf{J}, \quad (R_A)_{ij}=\tfrac{1}{2}\mathrm{tr}\tau^i A\tau^j A^\dagger .$$

Many qualitative features of the meson-baryon scattering matrix, including the relative sizes and signs of the partial-wave amplitudes, find a natural explanation in terms of this symmetry.[10,12–17] As recently shown by Braaten and Cai,[18] K spin can also account for certain qualitative features of baryon-baryon scattering, although the relation of these predictions to the large-N_c limit of QCD is less firm.

The $I=J$ baryons can be expanded in terms of states $|A\rangle$ that are singlets under $\mathbf{K}_A$ (see Sec. II):

$$\mathbf{K}_A|A\rangle=0 .$$

The advantage of this choice of basis is that in the large-N_c limit, the interactions of baryons become diagonal in A. This was first recognized in the context of the nonrelativistic quark model by Manohar.[6] In the Skyrme model, Mattis and Peskin[12] and, independently, Hayashi *et al.*[13] exploited K-spin symmetry to derive linear relations between the partial-wave amplitudes for the processes $\pi N\to\pi N$ and $\pi N\to\pi\Delta$, relations that express the isospin-$\frac{3}{2}$ amplitudes, for example, as linear combinations of the isospin-$\frac{1}{2}$ amplitudes with the same value of orbital angular momentum. These relations were subsequently generalized to the case of arbitrary meson spin and isospin.[14]

Most recently, within both the Skyrmion[17,1] and one-boson-exchange[2] frameworks, the linear relations of Ref. 14 have been shown to simplify dramatically when the partial-wave amplitudes are recast in terms of t-channel, rather than s-channel, quantum numbers (see Fig. 1). In the t-channel formulation, the meson-baryon collision is pictured in terms of boson exchange, whereas in the s-channel formulation, it is viewed as occurring via baryon resonance formation. The constraints due to K spin reduce to two simple rules in the t-channel picture, both valid to leading order in $1/N_c$.

(1) The $I_t=J_t$ rule (Refs. 1 and 2). The isospin of the exchanged state must equal its total angular momentum (spin + orbital).

(2) The proportionality rule (Ref. 1). For given $I_t=J_t$, meson-baryon amplitudes which differ only in the choice of initial and/or final $I=J$ baryon (e.g., a Δ instead of a nucleon), are proportional. The energy-independent constants of proportionality are $[(2R+1)(2R'+1)]^{1/2}$, where R and R' denote the initial and final spin-isospin representations of the baryon ($\frac{1}{2}$ for nucleons, $\frac{3}{2}$ for Δ's, etc.).

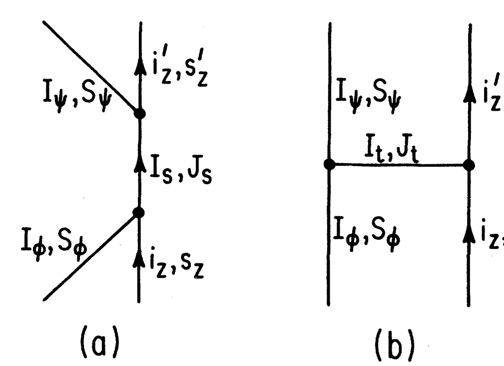

FIG. 1. (a) s-channel vs (b) t-channel meson-baryon scattering diagrams. Directed lines are baryons, undirected lines are mesons. The various quantum numbers are defined later in the text.

Summary of results. We now return to our main result, Eq. (1), and make it more precise. This will require some notation. As stated, we will be considering generic two-flavor meson-baryon collisions of the type

$$\phi + B \rightarrow \psi + B' .$$

ϕ and ψ stand for arbitrary mesons of spin S_ϕ and S_ψ, isospin I_ϕ and I_ψ, and parity P_ϕ and P_ψ (Ref. 19). B and B' denote $I=J$ baryons in the tower (2a) or (2b), depending on whether N_c is odd or even. A partial-wave amplitude for this process can be written as $\mathbf{T}_{I_s J_s SS'LL'RR'}$, with I_s and J_s the total s-channel isospin and angular momentum, S and S' the total spins of the initial and final meson-baryon systems, L and L' the initial and final orbital angular momenta, and R and R' the initial and final baryon spin-isospin representations. Parity ensures that $(-1)^L P_\phi = (-1)^{L'} P_\psi$.

We will show that, for each L and L', and to leading order in $1/N_c$, these scattering amplitudes are solutions to a matrix equation of the form

$$\mathbf{T}_{I_s J_s SS'LL'RR'} = \sum_{\tilde{I}_s \tilde{J}_s \tilde{S} \tilde{S}' \tilde{R} \tilde{R}'} \Pi(L,L')^{\tilde{I}_s \tilde{J}_s \tilde{S} \tilde{S}' \tilde{R} \tilde{R}'}_{I_s J_s SS'RR'} \cdot \mathbf{T}_{\tilde{I}_s \tilde{J}_s \tilde{S} \tilde{S}' LL' \tilde{R} \tilde{R}'} . \tag{4}$$

The matrix $\Pi(L,L')$ can be written compactly as an $18j$ *symbol of the second kind*:

$$\Pi(L,L')^{\tilde{I}_s \tilde{J}_s \tilde{S} \tilde{S}' \tilde{R} \tilde{R}'}_{I_s J_s SS'RR'} = k^{-1}[\tilde{I}_s][\tilde{J}_s]([R][R'][S][S'][\tilde{R}][\tilde{R}'][\tilde{S}][\tilde{S}'])^{1/2}(-1)^{I_s+J_s+S+S'-\tilde{I}_s-\tilde{J}_s-\tilde{S}-\tilde{S}'} \times \begin{vmatrix} \tilde{R} & & \tilde{S} & & \tilde{J}_s & & \tilde{S}' & & \tilde{R}' & & \tilde{I}_s & \\ & S_\phi & & L & & L' & & S_\psi & & I_\psi & & I_\phi \\ R & & S & & J_s & & S' & & R' & & I_s & \end{vmatrix} , \tag{5}$$

with $[R]$ short for $2R+1$, etc. The reader unfamiliar with this object will find it defined graphically in Fig. 2. Alternatively, it can be written as a product of six $6j$ symbols, as explained in Appendix A [Eq. (A10)].

The summations in Eq. (4) extend to infinity over all non-negative values (either integral or half-integral as appropriate, depending on whether N_c is even or odd), subject to the 12 triangle inequalities implicit in the $18j$ symbol, which can be read off from the vertices in Fig. 2. This divergent sum is compensated by the infinite normalization constant k in Eq. (5), defined by

$$k = \sum_R [R]^2 , \tag{6}$$

where R runs over the non-negative integers or half-integers as above. The infinities will cancel out in the end.

The remainder of the paper is organized as follows. Section II describes the construction of the large-N_c baryon as a superposition of the $\mathbf{K}_A$-invariant states $|A\rangle$, a construction that closely parallels the collective coordinate approach of the Skyrme model. The section is patterned on an elegant paper of Manohar's, which established the group-theoretic equivalence of the Skyrme and nonrelativistic quark models in the large-N_c limit.[6] Section III applies this formalism to the meson-baryon scattering matrix, and culminates in Eqs. (4) and (5). In Sec. IV we diagonalize the matrix $\Pi(L,L')$, and present the complete solution to Eq. (4). The main conclusions of Refs. 1 and 2, the $I_t = J_t$ rule and the proportionality rule, are recaptured, and the interpretation of $\Pi(L,L')$ as a projection operator is made manifest. Section V clarifies the relation between the present results and previous work on the subject. The projection-equation approach is also extended to operators other than the meson-baryon $\mathbf{T}$ matrix.

Most of our mathematical apparatus is stored in two appendixes. Appendix A is designed as a brief, self-contained introduction to the higher j symbols, material that (to the best of our knowledge) is not otherwise available in predigested form. All the identities needed in Secs. III–V are derived. Appendix B contains some technical details necessary for the proper normalization of the large-N_c baryon.

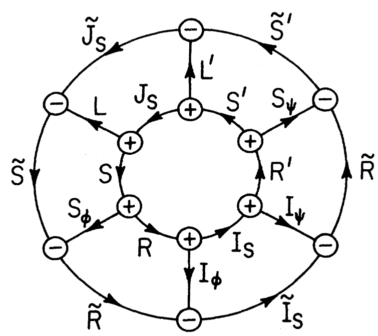

FIG. 2. The $18j$ symbol of the second kind, using the graphical notation of Yutsis, Levinson, and Vanagas (Ref. 20). Each line is labeled by an angular momentum j and has an implicit magnetic quantum number m. Each vertex represents a $3j$ symbol $\begin{pmatrix} j_1 & j_2 & j_3 \\ \pm m_1 & \pm m_2 & \pm m_3 \end{pmatrix}$, with the j's and m's ordered (counter)clockwise if the orientation of the vertex is negative (positive). An arrow pointing toward (away from) a vertex means that the corresponding m in the $3j$ symbol should have a minus (plus) sign. Regardless of arrow direction, each internal line comes with a factor $(-1)^{j-m}$. A sum on all m's associated with internal lines is implied.

II. BARYONS IN THE LARGE-N_c QUARK MODEL

In this section we review Manohar's construction of the baryon wave function in the two-flavor nonrelativistic quark model generalized to arbitrary values of N_c (Ref. 6). Consider the quark state

$$\psi^0=\frac{1}{\sqrt{2}}|u\downarrow-d\uparrow\rangle\ .$$

In component form,

$$\psi^0_{i_z s_z}=\frac{i}{\sqrt{2}}(\tau_2)_{i_z,s_z}\ .$$

This state is constructed to be a K-spin singlet:

$$\mathbf{K}\psi^0=(\mathbf{I}+\mathbf{J})\psi^0=0\ . \qquad (7)$$

Following Skyrme-model nomenclature, one might refer to ψ^0 as a "hedgehog quark state." The N_c-fold tensor product

$$\psi^0\otimes\cdots\otimes\psi^0$$

is the quark-model analog of the hedgehog Skyrmion U_0, which, too, is annihilated by $\mathbf{K}$.

One can also consider rotated hedgehog quark states

$$\psi^{(A)}_{i_z s_z}=\frac{i}{\sqrt{2}}(A\tau_2)_{i_z,s_z}\ ,$$

with A an arbitrary SU(2) matrix. Their elementary properties are

$$\mathbf{K}_A\psi^{(A)}=0\ , \qquad (8)$$

$$\psi^{(-A)}=-\psi^{(A)}\ , \qquad (9)$$

and

$$\langle\psi^{(A)}|\psi^{(B)}\rangle=\tfrac{1}{2}\mathrm{tr}\,A^{\dagger}B\ . \qquad (10)$$

Under general isospin and angular momentum transformations A_I and A_J, these states transform as

$$\psi^{(A)}_{i_z s_z}\to\frac{i}{\sqrt{2}}(A_I A\tau_2 A_J^T)_{i_z,s_z}=\frac{i}{\sqrt{2}}(A_I A A_J^{\dagger}\tau_2)_{i_z,s_z}=\psi^{(A_I A A_J^{\dagger})}_{i_z s_z}\ . \qquad (11)$$

The $\mathbf{K}_A$-invariant N_c-quark states $|A\rangle$ are defined by

$$|A\rangle=\psi^{(A)}\otimes\cdots\otimes\psi^{(A)}\ .$$

From Eqs. (8)–(10) it follows that

$$\mathbf{K}_A|A\rangle=0\ , \qquad (12)$$

$$|-A\rangle=(-1)^{N_c}|A\rangle\ , \qquad (13)$$

and

$$\langle A|B\rangle=(\tfrac{1}{2}\mathrm{tr}\,A^{\dagger}B)^{N_c}\ . \qquad (14)$$

Analogous states can be constructed in any model where $\mathbf{I}$ and $\mathbf{J}$ are defined. In Skyrmion physics, A is known as the *rotational collective coordinate* of the baryon and $|A\rangle$ represents the rotated Skyrmion $AU_0A^{\dagger}$.

We now explain how to express the physical $I=J$ baryons considered in Sec. I as quantum superpositions of the K-symmetry states $|A\rangle$. Actually, for finite N_c, this can be accomplished in an infinite number of ways, since the $|A\rangle$'s are overcomplete. [There are infinitely many of them, while the dimension of the space of N_c-quark states is only $(N_c+1)(N_c+2)(N_c+3)/6$.] The expansion presented here, due to Manohar,[6] is distinguished by the fact that, apart from normalization factors, the wave functions will be independent of N_c.

Let $|{}^{R}_{i_z s_z}\rangle$ stand for the positive-parity baryon state that transforms as

$$|R,i_z\rangle_{\mathrm{isospin}}\times|R,s_z\rangle_{\mathrm{spin}}\ .$$

We would like to write this state in the form

$$|{}^{R}_{i_z s_z}\rangle=\int_{\mathrm{SU}(2)}dA\,\chi^R_{i_z s_z}(A)|A\rangle\ . \qquad (15)$$

Here, $\chi^R_{i_z s_z}(A)$ is the wave function of the baryon (to be determined) in the group space spanned by A, and dA is the invariant measure on the group, normalized so that $\int dA=1$.

Under the general isospin and angular momentum transformations given in Eq. (11),

$$|{}^{R}_{i_z s_z}\rangle\to\int_{\mathrm{SU}(2)}dA\,\chi^R_{i_z s_z}(A)|A_I A A_J^{\dagger}\rangle=\int_{\mathrm{SU}(2)}dA\,\chi^R_{i_z s_z}(A_I^{\dagger} A A_J)|A\rangle\ , \qquad (16)$$

using the invariance of the group measure: $dA=d(A_I^{\dagger}AA_J)$. This will be the desired transformation law if

$$\chi^R_{i_z s_z}(A_I^{\dagger}AA_J)=\sum_{i_z',s_z'}\chi^{(R)}_{i_z' s_z'}(A)D^{(R)}_{i_z' i_z}(A_I)D^{(R)}_{s_z' s_z}(A_J)\ . \qquad (17)$$

Equation (17) is satisfied by the choice

$$\chi^R_{i_z s_z}(A)=c_R(N_c)D^{(R)}_{s_z i_z}(i\tau_2 A^{\dagger})\ , \qquad (18)$$

where

$$c_R(N_c)=2^{(N_c-1)/2}[R]\times\left\{\sum_{m=-R}^{R}\left[\binom{N_c}{N_c/2+m}-\frac{1}{4}\binom{N_c+2}{(N_c+2)/2+m}\right]\right\}^{-1/2}\ . \qquad (19)$$

As shown in Appendix B, this choice of normalization factor $c_R(N_c)$ ensures that the physical baryon states are orthonormal:

$$\langle{}^{R'}_{i_z' s_z'}|{}^{R}_{i_z s_z}\rangle=\int dA\int dB\,\chi^{R'*}_{i_z' s_z'}(A)\langle A|B\rangle\chi^R_{i_z s_z}(B)=\delta_{RR'}\delta_{i_z i_z'}\delta_{s_z s_z'}\ . \qquad (20)$$

For future reference, we note that, as $N_c\to\infty$,

$$c_R(N_c)\to\left[\frac{\pi N_c^3}{8}\right]^{1/4}\sqrt{[R]}\ . \qquad (21)$$

The K-symmetry wave functions (18) were first constructed within the framework of Skyrmion physics by Adkins, Nappi, and Witten.[7] Although these authors were interested in the large-N_c limit, the wave functions are actually valid for any value of N_c. In particular, for $N_c=3$, Eqs. (15) and (18) reproduce the familiar SU(6) expressions for the nucleon and Δ, as the reader can check.

Because the spin-isospin states (unlike the $\mathbf{K}_A$-invariant states) form a basis for the space of states, Eq. (15) has a unique inverse:

$$|A\rangle = \sum_{R=N_c/2,N_c/2-1,\ldots} \sum_{i_z,s_z} [R] c_R^{-2}(N_c) \chi^{R*}_{i_z s_z}(A) |^{R}_{i_z s_z}\rangle \ . \tag{22}$$

The easiest way to verify this formula is to show that $|A\rangle$ so defined indeed satisfies Eqs. (12)–(14) (see Appendix B).

Henceforth, we shall focus on the case $N_c \to \infty$. Two important simplifications occur in this limit which justify the use of the $\mathbf{K}_A$-invariant states $|A\rangle$. First, these states become linearly independent and form a bona fide basis for the infinite-dimensional state space; consequently, the wave functions $\chi^R_{i_z s_z}(A)$ given in Eq. (18) become unique. Second, the inner product $\langle A|B\rangle$ becomes more and more sharply peaked about the points $B=\pm A$; this is obvious from Eq. (14), since $|\frac{1}{2}\mathrm{tr}(A^\dagger B)|<1$ at all other points. To quantify this last statement we introduce the SU(2) δ function

$$\delta(A,B) = \sum_{R=0,\frac{1}{2},1,\ldots} \sum_{a,b} [R] D^{(R)}_{ab}(A) D^{(R)*}_{ab}(B) \ , \tag{23}$$

which satisfies

$$\int dB\, \delta(A,B) f(B) = f(A)$$

for any function f [as follows from Eq. (B4)]. It is easy to show that, in the large-N_c limit,

$$\langle A|B\rangle \to \left[\frac{\pi N_c^3}{8}\right]^{-1/2} \tfrac{1}{2}[\delta(A,B) + (-1)^{N_c}\delta(A,-B)] \ . \tag{24}$$

Equation (24) is an immediate consequence of Eqs. (18), (21), and (22). The effect of the phase $(-1)^{N_c}$ is to restrict the summation in Eq. (23) to either integral or half-integral values of R, depending on whether N_c is even or odd.

We can generalize the previous equation to matrix elements $\langle A|O|B\rangle$ of an arbitrary spin-isospin operator O. Suppose that O is expressible as a sum of operators that act on only n quarks at a time, with $n<N_c$. Manohar's key observation is that, in the large-N_c limit,

$$\langle A|O|B\rangle \underset{N_c\to\infty}{\sim} \langle A|O|A\rangle \frac{1}{2k}[\delta(A,B) + (-1)^{N_c}\delta(A,-B)] \ , \tag{25}$$

with

$$k = \tfrac{1}{2}[\delta(A,A) + (-1)^{N_c}\delta(A,-A)] \ . \tag{26}$$

We shall refer to this as the δ-function approximation. As before, it is valid because $\langle A|O|B\rangle$ contains a factor of $(\frac{1}{2}\mathrm{tr}A^\dagger B)^{N_c-n}$, which becomes more and more sharply peaked about the points $B=\pm A$ as $N_c\to\infty$ (or, more precisely, $N_c - n\to\infty$). The present definition of the infinite constant k coincides with the one given earlier in Eq. (6).

We can also consider matrix elements of O between physical baryon states. Changing from the spin-isospin to the $\mathbf{K}$-symmetry basis, one finds

$$\langle^{R'}_{i_z' s_z'}|O|^{R}_{i_z s_z}\rangle = \int dA \int dB\, \chi^{R'*}_{i_z' s_z'}(A) \langle A|O|B\rangle \chi^R_{i_z s_z}(B)$$
$$\underset{N_c\to\infty}{\sim} k^{-1} \int dA\, \chi^{R'*}_{i_z' s_z'}(A) \langle A|O|A\rangle \chi^R_{i_z s_z}(A) \ , \tag{27}$$

using (25) and (13). The δ-function approximation has reduced the double integral in Eq. (27) to a single integral. It is this reduction, Manohar observed, that makes the large-N_c nonrelativistic quark model outlined here group-theoretically equivalent to the Skyrme-model approach introduced by Adkins, Nappi, and Witten.[7] For, questions of normalization aside, the final expression in Eq. (27) is precisely what a Skyrme modeler would write for the physical matrix element of O.

In the next section we will examine the consequences

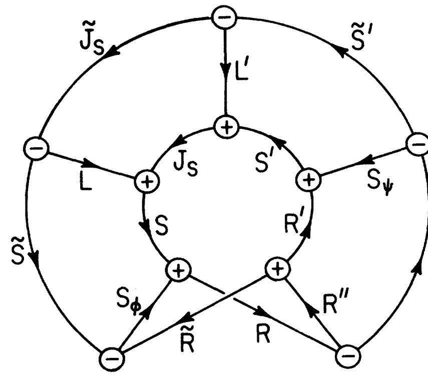

FIG. 3. The $15j$ symbol of the first kind, formed from the angular momentum quantum numbers characterizing the collision. See Fig. 2 for conventions.

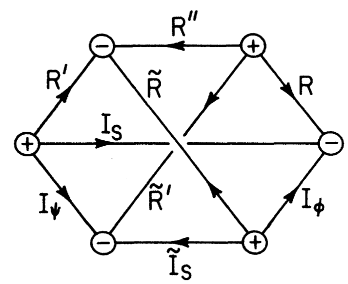

FIG. 4. The $9j$ symbol formed from the isospin quantum numbers characterizing the collision.

of applying this formalism to the meson-baryon scattering matrix. While the j-symbol manipulations render this section rather technical, the basic plan is simple: one changes from the spin-isospin to the **K**-symmetry basis, implements the δ-function approximation, then transforms back again to the spin-isospin basis.

III. DERIVATION OF THE PROJECTION EQUATION

We begin our derivation of Eqs. (4) and (5) by rewriting the scattering amplitude as a matrix element, and changing from a "JM"-type to a "$j_1 j_2$"-type basis in both isospin and angular momentum:

$$\begin{aligned}\mathbf{T}_{I_sJ_sSS'LL'RR'} &\equiv \left\langle \begin{smallmatrix} I_s & J_s \\ I_{sz} & J_{sz}\end{smallmatrix} R'S'L'I_\psi S_\psi \middle| \mathbf{T} \middle| \begin{smallmatrix} I_s & J_s \\ I_{sz} & J_{sz}\end{smallmatrix} RSLI_\phi S_\phi \right\rangle \\ &= \sum_{I_{\phi z} i_z L_z S_z s_z S_{\phi z}} \; \sum_{I_{\psi z} i'_z L'_z S'_z s'_z S_{\psi z}} \langle I_s I_{sz} | I_\phi R I_{\phi z} i_z \rangle \langle I_s I_{sz} | I_\psi R' I_{\psi z} i'_z \rangle \\ &\qquad \times \langle J_s J_{sz} | L S L_z S_z \rangle \langle J_s J_{sz} | L' S' L'_z S'_z \rangle \langle S S_z | R S_\phi s_z S_{\phi z} \rangle \langle S' S'_z | R' S_\psi s'_z S_{\psi z} \rangle \\ &\qquad \times \left\langle \begin{smallmatrix} R' & L' I_\psi & S_\psi \\ i'_z s'_z & L'_z I_{\psi z} & S_{\psi z} \end{smallmatrix} \middle| \mathbf{T} \middle| \begin{smallmatrix} R & L\, I_\phi & S_\phi \\ i_z s_z & L_z I_{\phi z} & S_{\phi z} \end{smallmatrix} \right\rangle . \end{aligned} \tag{28}$$

It will turn out to be useful to sum the right-hand side on I_{sz} and J_{sz} as well, a harmless step since **T** does not depend on them, remembering to divide by $[I_s][J_s]$ to compensate for the overcounting.

Next, we use Eq. (15) to switch to the **K**-symmetry basis:

$$\begin{aligned} \left\langle \begin{smallmatrix} R' & L' I_\psi & S_\psi \\ i'_z s'_z & L'_z I_{\psi z} & S_{\psi z} \end{smallmatrix} \middle| T \middle| \begin{smallmatrix} R & L\, I_\phi & S_\phi \\ i_z s_z & L_z I_{\phi z} & S_{\phi z} \end{smallmatrix} \right\rangle &= \int dA \int dB\, \chi^{R'*}_{i'_z s'_z}(A) \left\langle A \begin{smallmatrix} L' & I_\psi & S_\psi \\ L'_z & I_{\psi z} & S_{\psi z} \end{smallmatrix} \middle| \mathbf{T} \middle| B \begin{smallmatrix} L & I_\phi & S_\phi \\ L_z & I_{\phi z} & S_{\phi s} \end{smallmatrix} \right\rangle \chi^{R}_{i_z s_z}(B) \\ &\underset{N_c \to \infty}{\sim} k^{-1} \int dA\, \chi^{R'*}_{i'_z s'_z}(A) \left\langle A \begin{smallmatrix} L' & I_\psi & S_\psi \\ L'_z & I_{\psi z} & S_{\psi z} \end{smallmatrix} \middle| \mathbf{T} \middle| A \begin{smallmatrix} L & I_\phi & S_\phi \\ L_z & I_{\phi z} & S_{\phi z} \end{smallmatrix} \right\rangle \chi^{R}_{i_z s_z}(A) . \end{aligned} \tag{29}$$

In the crucial last step, we have used the δ-function approximation, Eq. (25). Transforming *back* to the spin-isospin basis with the help of Eqs. (21) and (22) we rewrite this last expression as

$$k^{-1} \left[\frac{\pi N_c^3}{8} \right]^{-1} \sum_{\tilde{R} \tilde{\imath}_z \tilde{s}_z} \; \sum_{\tilde{R}' \tilde{\imath}'_z \tilde{s}'_z} \left\langle \begin{smallmatrix} \tilde{R}' & L' & I_\psi & S_\psi \\ \tilde{\imath}'_z \tilde{s}'_z & L'_z & I_{\psi z} & S_{\psi z} \end{smallmatrix} \middle| \mathbf{T} \middle| \begin{smallmatrix} \tilde{R} & L & I_\phi & S_\phi \\ \tilde{\imath}_z \tilde{s}_z & L_z & I_{\phi z} & S_{\phi z} \end{smallmatrix} \right\rangle \int dA\, \chi^{R'*}_{i'_z s'_z}(A) \chi^{\tilde{R}'}_{\tilde{\imath}'_z \tilde{s}'_z}(A) \chi^{\tilde{R}*}_{\tilde{\imath}_z \tilde{s}_z}(A) \chi^{R}_{i_z s_z}(A) , \tag{30}$$

removing the A dependence from the matrix element. The integration of the four wave functions, defined in Eqs. (18) and (21), is accomplished with the help of the standard D-function identities

$$D^{(R_1)}_{a_1 b_1}(A) D^{(R_2)}_{a_2 b_2}(A) = \sum_{Rab} D^{(R)}_{ab}(A) \langle Ra | R_1 R_2 a_1 a_2 \rangle \langle Rb | R_1 R_2 b_1 b_2 \rangle \tag{31}$$

and Eq. (B4); the result is

$$\frac{\pi N_c^3}{8} \sum_{R'' I''_z S''_z} ([R][R'][\tilde{R}][\tilde{R}'])^{1/2} [R'']^{-1} \langle R'' I''_z | R \tilde{R}' i_z \tilde{\imath}'_z \rangle \langle R'' I''_z | R' \tilde{R} i'_z \tilde{\imath}_z \rangle \langle R'' S''_z | R \tilde{R}' s_z \tilde{s}'_z \rangle \langle R'' S''_z | R' \tilde{R} s'_z \tilde{s}_z \rangle . \tag{32}$$

Note that the explicit N_c dependence cancels out between Eqs. (30) and (32), as it must. A further step is to project the matrix element in (30) back onto partial-wave amplitudes $\mathbf{T}_{\tilde{I}_s \tilde{J}_s \tilde{S}\tilde{S}' LL' \tilde{R}\tilde{R}'}$, using the inverse of Eq. (28).

Putting these various pieces together, the attentive reader will verify that we have already arrived at a matrix equation of the form (4), where, at this stage, $\Pi(L,L')$ is expressed rather clumsily as a product of 16 Clebsch-Gordan (CG) coefficients, summed on all 24 magnetic quantum numbers, and on the dummy index R''. This unwieldy expression for $\Pi(L,L')$ can be simplified as follows. Of the 16 CG coefficients, the 10 that contain angular momentum quantities organize themselves into the graph shown in Fig. 3, which defines a $15j$ symbol of the first kind (see Appendix A):

$$\begin{aligned} \frac{1}{[J_s]} \sum \big(&\langle J_s J_{sz} | L S L_z S_z \rangle \langle J_s J_{sz} | L' S' L'_z S'_z \rangle \langle S S_z | R S_\phi s_z S_{\phi z} \rangle \langle S' S'_z | R' S_\psi s'_z S_{\psi z} \rangle \langle \tilde{J}_s \tilde{J}_{sz} | L \tilde{S} L_z \tilde{S}_z \rangle \langle \tilde{J}_s \tilde{J}_{sz} | L' \tilde{S}' L'_z \tilde{S}'_z \rangle \\ &\times \langle \tilde{S} \tilde{S}_z | \tilde{R} S_\phi \tilde{s}_z S_{\phi z} \rangle \langle \tilde{S}' \tilde{S}'_z | \tilde{R}' S_\psi \tilde{s}'_z S_{\psi z} \rangle \langle R'' S''_z | R \tilde{R}' s_z \tilde{s}'_z \rangle \langle R'' S''_z | R' \tilde{R} s'_z \tilde{s}_z \rangle \big) \\ &= (-1)^{J_s - \tilde{J}_s + R'' - \tilde{R} - \tilde{R}' + S + S' - \tilde{S} - \tilde{S}'} [R''][\tilde{J}_s] ([S][S'][\tilde{S}][\tilde{S}'])^{1/2} \begin{Bmatrix} \tilde{R} & & \tilde{S} & & \tilde{J}_s & & \tilde{S}' & & \tilde{R}' & \\ & S_\phi & & L & & L' & & S_\psi & & R'' \\ R & & S & & J_s & & S' & & R' & \end{Bmatrix} . \end{aligned} \tag{33}$$

The summation in (33) extends over the 15 magnetic quantum numbers that appear. Likewise, the six CG coefficients containing isospin quantities, when summed over the nine magnetic quantum numbers involved, yield a Wigner 9j symbol (Fig. 4):

$$\frac{1}{[I_s]}\sum \langle I_s I_{sz}|I_\phi R I_{\phi z} i_z\rangle\langle I_s I_{sz}|I_\psi R' I_{\psi z} i_z'\rangle\langle \tilde I_s \tilde I_{sz}|I_\phi \tilde R I_{\phi z}\tilde\iota_z\rangle\langle \tilde I_s \tilde I_{sz}|I_\psi \tilde R' I_{\psi z}\tilde\iota_z'\rangle\langle R'' I_z''|R\tilde R' i_z \tilde\iota_z'\rangle\langle R'' I_z''|R'\tilde R i_z'\tilde\iota_z\rangle$$
$$=(-1)^{\tilde R-\tilde R'+I_\phi-I_\psi}[R''][\tilde I_s]\begin{Bmatrix}\tilde R & R' & R''\\ \tilde I_s & I_\psi & \tilde R'\\ I_\phi & I_s & R\end{Bmatrix}. \tag{34}$$

These manipulations have reduced $\Pi(L,L')$ to a much neater form:

$$\Pi(L,L')^{\tilde I_s\tilde J_s\tilde S\tilde S'\tilde R\tilde R'}_{I_sJ_sSS'RR'}=k^{-1}\sum_{R''}(-1)^{J_s-\tilde J_s+R''+2\tilde R'+S+S'-\tilde S-\tilde S'+I_\phi-I_\psi}$$
$$\times[R''][\tilde I_s][\tilde J_s]([R][R'][S][S'][\tilde R][\tilde R'][\tilde S][\tilde S'])^{1/2}$$
$$\times\begin{Bmatrix}\tilde R & & \tilde S & & \tilde J_s & & \tilde S' & & \tilde R' & \\ & S_\phi & & L & & L' & & S_\psi & & R''\\ R & & S & & J_s & & S' & & R' & \end{Bmatrix}\begin{Bmatrix}\tilde R & R' & R''\\ \tilde I_s & I_\psi & \tilde R'\\ I_\phi & I_s & R\end{Bmatrix}. \tag{35}$$

Our maximally compact result (5), involving a single 18j symbol, emerges directly from the remaining summation on R'', thanks to the identity (A16). This completes the derivation of Eqs. (4) and (5).

IV. SOLUTION OF THE PROJECTION EQUATION

We have shown that, to leading order in large N_c, the meson-baryon scattering amplitudes satisfy the matrix equation (4). In this section we solve this equation by diagonalizing the matrix $\Pi(L,L')$. In the process, it will become evident that $\Pi(L,L')$ is, as claimed, a projection operator.

The key to solving Eq. (4) is the change of variables

$$\mathbf{T}_{I_tJ_tJ_\phi J_\psi LL'RR'}=\sum_{SS'I_sJ_s}[I_s][J_s]([J_\phi][J_\psi][S][S'])^{1/2}(-1)^{I_t+I_s+J_t+J_s+S_\phi+S_\psi+S+S'+J_\phi+I_\psi}$$
$$\times\begin{Bmatrix}R' & R & I_t\\ I_\phi & I_\psi & I_s\end{Bmatrix}\begin{Bmatrix}R' & R & J_t\\ J_\phi & J_\psi & J_s\end{Bmatrix}\begin{Bmatrix}J_s & J_\phi & R\\ S_\phi & S & L\end{Bmatrix}\begin{Bmatrix}J_s & J_\psi & R'\\ S_\psi & S' & L'\end{Bmatrix}\mathbf{T}_{I_sJ_sSS'LL'RR'} \tag{36a}$$

and the inverse relation

$$\mathbf{T}_{I_sJ_sSS'LL'RR'}=\sum_{J_\phi J_\psi I_tJ_t}[I_t][J_t]([J_\phi][J_\psi][S][S'])^{1/2}(-1)^{I_t+I_s+J_t+J_s+S_\phi+S_\psi+S+S'+J_\phi+I_\psi}$$
$$\times\begin{Bmatrix}R' & R & I_t\\ I_\phi & I_\psi & I_s\end{Bmatrix}\begin{Bmatrix}R' & R & J_t\\ J_\phi & J_\psi & J_s\end{Bmatrix}\begin{Bmatrix}J_s & J_\phi & R\\ S_\phi & S & L\end{Bmatrix}\begin{Bmatrix}J_s & J_\psi & R'\\ S_\psi & S' & L'\end{Bmatrix}\mathbf{T}_{I_tJ_tJ_\phi J_\psi LL'RR'}\,. \tag{36b}$$

Equation (36), introduced in Ref. 1, has an important physical meaning: it is the crossing relation that allows one to pass back and forth from an s-channel to a t-channel description of the meson-baryon collision, in the limit that the baryons are considered infinitely heavy (as they are in large N_c). I_t and J_t are the total isospin and angular momentum of the exchanged state (see Fig. 1), and J_ϕ and J_ψ denote the "total meson angular momenta":

$$\mathbf{J}_\phi=\mathbf{S}_\phi+\mathbf{L},\quad \mathbf{J}_\psi=\mathbf{S}_\psi+\mathbf{L}'\,.$$

Implementing the crossing relation (36) in Eq. (4) leads to considerable simplification, thanks to the identities (A17), (A15), and (A6). After some algebra, one obtains the simple matrix equation

$$\mathbf{T}_{I_tJ_tJ_\phi J_\psi LL'RR'}=\sum_{\tilde I_t\tilde J_t\tilde J_\phi\tilde J_\psi\tilde R\tilde R'}\Pi(L,L')^{\tilde I_t\tilde J_t\tilde J_\phi\tilde J_\psi\tilde R\tilde R'}_{I_tJ_tJ_\phi J_\psi RR'}\cdot\mathbf{T}_{\tilde I_t\tilde J_t\tilde J_\phi\tilde J_\psi LL'\tilde R\tilde R'}\,, \tag{37}$$

where

$$\Pi(L,L')^{\tilde I_t\tilde J_t\tilde J_\phi\tilde J_\psi\tilde R\tilde R'}_{I_tJ_tJ_\phi J_\psi RR'}=\delta_{I_tJ_t}\delta_{\tilde I_t\tilde J_t}\delta_{I_t\tilde I_t}\delta_{J_\phi\tilde J_\phi}\delta_{J_\psi\tilde J_\psi}\frac{([R][R'][\tilde R][\tilde R'])^{1/2}}{k[I_t]}\Delta(R,R',I_t)\Delta(\tilde R,\tilde R',I_t)\Delta(I_\phi,I_\psi,I_t)\Delta(J_\phi,J_\psi,I_t)\,. \tag{38}$$

The Δ function in Eq. (38) equals 1 if its arguments sum to an integer and obey the triangle inequalities, and zero otherwise. It follows from the Kronecker δ's that all t-channel amplitudes with $I_t \neq J_t$ are annihilated by $\Pi(L,L')$, and must therefore vanish to leading order in $1/N_c$.

To complete the solution of the matrix equation we carry out the trivial sums in Eq. (37), obtaining

$$\mathbf{T}_{I_t J_t J_\phi J_\psi L L' R R'} = \delta_{I_t J_t} \Delta(R,R',I_t)\Delta(I_\phi,I_\psi,I_t)\Delta(J_\phi,J_\psi,J_t)([R][R'])^{1/2}\hat{\mathbf{T}}_{I_t J_\phi J_\psi L L'} , \tag{39}$$

where the "reduced amplitudes" $\hat{\mathbf{T}}$ are independent of R and R':

$$\hat{\mathbf{T}}_{I_t J_\phi J_\psi L L'} = \frac{1}{k[I_t]} \sum_{\tilde{R}\tilde{R}'} \Delta(\tilde{R},\tilde{R}',I_t)([\tilde{R}][\tilde{R}'])^{1/2} \mathbf{T}_{I_t I_t J_\phi J_\psi L L' \tilde{R}\tilde{R}'} . \tag{40}$$

One can check the consistency of these two equations by inserting (39) into the right-hand side of (40). The sum on $\tilde{R}$ and $\tilde{R}'$ then yields

$$\sum_{\tilde{R},\tilde{R}'} [\tilde{R}][\tilde{R}']\Delta(\tilde{R},\tilde{R}',I_t) = k[I_t] , \tag{41}$$

which follows from the identity

$$\sum_x [x]\Delta(x,y,z) = [y][z] \tag{42}$$

and the definition (6) of the infinite constant k. We are left with the reassuring tautology $\hat{\mathbf{T}} = \hat{\mathbf{T}}$.

Equation (39)—with an arbitrary choice of reduced amplitudes $\hat{\mathbf{T}}$—therefore constitutes the general solution to the t-channel matrix equation (37). Likewise, using the crossing relation (36b), we can finally write down the complete solution to the s-channel equation (5):

$$\mathbf{T}_{I_s J_s S S' L L' R R'} = \sum_{J_\phi J_\psi I_t} [I_t]^2([J_\phi][J_\psi][S][S'][R][R'])^{1/2}(-1)^{I_s + J_s + S_\phi + S_\psi + S + S' + J_\phi + I_\psi} \times \begin{Bmatrix} R' & R & I_t \\ I_\phi & I_\psi & I_s \end{Bmatrix} \begin{Bmatrix} R' & R & I_t \\ J_\phi & J_\psi & J_s \end{Bmatrix} \begin{Bmatrix} J_s & J_\phi & R \\ S_\phi & S & L' \end{Bmatrix} \begin{Bmatrix} J_s & J_\psi & R' \\ S_\psi & S' & L' \end{Bmatrix} \hat{\mathbf{T}}_{I_t J_\phi J_\psi L L'} . \tag{43}$$

In the course of obtaining these results, we have also arrived at a pleasing physical interpretation of the matrix $\Pi(L,L')$. It is the projection operator onto the space of scattering amplitudes that obey the two principal conclusions of Refs. 1 and 2: the $I_t = J_t$ rule, $T_{I_t J_t J_\phi J_\psi L L' R R'} \propto \delta_{I_t J_t}$, and the proportionality rule, $T_{I_t J_t J_\phi J_\psi L L' R R'} \propto \sqrt{[R][R']}$.

V. REMARKS

In this section we clarify the connection between the projection-equation approach developed here and previous work on the meson-baryon system. We also describe how our methods can be applied to matrix elements of operators *other* than the meson-baryon **T** matrix.

Reduced amplitudes. In Eqs. (39) and (43) we have expressed the partial-wave amplitudes for meson-baryon scattering in terms of a smaller set of "$I_t = J_t$ reduced amplitudes" $\hat{\mathbf{T}}_{I_t J_\phi J_\psi L L'}$. In contrast, in Skyrme-model treatments of meson-baryon scattering,[10,12–17] the partial-wave amplitudes are expanded in a different set of quantities $\mathcal{T}_{K\tilde{K}\tilde{K}'LL'}$, which we shall refer to as "**K**-conserving reduced amplitudes." They are the **T**-matrix elements for the *unphysical* process

$$\phi + U_0 \rightarrow \psi + U_0 ,$$

where U_0 is the hedgehog Skyrmion given in Eq. (3). In the present context, the subscript K is the quantum number for the vector sum of the mesons' isospin and total (spin + orbital) angular momentum:

$$\mathbf{K} = \mathbf{I}_\phi + \mathbf{L} + \mathbf{S}_\phi = \mathbf{I}_\psi + \mathbf{L}' + \mathbf{S}_\psi .$$

It is conserved in this unphysical process, because of the **K** invariance of U_0. The subscripts $\tilde{K}$ and $\tilde{K}'$ are quantum numbers for the nonconserved intermediate quantities

$$\tilde{\mathbf{K}} = \mathbf{I}_\phi + \mathbf{L}, \quad \tilde{\mathbf{K}}' = \mathbf{I}_\psi + \mathbf{L}' .$$

The physical s- and t-channel partial-wave amplitudes characterizing the physical collisions

$$\phi + B \rightarrow \psi + B'$$

are expressed in terms of the **K**-conserving reduced amplitudes as follows:[14,1]

$$\mathbf{T}_{I_sJ_sSS'LL'RR'}=\sum_{K\tilde{K}\tilde{K}'}[K]([R][R'][S][S'][\tilde{K}][\tilde{K}'])^{1/2}\begin{Bmatrix}L & I_\phi & \tilde{K}\\ S & R & S_\phi\\ J_s & I_s & K\end{Bmatrix}\begin{Bmatrix}L' & I_\psi & \tilde{K}'\\ S' & R' & S_\psi\\ J_s & I_s & K\end{Bmatrix}\mathcal{T}_{K\tilde{K}\tilde{K}'LL'} \tag{44}$$

and

$$\mathbf{T}_{I_tJ_tJ_\phi J_\psi LL'RR'}=\delta_{I_tJ_t}\sum_{K\tilde{K}\tilde{K}'}[I_t]^{-1}[K]([R][R'][J_\phi][J_\psi][\tilde{K}][\tilde{K}'])^{1/2}(-1)^{I_t+J_\psi-I_\psi+K+\tilde{K}+\tilde{K}'+L+L'}$$
$$\times\begin{Bmatrix}J_\phi & I_\phi & K\\ I_\psi & J_\psi & I_t\end{Bmatrix}\begin{Bmatrix}J_\phi & I_\phi & K\\ \tilde{K} & S_\phi & L\end{Bmatrix}\begin{Bmatrix}J_\psi & I_\psi & K\\ \tilde{K}' & S_\psi & L'\end{Bmatrix}\mathcal{T}_{K\tilde{K}\tilde{K}'LL'} . \tag{45}$$

The relation between the two sets of reduced amplitudes can be seen by comparing Eqs. (39) and (45). One finds

$$\hat{\mathbf{T}}_{I_tJ_\phi J_\psi LL'}=\sum_{K\tilde{K}\tilde{K}'}[I_t]^{-1}[K]([J_\phi][J_\psi][\tilde{K}][\tilde{K}'])^{1/2}(-1)^{I_t+J_\psi-I_\psi+K+\tilde{K}+\tilde{K}'+L+L'}$$
$$\times\begin{Bmatrix}J_\phi & I_\phi & K\\ I_\psi & J_\psi & I_t\end{Bmatrix}\begin{Bmatrix}J_\phi & I_\phi & K\\ \tilde{K} & S_\phi & L\end{Bmatrix}\begin{Bmatrix}J_\psi & I_\psi & K\\ \tilde{K}' & S_\psi & L'\end{Bmatrix}\mathcal{T}_{K\tilde{K}\tilde{K}'LL'} , \tag{46a}$$

and conversely, using (A6),

$$\mathcal{T}_{K\tilde{K}\tilde{K}'LL'}=\sum_{I_tJ_\phi J_\psi}[I_t]^2([J_\phi][J_\psi][\tilde{K}][\tilde{K}'])^{1/2}(-1)^{I_t+J_\psi-I_\psi+K+\tilde{K}+\tilde{K}'+L+L'}$$
$$\times\begin{Bmatrix}J_\phi & I_\phi & K\\ I_\psi & J_\psi & I_t\end{Bmatrix}\begin{Bmatrix}J_\phi & I_\phi & K\\ \tilde{K} & S_\phi & L\end{Bmatrix}\begin{Bmatrix}J_\psi & I_\psi & K\\ \tilde{K}' & S_\psi & L'\end{Bmatrix}\hat{\mathbf{T}}_{I_tJ_\phi J_\psi LL'} . \tag{46b}$$

These are finite sums, constrained by the triangle inequalities implicit in the $6j$ symbols.

Although the $\mathcal{T}$'s and the $\hat{\mathbf{T}}$'s are equivalent [as Eq. (46) makes clear], each has its conceptual advantage. The **K**-conserving reduced amplitudes are particularly well suited to model-dependent numerical calculations: they emerge directly from a phase-shift analysis carried out in the hedgehog Skyrmion background.[10,13,15,16] However, their physical interpretation is not straightforward, because K, $\tilde{K}$, and $\tilde{K}'$ involve sums of isospin and angular momentum quantities. In contrast, the $I_t=J_t$ reduced amplitudes depend on I_t, J_ϕ, and J_ψ, which have direct physical interpretation as isospin or angular momentum quantum numbers. Furthermore, the simplest expression for the partial-wave amplitudes, Eq. (39), is defined in terms of them. Finally, using the $I_t=J_t$ reduced amplitudes, it is very simple to count the number $\mathcal{N}_{LL'RR'}$ of linearly independent amplitudes—and consequently the number of model-independent linear relations—for given values of orbital angular momenta L and L' and baryon representations R and R'. As is clear from Eq. (39), $\mathcal{N}_{LL'RR'}$ is just the number of distinct $\hat{\mathbf{T}}$'s:

$$\mathcal{N}_{LL'RR'}=\sum_{I_tJ_\phi J_\psi}\Delta(R,R',I_t)\Delta(I_\phi,I_\psi,I_t)\Delta(J_\phi,J_\psi,I_t)\times\Delta(J_\phi,S_\phi,L)\Delta(J_\psi,S_\psi,L') . \tag{47}$$

We know of no obvious parallel for this counting rule in terms of the **K**-invariant reduced amplitudes $\mathcal{T}_{K\tilde{K}\tilde{K}'LL'}$.

Kinematic limitations. In Refs. 1 and 2, the $I_t=J_t$ rule and the proportionality rule were shown to be subject to a kinematic constraint: the momentum transferred to the baryon during the collision must have magnitude of order N_c^0, rather than N_c. These papers used Skyrmion and one-boson-exchange techniques, respectively. We would like to understand how such a constraint might come about in the large-N_c nonrelativistic quark model, too. This suggests that we reexamine the validity of the δ-function approximation, Eq. (25), in the event that the generic spin-isospin operator O carries momentum $\mathbf{q}$.

As always, we restrict ourselves to the case that the final baryon (as the initial baryon) is a member of the $I=J$ tower (2); that is, the N_c quarks are in ground-state S-wave orbitals, with no vibrational or rotational modes excited. In this absence of relative quark motion, the kinematic effect of the operator $O(\mathbf{q})$ is simply to transfer equal momentum $\mathbf{q}/N_c$ to each of the N_c quarks. The matrix element $\langle A|O(\mathbf{q})|B\rangle$ will then be proportional to a kinematic factor $\langle\psi(\mathbf{q}/N_c)|\psi(\mathbf{0})\rangle^{N_c}$, where $\langle\psi(\mathbf{q}/N_c)|\psi(\mathbf{0})\rangle$ is the overlap of a single quark at rest with a single quark traveling with momentum $\mathbf{q}/N_c$. A back-of-the-envelope estimate of this overlap can be had from a Gaussian ansatz for the quark wave functions; one finds

$$\langle\psi(\mathbf{q}/N_c)|\psi(\mathbf{0})\rangle\sim\exp(-\text{const}\times q^2/N_c^2)$$

and hence

$$\langle A|O(\mathbf{q})|B\rangle\sim\exp(-\text{const}\times q^2/N_c) .$$

If q is held fixed as $N_c\to\infty$, this kinematic factor approaches unity, and the derivation of the δ-function approximation given in Sec. II, together with all subsequent developments, carries over unchanged. On the other hand, if q is itself of order N_c, the kinematic factor gives an exponential suppression to the matrix element, and a leading-order large-N_c analysis becomes meaningless. Similar arguments can be found in Sec. 8.3 of Ref. 5.

Projection equation generalized. Finally, we note that the projection equation formulation advocated here is by no means restricted to the meson-baryon **T** matrix. Consider, once again, Eq. (27):

$$\langle {}^{R'}_{i_z's_z'}|O|{}^{R}_{i_zs_z}\rangle \underset{N_c\to\infty}{\sim} k^{-1}\int dA\, \chi^{R'*}_{i_z's_z'}(A)\langle A|O|A\rangle\chi^{R}_{i_zs_z}(A) \,.$$

Transforming the right-hand side back onto the spin-isospin basis [Eq. (22)] and carrying out the integral over SU(2) [Eq. (32)], one obtains the projection equation

$$\langle {}^{R'}_{i_z's_z'}|O|{}^{R}_{i_zs_z}\rangle = \sum_{\tilde R\tilde R'\tilde\imath_z\tilde\imath_z'\tilde s_z\tilde s_z'} \Pi^{\tilde R\tilde R'\tilde\imath_z\tilde\imath_z'\tilde s_z\tilde s_z'}_{RR'i_zi_z's_zs_z'}\cdot\langle {}^{\tilde R'}_{\tilde\imath_z'\tilde s_z'}|O|{}^{\tilde R}_{\tilde\imath_z\tilde s_z}\rangle \,, \tag{48}$$

where

$$\Pi^{\tilde R\tilde R'\tilde\imath_z\tilde\imath_z'\tilde s_z\tilde s_z'}_{RR'i_zi_z's_zs_z'} = k^{-1}\sum_{JM_1M_2}([R][R'][\tilde R][\tilde R'])^{1/2}[J]^{-1} \times\langle JM_1|R\tilde R'i_z\tilde\imath_z'\rangle\langle JM_1|R'\tilde R i_z'\tilde\imath_z\rangle\langle JM_2|R\tilde R's_z\tilde s_z'\rangle\langle JM_2|R'\tilde R s_z'\tilde s_z\rangle \,. \tag{49}$$

That Π is, in fact, a projection operator, satisfying the idempotency condition

$$\sum_{\tilde R\tilde R'\tilde\imath_z\tilde\imath_z'\tilde s_z\tilde s_z'} \Pi^{\tilde R\tilde R'\tilde\imath_z\tilde\imath_z'\tilde s_z\tilde s_z'}_{RR'i_zi_z's_zs_z'}\cdot\Pi^{\hat R\hat R'\hat\imath_z\hat\imath_z'\hat s_z\hat s_z'}_{\tilde R\tilde R'\tilde\imath_z\tilde\imath_z'\tilde s_z\tilde s_z'} = \Pi^{\hat R\hat R'\hat\imath_z\hat\imath_z'\hat s_z\hat s_z'}_{RR'i_zi_z's_zs_z'} \,, \tag{50}$$

can be verified with the help of Eqs. (A14), (A13), (42), and (6).

Many large-N_c group-theoretic results, of the sort that have permeated the Skyrme-model literature over the past few years,[21] can be elegantly reformulated as projection equations of the form of Eq. (48). Our principal result, Eqs. (4) and (5), stands as an important special case in which the angular momentum and isospin-preserving qualities of **T**, as well as its transformation properties necessary for meson annihilation and creation, have been fully exploited.

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APPENDIX A: EVERYTHING YOU ALWAYS WANTED TO KNOW ABOUT THE HIGHER *j* SYMBOLS

In this appendix we give a short self-contained introduction to the lost art of the higher j symbols. We will probe only as far as is necessary to derive the identities used in this paper. The reader whose appetite is whet is referred to the comprehensive treatise of Yutsis, Levinson, and Vanagas.[30] The notation $[x]\equiv 2x+1$ will be used throughout.

The basic building blocks of the subject are the $3j$ symbols, which are proportional to the usual Clebsch-Gordan coefficients:

$$\begin{bmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{bmatrix} = \frac{(-1)^{j_1-j_2-m_3}}{\sqrt{[j_3]}}\langle j_1j_2m_1m_2|j_3-m_3\rangle \,. \tag{A1}$$

Under an odd permutation of the columns, or under the simultaneous negation of the m's, $3j$ symbols are multiplied by the phase $(-1)^{j_1+j_2+j_3}$. They satisfy the orthogonality relations

$$\sum_{j_3m_3}[j_3]\begin{bmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{bmatrix}\begin{bmatrix} j_1 & j_2 & j_3 \\ m_1' & m_2' & m_3 \end{bmatrix} = \delta_{m_1m_1'}\delta_{m_2m_2'} \tag{A2}$$

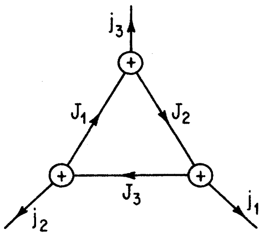
and

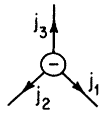
$$\sum_{m_1m_2}\begin{bmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{bmatrix}\begin{bmatrix} j_1 & j_2 & j_3' \\ m_1 & m_2 & m_3' \end{bmatrix} = [j_3]^{-1}\delta_{j_3j_3'}\delta_{m_3m_3'}\Delta(j_1,j_2,j_3) \,. \tag{A3}$$

The function $\Delta(j_1,j_2,j_3)$ equals 1 if its arguments sum to an integer and obey the triangle inequalities, and zero otherwise.

A $3j$ symbol can be pictured as an oriented vertex at which three directed lines meet. Products of $3j$ symbols summed over magnetic quantum numbers can be represented by graphs, using the conventions of Yutsis *et al.*[20] laid forth in the caption to Fig. 2. Note that, with these conventions, reversing an arrow on an internal leg in a graph gives a phase $(-1)^{2j}$, while reversing the three arrows associated with a vertex, or switching the orientation of a vertex, multiplies the associated $3j$ symbol by $(-1)^{j_1+j_2+j_3}$. In physical applications, it often happens that one of the j's in a graph is zero; up to phases, this is tantamount to removing that line from the diagram.

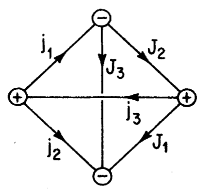
The simplest nontrivial rotationally invariant symbol, with no dependence on magnetic quantum numbers (the m's), is the $6j$ symbol, defined as



$=$


$$\times \begin{Bmatrix} j_1 & j_2 & j_3 \\ J_1 & J_2 & J_3 \end{Bmatrix} . \qquad \text{(A4)}$$

Alternatively, using Eq. (A3), one can tie together the three loose ends in (A4) with a $3j$ symbol and obtain

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ J_1 & J_2 & J_3 \end{Bmatrix} =$$


$$. \qquad \text{(A.5)}$$

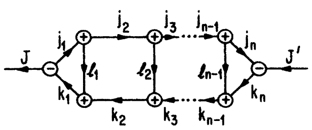
The $6j$ symbol is invariant under a permutation of columns, and under the simultaneous interchange of two elements from the top row with the corresponding elements from the bottom row. It also satisfies an orthogonality relation

$$\sum_x [x] \begin{Bmatrix} j_1 & j_2 & x \\ J_1 & J_2 & J_3 \end{Bmatrix} \begin{Bmatrix} j_1 & j_2 & x \\ J_1 & J_2 & J_3' \end{Bmatrix}$$

$$= [J_3]^{-1} \delta_{J_3 J_3'} \Delta(j_1, J_2, J_3) \Delta(j_2, J_1, J_3) , \qquad \text{(A6)}$$

with Δ as in (A3).

By applying Eq. (A.4) iteratively, it is easy to establish the identity



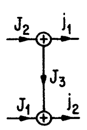
$$= [J]^{-1} \delta_{JJ'} \delta_{MM'} (-1)^{j_1 + \cdots + j_{n-1} - j_n + k_1 + \cdots + k_n + (n+1)J + M}$$

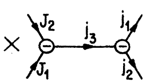
$$\times \begin{Bmatrix} j_1 & k_1 & J \\ k_2 & j_2 & l_1 \end{Bmatrix} \begin{Bmatrix} j_2 & k_2 & J \\ k_3 & j_3 & l_2 \end{Bmatrix} \cdots \begin{Bmatrix} j_{n-1} & k_{n-1} & J \\ k_n & j_n & l_{n-1} \end{Bmatrix} . \qquad \text{(A7)}$$

A second useful corollary of Eq. (A4) comes from multiplying both sides of the equation by

$$[j_3] \begin{pmatrix} J_1 & J_2 & j_3 \\ M_1' & -M_2' & m_3 \end{pmatrix}$$

and, using Eq. (A2),



$$= (-1)^{2J_2} \sum_{j_3} [j_3] \begin{Bmatrix} j_1 & j_2 & j_3 \\ J_1 & J_2 & J_3 \end{Bmatrix} \times$$


$$. \qquad \text{(A8)}$$

As is clear from these diagrams, Eq. (A8) has a natural particle physics interpretation as a *crossing relation* that enables one to pass between an s-channel and a t-channel description of a scattering event.

The $(3n)$-j symbols of the first and second kind,

$$\begin{bmatrix} j_1 & & \cdots & j_{n-1} & & j_n & \\ & l_1 & \cdots & & l_{n-1} & & l_n \\ k_1 & & \cdots & k_{n-1} & & k_n & \end{bmatrix}$$

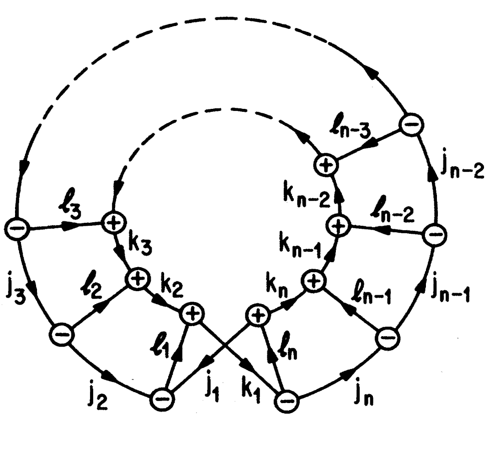


FIG. 5. The $(3n)$-j symbol of the first kind.

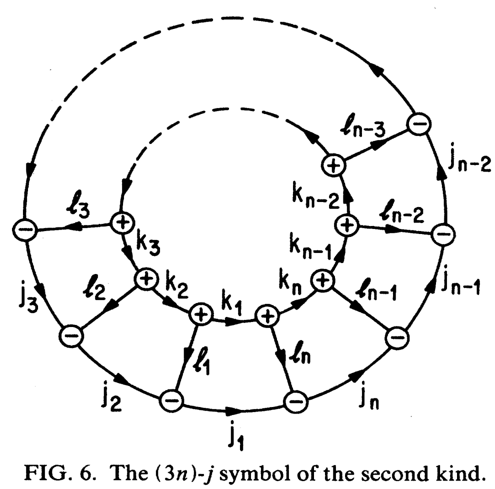


FIG. 6. The $(3n)$-j symbol of the second kind.

and

$$\begin{bmatrix} j_1 & & \cdots & j_{n-1} & & j_n & \\ & l_1 & \cdots & & l_{n-1} & & l_n \\ k_1 & & \cdots & k_{n-1} & & k_n & \end{bmatrix},$$

are defined, respectively, in Figs. 5 and 6.[22] They can both be expressed as a product of n $6j$ symbols, as follows. First, one applies the crossing relation (A8) to any one of the "rungs." Next, one uses the identity (A7). The results of this calculation are

$$\begin{Bmatrix} j_1 & & \cdots & j_{n-1} & & j_n & \\ & l_1 & \cdots & & l_{n-1} & & l_n \\ k_1 & & \cdots & k_{n-1} & & k_n & \end{Bmatrix} = \sum_x [x](-1)^{\sigma+(n-1)x} \begin{Bmatrix} j_1 & k_1 & x \\ k_2 & j_2 & l_1 \end{Bmatrix} \cdots \begin{Bmatrix} j_{n-1} & k_{n-1} & x \\ k_n & j_n & l_{n-1} \end{Bmatrix} \begin{Bmatrix} j_n & k_n & x \\ j_1 & k_1 & l_n \end{Bmatrix} \tag{A9}$$

and

$$\begin{bmatrix} j_1 & & \cdots & j_{n-1} & & j_n & \\ & l_1 & \cdots & & l_{n-1} & & l_n \\ k_1 & & \cdots & k_{n-1} & & k_n & \end{bmatrix} = \sum_x [x](-1)^{\sigma+nx} \begin{Bmatrix} j_1 & k_1 & x \\ k_2 & j_2 & l_1 \end{Bmatrix} \cdots \begin{Bmatrix} j_{n-1} & k_{n-1} & x \\ k_n & j_n & l_{n-1} \end{Bmatrix} \begin{Bmatrix} j_n & k_n & x \\ k_1 & j_1 & l_n \end{Bmatrix}, \tag{A10}$$

with σ the sum of the $3n$ entries. The various symmetries of these symbols (up-down flip, left-right flip, cyclic permutations) are obvious from Figs. 5 and 6, and also follow immediately from Eqs. (A9) and (A10).

The case $n=3$ merits special attention. The highly symmetric $9j$ symbol of the first kind (Fig. 7) is equivalent to the well-known Wigner $9j$ symbol:

$$\begin{Bmatrix} j_1 & & j_2 & & j_3 & \\ & l_1 & & l_2 & & l_3 \\ k_1 & & k_2 & & k_3 & \end{Bmatrix} = \begin{Bmatrix} l_1 & j_1 & j_2 \\ k_1 & l_3 & j_3 \\ k_2 & k_3 & l_2 \end{Bmatrix}. \tag{A11}$$

The Wigner $9j$ symbol is invariant under reflection about either diagonal, and picks up a phase of $(-1)^\sigma$ under an odd permutation of rows or columns. Thanks to these symmetries, the $9j$ symbol can be expanded as a product of three $6j$ symbols in six distinct ways. It satisfies three useful identities which the reader should have no trouble proving using properties of $3j$ and $6j$ symbols discussed above:

$$\sum_x [x] \begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \\ x & k_2 & k_3 \end{Bmatrix} \begin{Bmatrix} x & k_2 & k_3 \\ k_1 & j_1 & l_1 \end{Bmatrix} = (-1)^{2k_1} \begin{Bmatrix} j_1 & j_2 & j_3 \\ l_3 & k_3 & k_1 \end{Bmatrix} \begin{Bmatrix} l_1 & l_2 & l_3 \\ j_2 & k_1 & k_2 \end{Bmatrix}, \tag{A12}$$

$$\sum_{J_{13}J_{24}} [J_{13}][J_{24}] \begin{Bmatrix} j_1 & j_2 & J_{12} \\ j_3 & j_4 & J_{34} \\ J_{13} & J_{24} & J \end{Bmatrix} \begin{Bmatrix} j_1 & j_2 & J'_{12} \\ j_3 & j_4 & J'_{34} \\ J_{13} & J_{24} & J \end{Bmatrix} = [J_{12}]^{-1}[J_{34}]^{-1}\delta_{J_{12}J'_{12}}\delta_{J_{34}J'_{34}}\Delta(j_1,j_2,J_{12})\Delta(j_3,j_4,J_{34})\Delta(J_{12},J_{34},J), \tag{A13}$$

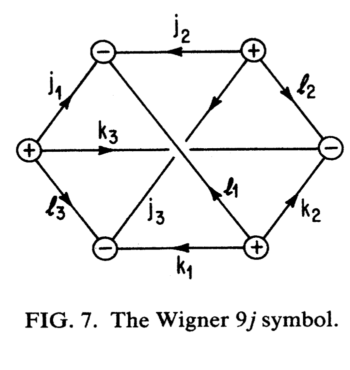


FIG. 7. The Wigner $9j$ symbol.

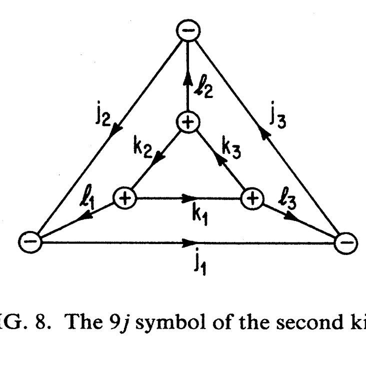
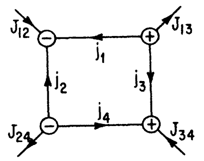
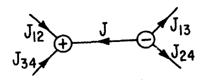


FIG. 8. The $9j$ symbol of the second kind.

and



$$=\sum_J [J] \begin{Bmatrix} j_1 & j_2 & J_{12} \\ j_3 & j_4 & J_{34} \\ J_{13} & J_{24} & J \end{Bmatrix} \times \qquad \text{(A14)}$$



In contrast, as is obvious from Fig. 8 and Eq. (A4), the $9j$ symbol of the second kind is not really a new symbol at all, but rather a product of two $6j$ symbols:

$$\begin{bmatrix} j_1 & & j_2 & & j_3 & \\ & l_1 & & l_2 & & l_3 \\ k_1 & & k_2 & & k_3 & \end{bmatrix} = (-1)^{2j_1+2k_1} \begin{Bmatrix} l_1 & l_2 & l_3 \\ k_3 & k_1 & k_2 \end{Bmatrix} \begin{Bmatrix} l_1 & l_2 & l_3 \\ j_3 & j_1 & j_2 \end{Bmatrix} . \qquad \text{(A15)}$$

In Secs. III and IV, respectively, we make use of two identities relating $(3n)-j$ symbols to $3(n\pm1)-j$ symbols:

$$\sum_x [x](-1)^x \begin{bmatrix} j_1 & & \cdots & j_{n-1} & & j_n & \\ & l_1 & \cdots & & l_{n-1} & & x \\ k_1 & & \cdots & k_{n-1} & & k_n & \end{bmatrix} \begin{Bmatrix} j_1 & k_n & x \\ j_{n+1} & l_n & j_n \\ l_{n+1} & k_{n+1} & k_1 \end{Bmatrix} \qquad \text{(A16)}$$

$$= (-1)^{j_{n+1}+k_{n+1}+l_n+l_{n+1}} \begin{bmatrix} j_1 & & \cdots & j_n & & j_{n+1} & \\ & l_1 & \cdots & & l_n & & l_{n+1} \\ k_1 & & \cdots & k_n & & k_{n+1} & \end{bmatrix}$$

and

$$\sum_x [x] \begin{bmatrix} j_1 & & \cdots & j_{n-1} & & x & \\ & l_1 & \cdots & & l_{n-1} & & l_n \\ k_1 & & \cdots & k_{n-1} & & k_n & \end{bmatrix} \begin{Bmatrix} j_1 & l_n & x \\ l_{n-1} & j_{n-1} & j_n \end{Bmatrix}$$

$$= (-1)^{j_1-k_1+j_{n-1}-k_{n-1}} \begin{bmatrix} j_1 & & \cdots & j_{n-2} & & j_{n-1} & \\ & l_1 & \cdots & & l_{n-2} & & j_n \\ k_1 & & \cdots & k_{n-2} & & k_{n-1} & \end{bmatrix} \begin{Bmatrix} k_1 & l_n & k_n \\ l_{n-1} & k_{n-1} & j_n \end{Bmatrix} . \qquad \text{(A17)}$$

Equation (A16) is a straightforward consequence of Eqs. (A9)–(A12), while Eq. (A17) follows directly from Eqs. (A10) and (A15). This completes our discussion of the higher j symbols.

APPENDIX B: NORMALIZATION OF BARYON WAVE FUNCTION

In this appendix we derive the expression (19) for the baryon wave-function normalization factor $c_R(N_c)$. It is fixed by the orthonormality condition (20):

$$\langle {}^{R'}_{i_z's_z'} | {}^{R}_{i_zs_z} \rangle = c_R^2(N_c) \int dA \int dB\, D^{(R')*}_{i_z's_z'}(i\tau_2 A^\dagger) \times D^{(R)}_{i_zs_z}(i\tau_2 B^\dagger)(\tfrac{1}{2}\mathrm{tr}\, A^\dagger B)^{N_c}$$

$$= \delta_{RR'}\delta_{i_zi_z'}\delta_{s_zs_z'} , \qquad \text{(B1)}$$

where the group-invariant measure dA is normalized to $\int dA = 1$.

We first argue that the inner product is, in fact, proportional to the three Kronecker δ's. Changing integration variables to $\tilde{A} = AB^\dagger$ and $\tilde{B} = -iB\tau_2$, and using the elementary properties of D functions

$$D^{(R)}(A_1A_2) = D^{(R)}(A_1)\cdot D^{(R)}(A_2) , \qquad \text{(B2)}$$

$$D^{(R)}(A^\dagger) = D^{(R)\dagger}(A) , \qquad \text{(B3)}$$

and

$$\int dA\, D^{(R')*}_{a'b'}(A) D^{(R)}_{ab}(A) = \frac{1}{2R+1}\delta_{RR'}\delta_{aa'}\delta_{bb'} , \qquad \text{(B4)}$$

one confirms that the integral in (B1) is proportional to $\delta_{RR'}\delta_{i_zi_z'}$. Alternatively, changing integration variables to $\tilde{A} = B^\dagger A$ and $\tilde{B} = B$, one finds that the integral is proportional to $\delta_{RR'}\delta_{s_zs_z'}$, completing the argument.

We now set $R=R'$, $i_z = i_z'$, and $s_z = s_z'$ in (B1) and sum on i_z and s_z:

$$(2R+1)^2 = c_R^2(N_c) \int dA \int dB\, \mathrm{tr} D^{(R)}(AB^\dagger)(\tfrac{1}{2}\mathrm{tr}\, A^\dagger B)^{N_c}$$

$$= c_R^2(N_c) \int dA\, \mathrm{tr} D^{(R)}(A)(\tfrac{1}{2}\mathrm{tr}\, A^\dagger)^{N_c} . \qquad \text{(B5)}$$

The second equality follows from the invariance of the measure, $d(AB^\dagger) = dA$. To compute the integral over A we will use a convenient parametrization of SU(2):

$$A = \cos f + i \sin f[\cos\theta\tau_3 + \sin\theta(\cos\phi\tau_1 + \sin\phi\tau_2)]$$

$$= e^{-i\phi\tau_3/2} e^{-i\theta\tau_2/2} e^{if\tau_3} e^{i\theta\tau_2/2} e^{i\phi\tau_3/2} . \qquad \text{(B6)}$$

The range of parameters is

$$0 \le f \le \pi, \quad 0 \le \theta \le \pi, \quad 0 \le \phi \le 2\pi .$$

The measure dA must be proportional to $(\det g_{ab})^{1/2}$, where

$$g_{ab} = \tfrac{1}{2}\mathrm{tr}(\partial_a A^\dagger \partial_b A)$$

defines the natural metric on the group space. A little algebra reveals three nonvanishing components:

$$g_{ff} = 1, \quad g_{\theta\theta} = \sin^2 f, \quad g_{\phi\phi} = \sin^2 f \sin^2\theta .$$

The properly normalized measure is therefore

$$\int dA \equiv \frac{1}{2\pi^2}\int_0^\pi df \sin^2 f \int_{-1}^1 d(\cos\theta)\int_0^{2\pi} d\phi . \quad \text{(B7)}$$

The advantage of this parametrization is that the traces are particularly simple:

$$\mathrm{tr}D^{(R)}(A) = \mathrm{tr}D^{(R)}(e^{if\tau_3}) = \sum_{m=-R}^{R} e^{2ifm} , \quad \text{(B8a)}$$

$$\mathrm{tr}\, A^\dagger = \mathrm{tr}(e^{-if\tau_3}) = 2\cos f . \quad \text{(B8b)}$$

The normalization equation is then

$$(2R+1)^2 = c_R^2(N_c)\frac{2}{\pi}\int_0^\pi df \sin^2 f \left|\sum_{m=-R}^{R} e^{2ifm}\right| \cos^{N_c} f$$
$$= c_R^2(N_c)\sum_{m=-R}^{R}\frac{1}{\pi}\int_{-\pi}^{\pi} df\, e^{2ifm}(\cos^{N_c} f - \cos^{N_c+2} f) , \quad \text{(B9)}$$

whereupon our claimed result, Eq. (19), follows immediately from a binomial expansion of the cosines.

Using the methods outlined here, the reader should have no difficulty verifying the normalization of the inverse relation, Eq. (22).

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[22]There is a mistake in Fig. 17.1 of Ref. 20: the directions of the arrows for the lines $l_1, l_2, \ldots, l_n$ should be reversed.