# Studies of confinement: How the gluon propagates

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The behavior of the gluon propagator in the Landau gauge is studied over distances from the scales probed in deep-inelastic scattering out to the size of a hadron—from the perturbative to the nonperturbative regimes—using the Schwinger-Dyson equations. The approximations needed to make the program tractable are both detailed and analyzed by considering two different calculational schemes, one of which takes into account more fully the complete structure of the crucial triple-gluon vertex. In both cases, the gluon propagator is found to be strongly enhanced like $1/p^4$ at low momenta $p$. We describe how this depends on $\Lambda$, the scale of QCD dynamics, and what this behavior means for the confinement potential. The use of a covariant gauge avoids difficulties inherent in axial gauges, where much previous work has been done, the differences with which are discussed.

## I. INTRODUCTION

The success of QCD in correlating the gamut of short-distance phenomena of hadron physics, from jet production in $e^+e^-$ annihilation to scaling violations in deep-inelastic scattering and the $p_T$ distribution of the $W$ and $Z$ bosons at the colliders, rightly makes it the accepted theory of the strong interactions. Yet away from the hard-scattering regions of such processes, it is at best a qualitative guide. All descriptions of hadron formation in $e^+e^-$ collisions, for instance, begin with the creation of a $q\bar{q}$ pair and the subsequent splitting off of gluons, whether soft, collinear, or hard, evolves according to the rules of perturbative QCD. However, the transformation of these colored partons, fractions of a femtometer apart, to the hadrons triggering macroscopic detectors must be modeled in terms of clusters, strings, and a plethora of fragmentation parameters. However phenomenologically successful the representations of this metamorphosis are, such modelings are not obviously detailed consequences of QCD itself. Clearly, if this is to be regarded as the theory of the strong interactions in something more than name, it must be capable of accurately describing such processes, solely in terms of the parameters of the theory itself, its scale $\Lambda$, and the quark masses (though for the lightest flavors these may be generated too). For this, it is necessary to understand the behavior of basic quantities of the theory, such as propagators and couplings not just over the short distances, when they are nearly free, but over the longer scales of the size of a hadron that are responsible for color neutralization and the fragmentation of partons, and the spectrum of hadrons themselves. The aim of this paper is to hammer out the calculational tools needed to work with continuum QCD at all distances. The Schwinger-Dyson equations provide the nonperturbative framework that permits such a study.[1-13] In this paper, we investigate the behavior of the gluon propagator in covariant gauges (the Landau gauge, in particular) in a world without quarks. This brings with it the triple-gluon coupling, which, together with the boson propagator, are the templates for the ambitious constructs outlined above, which form our eventual aim.

The Schwinger-Dyson equations, which embody the field equations of the theory, are an infinite set of nested relations, the exact solution of which is impossible. Simplifying assumptions are necessary to render the problem tractable. A perturbative approximation is the most commonly used and best known simplification. However, to go beyond perturbation theory and so be able to discuss issues of confinement, we must make some approximation that maintains the nonperturbative structure of these equations. We begin in this paper by considering just the behavior of the gluon propagator. The Schwinger-Dyson equation, detailed in Sec. II, involves not just the full (nonperturbative) gluon propagator itself, but the full triple- and quartic-gluon interactions too, as well as the propagator and coupling for the ghost. This means that the equation for the two-point function involves the form of the three- and four-point functions too. These in turn satisfy equations which involve yet higher-point functions and so on. However, a most important aspect of a gauge theory is that gauge invariance imposes relationships between Green's functions with different numbers of external legs. Such Ward identities (or rather Slavnov-Taylor identities in the case of a non-Abelian theory) thereby constrain the three-point function, for example, in terms of the propagator. By including only that part of the vertex so determined, the equation can now be solved by demanding self-consistency. It is natural to regard that part of an $n$-point Green's function which is determined in terms of lower-point Green's functions via the Slavnov-Taylor identity as the longitudinal part. This approximation, then, involves neglecting the transverse pieces. These parts are, of course, determined by the Schwinger-Dyson equation for the $n$-point function itself, which will involve the longitudinal part of the $(n+1)$-point function. This, in a similar way, is determined by a higher Slavnov-Taylor identity. This procedure introduced by Baker, Ball, and Zachariasen[3] in axial gauges allows, at least in principle, a gauge-invariant truncation of the Schwinger-Dyson equations. In covariant gauges, where ghosts enter, further assumptions are needed on how to treat these.[14] How we approximate the ghost is described in Sec. II.

Having set up the gluon equation, we solve it in Sec. III using the *Ansatz* for the full triple-gluon vertex that satisfies the appropriate Slavnov-Taylor identity. All the technicalities of rendering numerical integrals finite, handling infrared singularities, renormalizing the ultraviolet divergences that introduce, as usual, the QCD scale $\Lambda$, as well as actually solving the gluon equation are also detailed. The gluon propagator is found to be enhanced like $1/p^4$ at low momenta with a scale set by $\Lambda$. A brief report of these results has appeared previously in Ref. 8 where none of the crucial technicalities given here were discussed. We then go on in Sec. IV to investigate a proposal made by Mandelstam[4] that a far simpler *Ansatz* for the triple-gluon vertex, while not respecting gauge invariance, maintains the essential structure of the full equation. This simplified Schwinger-Dyson equation for the gluon propagator is solved and found to give very similar results to those using the full vertex of Sec. III. We comment on how and why this happens.

In Sec. V we discuss the implications these covariant gauge solutions for the full gluon propagator have for our understanding of confinement, the interquark potential and the Wilson area law and comment on the corresponding results in axial gauges found by Baker, Ball, and Zachariasen.[3]

The next step in this program is to include $n_f$ quark flavors in QCD. These not only enter in the Schwinger-Dyson equation for the fermion propagator itself but in quark-loop corrections to the boson propagator studied here. This requires the simultaneous solution of coupled integral equations—two in number for massless quarks. The fact that here we have shown that the far simpler Mandelstam treatment of the triple-gluon vertex gives a gluon propagator with very similar properties to that with a full vertex makes it natural to use this. The results of this are summarized in Ref. 11. This paper contains all the precursory details needed to solve the gluon equations embedded in that work.

In Sec. VI we briefly state our conclusions.

## II. THE GLUON EQUATION

The bare gluon propagator in a covariant gauge is composed of a transverse and a longitudinal piece:

$$\Delta_0^{\mu\nu}=\frac{1}{p^2}\left[\delta^{\mu\nu}-\frac{p^\mu p^\nu}{p^2}\right]+\xi\frac{p^\mu p^\nu}{p^4}\ , \qquad (2.1)$$

where the parameter $\xi$ specifies the gauge. The tensor structure of the full propagator is given analogously by

$$\Delta^{\mu\nu}(p)=\frac{\mathcal{G}(p)}{p^2}\left[\delta^{\mu\nu}-\frac{p^\mu p^\nu}{p^2}\right]+\xi\frac{p^\mu p^\nu}{p^4}\ , \qquad (2.2)$$

where a Ward identity ensures that it is only the transverse part that is renormalized and so the coefficient of the transverse part, viz., $\mathcal{G}(p)$, is known as the gluon renormalization function. The inverse of this propagator, $\Pi^{\mu\nu}(p)$, satisfies the Schwinger-Dyson equation depicted in Fig. 1, which involves the unknown triple- and quartic-gluon vertices, as well as ghost contributions. In order to derive a closed equation for the gluon renormalization function we have to make approximations.

The first step is to ignore those diagrams which involve quartic-gluon couplings. This can be seen as a first step in an iterative procedure to solve the hierarchy of the Schwinger-Dyson equations. Perhaps more realistically, we note that these diagrams contain an explicit factor of $\alpha_0^2$, where $\alpha_0=g_0^2/4\pi$ is the bare coupling. When we come to renormalize our equation (see Sec. III A) to remove the ultraviolet divergences, we will do this at some scale $\mu^2$, as usual. The bare couplings which appear in the unrenormalized equation will be replaced by the renormalized running coupling $\alpha(\mu)$. We shall choose $\mu^2$ to be in the perturbative regime so that $\alpha(\mu)$ will be a small number. Hence, those diagrams involving quartic couplings will be suppressed by a factor $\alpha(\mu)$ relative to the remaining terms. Of course, the value of the loop integral could swamp this suppression, but it seems natural to believe that the full triple-gluon vertex already contains the essence, if not all the details, of the confinement mechanism.

Next we consider the ghost contribution. Since we choose to work in the Landau gauge, the ghost kernels which appear in the Slavnov-Taylor identities for the triple-gluon vertex (and incidentally in the quark-gluon vertex too) will vanish as one of the external momenta goes to zero.[5,15] This is related to the transversality of the gluon propagator in this gauge. Ghost contributions are, of course, essential in preserving the transversality of the inverse propagator, but it is hoped that neglecting them will not significantly alter the analytic behavior of $\mathcal{G}(p)$. This is supported by the fact that in a one-loop perturbative calculation, the ghost loop makes a *numerically* small contribution to $\mathcal{G}(p)$, at least for the Landau

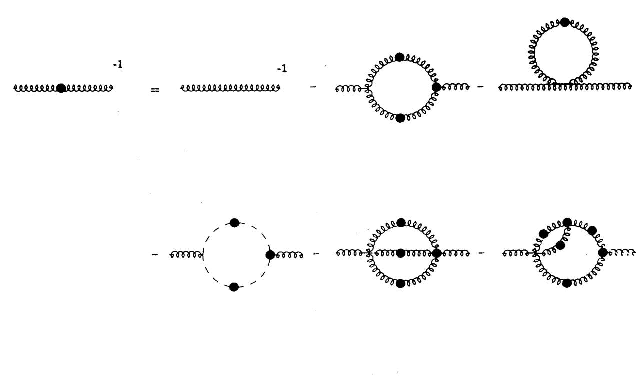

FIG. 1. The complete Schwinger-Dyson equation for the inverse gluon propagator with no fermions. The spiral lines represent gluons and the dashed lines ghosts. The dots denote full (as opposed to bare) propagators and vertices.

and other "sensible" gauges (in a related work[10] the ghost was treated as bare with very little difference).

Our equation now involves just the crucial triple-gluon vertex $\Gamma^{\mu\nu\rho}(p,q,r)$ in its full form. This is constrained by the Slavnov-Taylor identity. By setting the ghost kernels to zero, in keeping with their infrared vanishing (as mentioned above), this relation simplifies to

$$q^{\nu}\Gamma_{\mu\nu\rho}(p,q,r)=+\frac{1}{\mathcal{G}(p)}(p^2\delta^{\mu\rho}-p^{\mu}p^{\rho}) -\frac{1}{\mathcal{G}(r)}(r^2\delta^{\mu\rho}-r^{\mu}r^{\rho}) , \tag{2.3}$$

where $p,q,r$ are the incoming momenta in the three-gluon legs. Separating $\Gamma^{\mu\nu\rho}$ into two pieces, the longitudinal and transverse parts, where the transverse part is defined to vanish when contracted with any external momentum, we see that Eq. (2.3) only constrains the longitudinal part. This can then be "solved" to give[16,17]

$$\begin{aligned}\Gamma^{L}_{\mu\nu\rho}(p,q,r)=&\,\delta_{\mu\nu}\left[\frac{p_\rho}{\mathcal{G}(p)}-\frac{q_\rho}{\mathcal{G}(q)}\right]+\delta_{\nu\rho}\left[\frac{q_\mu}{\mathcal{G}(q)}-\frac{r_\mu}{\mathcal{G}(r)}\right]+\delta_{\rho\mu}\left[\frac{r_\nu}{\mathcal{G}(r)}-\frac{p_\nu}{\mathcal{G}(p)}\right]\\ &+\frac{1}{p^2-q^2}\left[\frac{1}{\mathcal{G}(p)}-\frac{1}{\mathcal{G}(q)}\right](p_\nu q_\mu-\delta_{\mu\nu}p\cdot q)(p_\rho-q_\rho)+\frac{1}{q^2-r^2}\left[\frac{1}{\mathcal{G}(q)}-\frac{1}{\mathcal{G}(r)}\right](q_\rho r_\nu-\delta_{\nu\rho}q\cdot r)(q_\mu-r_\mu)\\ &+\frac{1}{r^2-p^2}\left[\frac{1}{\mathcal{G}(r)}-\frac{1}{\mathcal{G}(p)}\right](r_\mu p_\rho-\delta_{\mu\rho}r\cdot p)(r_\nu-p_\nu) .\end{aligned} \tag{2.4}$$

In the next section we shall solve the gluon equation under the assumption that this on its own is a good approximation to the full triple-gluon vertex. This is motivated not only by the fact that it satisfies the Slavnov-Taylor identities, but also by the fact that the neglected transverse part vanishes when the external momenta go to zero. It should also be noted that the longitudinal part also gives the leading-order perturbative result at large momenta.

For the discussion in Sec. IV it is helpful to notice the seemingly trivial fact that if the renormalization function $\mathcal{G}(k)$ is more or less equal regardless of whether its argument is $p$, $q$, or $r$, then Eq. (2.4) for the vertex becomes

$$\Gamma^{L}_{\mu\nu\rho}(p,q,r)=\frac{\Gamma^{0}_{\mu\nu\rho}(p,q,r)}{\mathcal{G}(p)} , \tag{2.5}$$

where $\Gamma^{0}_{\mu\nu\rho}$ is the bare triple-gluon vertex. We shall see later that this apparently trivial approximation is in fact meaningful as first noted by Mandelstam.[4]

After these approximations, we have an implicit equation for $\mathcal{G}(p)$:

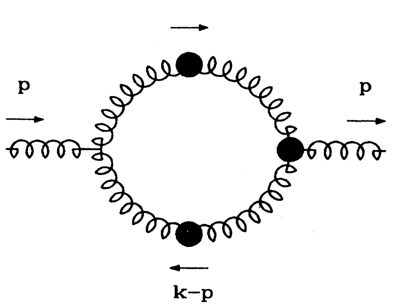


FIG. 2. The one-loop gluon contribution to the inverse gluon propagator of Fig. 1 with the momenta labeled.

$$\Pi^{\mu\nu}_{ab}=\Pi^{\mu\nu}_{0ab}-\rho^{\mu\nu}_{ab} , \tag{2.6}$$

where $\Pi$ represents the full inverse gluon propagator, $\Pi_0$ the bare one, and $\rho$ is the contribution to the gluon self-energy from the one-loop gluon diagrams depicted in Fig. 1. Since we will use an ultraviolet cutoff to regularize our integrals we need to avoid quadratic divergences, and these are absent from the term proportional to $p^{\mu}p^{\nu}$. Consequently, we can project out these $p^{\mu}p^{\nu}$ terms by contracting with the tensor

$$P^{\mu\nu}=\frac{1}{3p^2}(4p^{\mu}p^{\nu}-p^2\delta^{\mu\nu}) \tag{2.7}$$

to give a scalar equation. As the tadpole diagram in Fig. 1 is proportional to $\delta^{\mu\nu}$, its contribution will decouple from our equation. Thus the equation for $\mathcal{G}(p)$ receives contributions from just the gluon loop of Fig. 2. With the momenta labeled as in Fig. 2, we have

$$\begin{aligned}\frac{1}{\mathcal{G}(p)}=1+\frac{g_0^2C_A}{32\pi^4p^2}P_{\mu\nu}\int d^4k\,\Gamma_0^{\mu\alpha\delta}(-p,k,q)\Delta_{\alpha\beta}(k)\\ \times\Delta_{\gamma\delta}(q)\Gamma_L^{\beta\nu\gamma}(-k,p,-q) ,\end{aligned} \tag{2.8}$$

where $C_A$ is the Casimir constant equal to $N$ for an SU($N$) gauge theory. This is the equation we solve in Secs. III and IV using the *Ansätze* for $\Gamma_L$ of Eqs. (2.4) and (2.5), respectively.

## III. THE GLUON WITH A FULL TRIPLE-GLUON VERTEX

### A. The equation

Approximating the full triple-gluon vertex in Eq. (2.8) by the longitudinal part specified by the Slavnov-Taylor identity, viz., $\Gamma_L$ given by Eq. (2.4), the gluon equation becomes

$$\frac{1}{\mathcal{G}(p)}=1+\frac{C_A g_0^2}{96\pi^4 p^2}\int d^4k\left[\frac{\mathcal{G}(k)A(k,p)}{k^4q^2p^2}+\frac{\mathcal{G}(k)\mathcal{G}(q)}{\mathcal{G}(p)}\frac{B(k,p)}{k^4q^4}+\frac{\mathcal{G}(q)-\mathcal{G}(p)}{q^2-p^2}\frac{C(k,p)}{k^2q^2}\frac{\mathcal{G}(k)}{\mathcal{G}(p)}+\frac{\mathcal{G}(q)-\mathcal{G}(k)}{q^2-k^2}\frac{D(k,p)}{k^2q^2p^2}\right] \tag{3.1}$$

with $q=k-p$, where

$$\begin{aligned} A(k,p)&=48k^2(p\cdot k)^2-32k^2p^2p\cdot k-16(p\cdot k)^3 \\ &\quad -12k^4p^2+6k^2p^4+6p^2(p\cdot k)^2 , \\ B(k,p)&=38k^2p^2p\cdot k-25p^2(p\cdot k)^2-14k^2p^4 \\ &\quad +12p^4p\cdot k-13k^4p^2-2k^2(p\cdot k)^2+4(p\cdot k)^3 , \end{aligned} \tag{3.2}$$

$$\begin{aligned} C(k,p)&=24p^4+14k^2p^2+16(p\cdot k)^2-48p^2p\cdot k-6k^2p\cdot k , \\ D(k,p)&=12k^4p^2-48k^2(p\cdot k)^2+48(p\cdot k)^3 \\ &\quad +24k^2p^2p\cdot k-5k^2p^4-40p^2(p\cdot k)^2+9p^4p\cdot k . \end{aligned}$$

This is the equation we have to solve. As it stands the integrals have not only the usual ultraviolet divergences of perturbation theory, but potential infrared singularities too. These must be dealt with to give us a finite renormalized equation.

As we shall argue later, the only possible consistent behavior for $\mathcal{G}(p)$ as $p^2\to 0$ is $\mathcal{G}(p)\sim 1/p^2$. If $\mathcal{G}(p)$ is this singular, however, the integrals in Eq. (3.1) are potentially infrared divergent. In a related study in axial gauges,[3] such divergences were avoided by assuming that (i) the full axial gauge propagator has the same tensor structure as the bare one and (ii) that the coefficient of such an infrared enhanced term is independent of the choice of axial gauge. It is possible that if we could solve the complete Schwinger-Dyson equations in covariant gauges with no approximations or truncations, then such a term would not give rise to divergences. Nevertheless, as they do appear in our truncation, we must deal with them. We choose to treat them by defining $\mathcal{G}(p)$ by a "plus" prescription, which we define later in this section. Of course, this prescription is not determined by the theory, but put in "by hand"; the justification here will lie in the physically meaningful results we obtain. Nevertheless, one of the long-term goals of any analysis based upon the Schwinger-Dyson equations will be to eliminate these infrared divergences in a self-consistent way, entirely within the context of the theory.

Even when we have dealt with these divergences, we still obtain contributions violating the masslessness of the gluon, a condition demanded by gauge invariance. If $\mathcal{G}(p)$ has such an infrared singular term, on dimensional grounds it must be $A\mu^2/p^2$. Even after integration, the dimensionful quantity $\mu^2$ must be balanced, and the only quantity available is $p^2$. Thus we will have contributions to the right-hand side of Eq. (3.1) which behave like $1/p^2$. The condition for masslessness is

$$\lim_{p^2\to 0}\Pi^{\mu\nu}=0 , \tag{3.3}$$

where $\Pi^{\mu\nu}$ is the full inverse gluon propagator. Since the bare inverse gluon propagator obeys the condition Eq. (3.3), then all our integrals on the right-hand side of Eq. (3.1) must obey this as well. This is clearly not true of some of the terms which arise from integration of the infrared enhanced term. We must perform a subtraction to remove these terms. Formally, we may imagine adding some appropriate counterterms in the bare Lagrangian to achieve this.

The easiest way to preserve masslessness is to write $\mathcal{G}(p)=A\mu^2/p^2+\mathcal{G}_1(p)$, where $p^2/\mathcal{G}_1(p)\to 0$ as $p^2\to 0$. The entire contribution to the mass term comes from the enhanced term $A\mu^2/p^2$. Thus if in the right-hand side of Eq. (3.1) we set $\mathcal{G}(p)\equiv A\mu^2/p^2$ and subtract this from Eq. (3.1), then no mass term will arise. This mass renormalization, and the "plus" prescription treatment of the infrared divergences do not prejudge that $\mathcal{G}(p)$ does possess such an infrared-enhanced term, as we allow the possibility that the coefficient of this term can vanish. Finally, any terms on the right-hand side of Eq. (3.1) which are linear in $\mathcal{G}$ will have the entire contribution of the enhanced term subtracted by this mass renormalization. In these terms we can effectively set $\mathcal{G}\equiv\mathcal{G}_1$.

We still have the usual logarithmic ultraviolet divergences, which arise from the momentum loop integral at large $k^2$. To handle these we cast Eq. (3.1) in the form

$$\frac{1}{\mathcal{G}(p)}=1+\frac{C_A\alpha_0}{24\pi^3}\int d^4k\,\mathcal{K}(k,p)+\frac{C_A\alpha_0}{24\pi^3\mathcal{G}(p)}\int d^4k\,\mathcal{G}(k)\mathcal{L}(k,p) , \tag{3.4}$$

where

$$\begin{aligned} \mathcal{K}(k,p)&=\frac{\mathcal{G}(k)A(k,p)}{k^4q^2p^4}+\frac{\mathcal{G}(q)-\mathcal{G}(k)}{q^2-k^2}\frac{D(k,p)}{k^2q^2p^4}-\frac{\mathcal{G}(k)C(k,p)}{(q^2+\mu^2)k^2q^2p^2} , \\ \mathcal{L}(k,p)&=\frac{\mathcal{G}(q)B(k,p)}{k^4q^4p^2}+\frac{C(k,p)}{q^2-p^2}\left[\mathcal{G}(q)-\mathcal{G}(p)\left[\frac{p^2+\mu^2}{q^2+\mu^2}\right]\right]\frac{1}{k^2q^2p^2} . \end{aligned} \tag{3.5}$$

Here $\mathcal{K}(k,p)$ contains those terms which are linear in $\mathcal{G}$, whereas $\mathcal{L}(k,p)$ contains those terms which are quadratic in $\mathcal{G}$ in the numerator, with an explicit $1/\mathcal{G}(p)$ in the denominator.

In splitting the integral into these two terms, there is a slight technicality over the term proportional to $C(k,p)$, which involves the factor

$$[\mathcal{G}(k)/\mathcal{G}(p)][\mathcal{G}(q)-\mathcal{G}(p)]/(q^2-p^2) .$$

This term contains a part which belongs to $\mathcal{K}(k,p)$ and a part belonging to $\mathcal{L}(k,p)$. These will be renormalized differently, and so we must be careful not to introduce any singularities from the $q^2-p^2$ term in the denominator. This is done by writing

$$\frac{\mathcal{G}(k)}{\mathcal{G}(p)}\frac{\mathcal{G}(q)-\mathcal{G}(p)}{q^2-p^2}=\frac{\mathcal{G}(k)}{\mathcal{G}(p)}\frac{\mathcal{G}(q)}{q^2-p^2}\left[1-\frac{\mathcal{G}(p)}{\mathcal{G}(q)}\frac{p^2+\mu^2}{q^2+\mu^2}\right]$$
$$-\frac{\mathcal{G}(k)}{\mathcal{G}(p)}\frac{\mathcal{G}(p)}{q^2-p^2}\left[1-\frac{p^2+\mu^2}{q^2+\mu^2}\right]$$
$$=\frac{1}{q^2-p^2}\frac{\mathcal{G}(k)}{\mathcal{G}(p)}\left[\mathcal{G}(q)-\mathcal{G}(p)\frac{p^2+\mu^2}{q^2+\mu^2}\right]$$
$$-\frac{\mathcal{G}(k)}{q^2+\mu^2}\ . \tag{3.6}$$

Here we call the first term part of $\mathcal{L}(k,p)$ and the second term part of $\mathcal{K}(k,p)$. All we have done is to add and subtract a term which ensures both parts are individually finite at $p^2=q^2$. This term is obviously arbitrary, but we have chosen it so that it does not introduce any extra ultraviolet divergences, being finite by power counting. The momentum $\mu^2$ in Eq. (3.6) is also arbitrary, but for convenience we choose it equal to our renormalization scale.

Returning to Eq. (3.4), we define a renormalized gluon function $\mathcal{G}_R(p)$ by

$$\mathcal{G}(p)=Z(\kappa/\mu)\mathcal{G}_R(p)\ . \tag{3.7}$$

Here $\kappa$ is an ultraviolet cutoff which serves to make the integrals finite. For convenience, $\mathcal{K}(k,p),\mathcal{L}(k,p)$, of Eq. (3.6) with $\mathcal{G}$ replaced by $\mathcal{G}_R$ will be denoted by $\mathcal{K}_R(k,p),\mathcal{L}_R(k,p)$, respectively. We now define a renormalized running coupling constant by

$$g^2(\mu)=\frac{g_0^2Z(\kappa/\mu)}{1+\dfrac{g_0^2C_A}{96\pi^4}Z(\kappa/\mu)\int d^4k\,\mathcal{K}_R(k,\mu)}\ . \tag{3.8}$$

Using Eq. (3.4) we can also rewrite this as

$$g^2(\mu)=\frac{g_0^2Z^2(\kappa/\mu)\mathcal{G}_R(\mu)}{1-\dfrac{g_0^2C_A}{96\pi^4}Z^2(\kappa/\mu)\int d^4k\,\mathcal{G}_R(k)\mathcal{L}_R(k,\mu)}\ . \tag{3.9}$$

Inverting these two equations in terms of the bare coupling $g_0^2$ and substituting these into Eq. (3.4), using Eq. (3.7), gives, after a little rearranging,

$$\frac{\mathcal{G}_R(\mu)}{\mathcal{G}_R(p)}=1+\frac{\alpha(\mu)C_A}{24\pi^3}\int d^4k\,[\mathcal{K}_R(k,p)-\mathcal{K}_R(k,\mu)]$$
$$+\frac{\alpha(\mu)C_A}{\mathcal{G}_R(p)24\pi^3}$$
$$\times\int d^4k\,\mathcal{G}_R(k)[\mathcal{L}_R(k,p)-\mathcal{L}_R(k,\mu)] \tag{3.10}$$

in which both the integrals are ultraviolet finite. This is now the equation we solve for $\mathcal{G}_R(p)$ by requiring self-consistency.

To solve this (see Sec. III D), we will need a more explicit form for the running coupling, in order to determine the behavior of $\mathcal{G}_R(p)$ at large $p^2$. We first invert Eq. (3.8) to give

$$\frac{Z(\kappa/\mu)}{g^2(\mu)}=\frac{1}{g_0^2}+\frac{C_A}{96\pi^4}\int d^4k\,\mathcal{K}(k,\mu)\ . \tag{3.11}$$

Subtracting Eq. (3.11) from the same equation evaluated at $p$ gives

$$\frac{Z(\kappa/p)}{g^2(p)}=\frac{Z(\kappa/\mu)}{g^2(\mu)}$$
$$+Z(\kappa/\mu)\frac{C_A}{96\pi^4}\int d^4k[\mathcal{K}_R(k,p)-\mathcal{K}_R(k,\mu)]\ . \tag{3.12}$$

If we take the definition of Eq. (3.7), and instead renormalize at $p^2$ we would have

$$\mathcal{G}(p)=Z(\kappa/p)\mathcal{G}_{R_p}(p)\ , \tag{3.13}$$

where for clarity we have written the subscript $R_p$ to denote that $p^2$ is our renormalization scale. However we must have $\mathcal{G}_{R_p}(p)=\mathcal{G}_{R_\mu}(\mu)$, and we rewrite Eq. (3.13) as

$$\mathcal{G}(p)=Z(\kappa/p)\mathcal{G}_{R_\mu}(\mu)\ . \tag{3.14}$$

Then from the equality of Eqs. (3.7) and (3.14) we can write

$$\frac{Z(\kappa/p)}{Z(\kappa/\mu)}=\frac{\mathcal{G}_R(p)}{\mathcal{G}_R(\mu)}\ , \tag{3.15}$$

where all quantities are renormalized at the scale $\mu^2$. This allows us to simply rewrite Eq. (3.12) as

$$\frac{1}{g^2(p)}=\frac{\mathcal{G}_R(\mu)}{\mathcal{G}_R(p)}\left[\frac{1}{g^2(\mu)}\right.$$
$$\left.+\frac{C_A}{96\pi^4}\int d^4k[\mathcal{K}_R(k,p)-\mathcal{K}_R(k;\mu)]\right] \tag{3.16}$$

as an equation for the renormalized running coupling constant.

### B. Consistent asymptotic behavior of $\mathcal{G}_R(p)$

Because of the complicated structure of our equation for the gluon renormalization function $\mathcal{G}_R(p)$, Eq. (3.10), an analytic solution is not possible, and we attempt a numerical study. Before doing this, we investigate the possible asymptotic behavior for both small and large $p^2$ that we should build into our numerical solution. In general we will take a trial input function $\mathcal{G}_{\rm in}(p)$, and substitute this into the right-hand side of our equation, Eq. (3.10). After performing the integrations we obtain an output function $1/\mathcal{G}_{\rm out}(p)$, to be compared to the reciprocal of the input function. We allow our trial function to depend on a number of parameters, and we vary these until we find good agreement between input and output over a suitable range of $p^2$.

All of the integrals in Eq. (3.10) give rise to dimensionless quantities, and all contain the factor $1/k^2q^2$ in their denominator. This allows us to derive the following general results.

(a) If $\mathcal{G}_{\text{in}}(p)\to\text{const}$, as $p^2\to 0$, then the output of the right-hand side of Eq. (3.10) is easily seen to give

$$\frac{1}{\mathcal{G}_{\text{out}}}=1+\text{const}\times\ln(p^2/\mu^2)\ .$$

(b) If $\mathcal{G}_{\text{in}}(p)\sim\ln p^2$ as $p^2\to 0$, then the right-hand side gives

$$\frac{1}{\mathcal{G}_{\text{out}}}=1+\text{const}\times\ln^2(p^2/\mu^2)\ .$$

(c) These logarithms may sum to a power. If we let $\mathcal{G}_{\text{in}}(p)\to(\mu^2/p^2)^\eta$ as $p^2\to 0$ with $\eta<1$, then all the integrals are infrared finite, and we would obtain

$$\frac{1}{\mathcal{G}_{\text{out}}}=1+\text{const}\times\left[\frac{\mu^2}{p^2}\right]^\eta ,$$

where this is purely on dimensional grounds, to balance the dimensionful quantity $\mu^2$.

In all of these cases it is not possible to obtain agreement between input and output. If, however, we consider the case $\mathcal{G}_{\text{in}}\to\mu^2/p^2$, and we use some regularization technique to deal with the infrared divergences, then we would find by the same dimensional arguments that

$$\frac{1}{\mathcal{G}_{\text{out}}}=1+C\mu^2/p^2\ ,$$

where $C$ is a constant. Unlike the terms in (c) above, this will violate our masslessness condition, and so must be subtracted. Then if the higher-order terms in $p^2$ of $\mathcal{G}(p)$ generate a contribution to the right-hand side of which cancels the explicit factor of 1, then the right-hand side of the equation would vanish as $p^2\to 0$. This is exactly the behavior of the left-hand side, and we have the possibility of finding agreement. If the next-order-$p^2$ corrections to $\mathcal{G}_{\text{in}}(p)$ behaved like a constant, a logarithm, or a power as in (c) above, but with $0<\eta<1$, then this would give rise to inconsistent behavior, as the left-hand side of Eq. (3.10), i.e., the reciprocal of the input function *should vanish* at small $p^2$. We therefore will attempt to find solutions which behave like

$$\mathcal{G}_{\text{in}}=A\mu^2/p^2+B(p^2/\mu^2)^\nu,\quad \nu>0 \tag{3.17}$$

as $p^2\to 0$.

We can now use Eqs. (3.10) and (3.16) to determine the asymptotic form of our equation for large $p^2$. By writing $\mathcal{G}_R=1+O(\alpha(\mu))$ then if we work only to $O(\alpha(\mu))$, we can simply set $\mathcal{G}_R=1$ on the right-hand side of Eq. (3.10). However, if we set $\mathcal{G}_R=1$ in our *Ansatz* for the full triple-gluon vertex Eq. (2.4), it reduces to the bare vertex. Thus the leading term from this diagram is simply that of perturbation theory and we have

$$\frac{1}{\mathcal{G}_R(p)}=1+\frac{\alpha(\mu)}{4\pi}\gamma_0\ln\left[\frac{p^2}{\mu^2}\right] , \tag{3.18}$$

where calculation yields $\gamma_0=\frac{7}{3}C_A$. Similarly, extracting the leading behavior of the renormalized running coupling, Eq. (3.16), we obtain

$$\frac{1}{\alpha(p)}=\frac{\mathcal{G}_R(\mu)}{\mathcal{G}_R(p)}\left[\frac{1}{\alpha(\mu)}+\frac{25C_A}{48\pi}\ln\left[\frac{p^2}{\mu^2}\right]\right] . \tag{3.19}$$

Using Eq. (3.18) to expand $\mathcal{G}_R(p)$, we obtain

$$\frac{1}{\alpha(p)}=\frac{1}{\alpha(\mu)}+\frac{\beta_0}{4\pi}\ln\left[\frac{p^2}{\mu^2}\right] , \tag{3.20}$$

where $\beta_0=\frac{53}{12}C_A$. This is to be compared with the familiar $\beta_0$ of perturbation theory, viz., $\frac{11}{3}C_A$, the difference arising partly from our neglect of the ghost contribution.

The standard renormalization-group argument allows us to rewrite Eq. (3.20) as

$$\alpha(p)=\frac{4\pi}{\beta_0\ln(p^2/\Lambda^2)} \tag{3.21}$$

introducing the QCD scale parameter $\Lambda$. Any dependence of Eq. (3.10) on $\mu$ can be reexpressed in terms of this intrinsic scale of the theory. The standard renormalization-group argument allows us to rewrite the asymptotic form for $\mathcal{G}_R(p)$ as

$$\mathcal{G}_R(p)\sim\mathcal{G}_R(\mu)\left[\frac{\alpha(p)}{\alpha(\mu)}\right]^{\gamma_0/\beta_0}\sim[\ln(p^2/\mu^2)]^{-\gamma_0/\beta_0} , \tag{3.22}$$

where $\gamma_0/\beta_0=\frac{28}{53}$. We have thus deduced in Eqs. (3.17) and (3.22) the possible analytic behavior for both small and large $p^2$. Before we proceed with our numerical study which will show that self-consistent solutions to Eq. (3.10) do exist with this behavior, there are some technicalities with which we have to deal.

### C. Technical details

In order to calculate the numerical integrals in Eq. (3.10), a number of technical details must be specified. The most important of these is the definition and implementation of the "plus" prescription, which we use to make our potentially infrared divergent integrals finite. The basic definition is that

$$\left[\frac{A\mu^2}{k^2}\right]_+=\frac{A\mu^2}{k^2}\quad\text{for }\infty>k^2>0 \tag{3.23}$$

and that in the neighborhood of $k^2=0$ it is a distribution that satisfies

$$\int_0^\infty dk^2\left[\frac{A\mu^2}{k^2}\right]_+S(k,p)=\int_0^{p^2}dk^2\left[\frac{A\mu^2}{k^2}\right][S(k,p)-S(0,p)]+\int_{p^2}^\infty dk^2\left[\frac{A\mu^2}{k^2}\right]S(k,p) \tag{3.24}$$

for any nonzero $p$ and nonsingular $S(k,p)$. For those integrals in which the angular integrations can be performed analytically, the implementation of this prescription is easy. Unfortunately for many of the integrals we have to deal with, namely, the $\mathcal{L}(k,p)$ terms, the angular integrations cannot all be performed analytically, and must be computed numerically. Thus writing

$$S(k,p)=\int_0^\pi d\psi \sin^2\psi\, s(k,p,\cos\psi) \tag{3.25}$$

we define

$$S(0,p)=\lim_{k^2\to 0}\int_0^\pi d\psi \sin^2\psi\, s(k,p,\cos\psi) \ . \tag{3.26}$$

For many of these terms the limit and the integral sign in Eq. (3.26) can be interchanged to give us $s(0,p,\cos\psi)$, for other terms more work remains to be done. We illustrate this and other problems by means of some examples.

First of all, some of the terms are infrared finite, and for these, the subtraction in Eq. (3.24) is obviously irrelevant, $S(0,p)$ being zero. The terms from $\mathcal{L}(k,p)$, Eq. (3.5) in Eq. (3.10), are proportional to $\mathcal{G}(k)\mathcal{G}(q)$, and either of these two can be infrared enhanced, the contribution where both are enhanced being subtracted by our mass renormalization (see Sec. III A). By transforming integration variables $k\to q$ we can always make the enhanced term $A\mu^2/k^2$. From Eq. (3.2) and its transform when $k\to q$, we can read off the kinds of terms with which we have to deal for integrals over $B(k,p)$, for example. With the measure $dk^2 d\psi \sin^2\psi$ the potentially infrared divergent terms are

$$\begin{gathered} s(k,p,\cos\psi)\sim \frac{p^2k\cdot p}{q^2k^2}\mathcal{G}_1(q)\ , \\ \frac{(k\cdot p)^2}{q^4k^2}\mathcal{G}_1(q),\quad \frac{p^2}{q^4}\mathcal{G}_1(q)\ , \end{gathered} \tag{3.27}$$

where $\mathcal{G}_1$ is the nonenhanced part of $\mathcal{G}$. The second two of these are relatively easy to handle giving

$$s(0,p,\cos\psi)\sim \frac{\cos^2\psi}{p^2}\mathcal{G}_1(p),\quad \frac{1}{p^2}\mathcal{G}_1(p)\ . \tag{3.28}$$

To calculate the actual value of the integrand at $k^2=0$, we can Taylor expand $s(k,p,\cos\psi)$, subtracting the leading term $s(0,p,\cos\psi)$. Thus for the second of the terms in Eq. (3.27) we obtain

$$\begin{aligned} s(k,p,\cos\psi)=\frac{\cos^2\psi}{p^2}[&\mathcal{G}_1(p)+(k^2-2k\cdot p)\mathcal{G}_1'(p) \\ &+2(k\cdot p)^2\mathcal{G}_1''(p)] \\ &\times\left[1-2\frac{k^2}{p^2}+4\frac{k\cdot p}{p^2}+\frac{12(k\cdot p)^2}{p^4}\right]+O(k^3)\ , \end{aligned} \tag{3.29}$$

where a prime denotes differentiation with respect to $p^2$. Here we can expand the brackets, subtract the leading term [Eq. (3.28)], and use the fact that terms odd in $\cos\psi$ will integrate to zero over $\psi$.

We perform our numerical integrations by a Simpson's rule method, increasing the number of integration points until the answer is stable to within 0.1%, as well as demanding similar numerical agreement with the analytically calculable answer when $\mathcal{G}_1\equiv 1$. To implement the plus prescription, Eq. (3.24), numerically, the integral from 0 to $p^2$ is split into two regions: 0 to $\epsilon p^2$ and $\epsilon p^2$ to $p^2$. Over the second region, away from $k^2=0$ we simply use $s(k,p,\cos\psi)-s(0,p,\cos\psi)$ in our integrand. Over the first region we use the Taylor expansion to $O(k^2)$, subtracting the leading term. By using $\epsilon=10^{-4}$ we maintain our numerical accuracy.

The first of the terms in Eq. (3.27) is a little more difficult, itself appearing divergent as $k^2\to 0$, leaving $s(0,p,\cos\psi)$ apparently undefined. In this case the limit and integral sign in Eq. (3.26) cannot be naively interchanged. Taylor expanding $s(k,p,\cos\psi)$ we obtain

$$\begin{aligned} s(k,p,\cos\psi)&=\frac{p^2k\cdot p}{q^4k^2}\mathcal{G}_1(q) \\ &=\frac{p^2k\cdot p}{p^4k^2}[\mathcal{G}_1(p)+(k^2-2k\cdot p)\mathcal{G}_1'(p)] \\ &\quad\times\left[1-2\frac{k^2-2k\cdot p}{p^2}\right]+O(k)\ . \end{aligned} \tag{3.30}$$

The potentially divergent leading term is odd in $\cos\psi$ and so will vanish on performing the $\psi$ integration. Thus from Eq. (3.30) we obtain

$$s(0,p,\cos\psi)=\cos^2\psi\left[4\frac{\mathcal{G}_1(p)}{p^2}-2\mathcal{G}_1'(p)\right]\ . \tag{3.31}$$

Once again we can split the 0 to $p^2$ integration into two regions, using the Taylor expansion with the leading term Eq. (3.30) subtracted for $0<k^2<\epsilon p^2$, and with the full $s(k,p,\cos\psi)-s(0,p,\cos\psi)$ for $\epsilon p^2<k^2<p^2$.

The only other potentially divergent terms in Eq. (3.2) are those from $C(k,p)$ of Eq. (3.5). With $\mathcal{G}(k)$ enhanced, all the terms are finite by counting powers of $k$ except for the term proportional to $p^4$ for which

$$s(k,p,\cos\psi)=\frac{24p^4}{q^2p^2}\left[\mathcal{G}_1(q)-\frac{p^2+\mu^2}{q^2+\mu^2}\mathcal{G}_1(p)\right]\frac{1}{q^2-p^2}\ . \tag{3.32}$$

By Taylor expanding we can easily obtain

$$s(0,p,\cos\psi)=24\left[\frac{\mathcal{G}_1(p)}{p^2+\mu^2}+\mathcal{G}'(p)\right] \tag{3.33}$$

and we use the same treatment of the numerical integrals as outlined above. For the case where $\mathcal{G}(q)$ is enhanced, we can transform variables $k\to q$ for the $C(k,p)$ terms. It can easily be checked that $C(k,p)$ transforms into the quantity

$$8k^2p^2+4(k\cdot p)^2+6k^2k\cdot p+6p^2k\cdot p \tag{3.34}$$

and all terms are explicitly infrared finite because of sufficient powers of $k$ in the numerator. This completes our discussion of the plus prescription and its implementation.

We now turn to the ultraviolet region of our integrals, where we must implement the renormalization procedure outlined in Sec. III A. By transforming the integration variable in the region $p^2<k^2<\infty$ to $1/k^2$, we clearly obtain a finite integration region that is suitable for numerical evaluation, at least for those integrals which are ultraviolet finite. For those which diverge, however, we must make sure our renormalization procedure cancels these divergences *before* we perform our numerical integrations. The terms involving the enhanced infrared behavior for $\mathcal{G}$ are all ultraviolet finite by power counting, because of the extra power of $k^2$ in the denominator from this term. Thus we are left to deal with the terms where $\mathcal{G}\equiv\mathcal{G}_1$. We illustrate how this is done by considering the $A(k,p)$ and $B(k,p)$ terms of Eqs. (3.1) and (3.2). Some of these appear quadratically divergent by power counting, but this divergence will cancel between compensating terms leaving only a logarithmic divergence. Other terms appear linearly divergent, but here the angular integrations will make them only logarithmically divergent again. Since the term proportional to $A(k,p)$ is linear in $\mathcal{G}$, we can analytically perform the angular integrations to obtain

$$\int d^4k \frac{\mathcal{G}_1(k)A(k,p)}{k^4q^2p^4} = 2\pi^2\int_0^{p^2} dk^2\frac{\mathcal{G}_1(k)}{k^2p^4}\left[\frac{5k^6}{p^2}-\frac{37k^4}{4}+\frac{15k^2p^2}{4}\right] + 2\pi^2\int_{p^2}^{\kappa^2} dk^2\frac{\mathcal{G}_1(k)}{k^2p^4}\left[-\frac{p^4}{4}-\frac{p^6}{4k^2}\right] . \quad (3.35)$$

Here we can see the explicit cancellation of the quadratically divergent terms and the absence of linearly divergent pieces. This leaves a logarithmically divergent integral, where for the moment we have retained the ultraviolet cutoff $\kappa^2$. The renormalization procedure essentially consists of subtracting Eq. (3.35) evaluated at $p^2=\mu^2$, from itself [see Eq. (3.10)]. The logarithmically divergent term in the second of the integrals of Eq. (3.35) is independent of $p^2$, and so will cancel in this subtraction, leaving us with a finite integral. For the finite terms in Eq. (3.35), our renormalization consists of a finite subtraction. The integrals in Eq. (3.35) are the easiest case to consider, because we have been able to perform the angular integrations explicitly.

We now turn to the $B(k,p)$ terms in Eqs. (3.1) and (3.2). There are eight powers of $k$ in the denominator, so that with the four powers of $k$ from the integration measure $d^4k$, the only divergent terms are those proportional to $-13k^4-2k^2(k\cdot p)^2/p^2$. After our ultraviolet subtraction we have to calculate the integral

$$\int d^4k\, \mathcal{G}_1(k)(-13-2\cos^2\psi)\left[\frac{\mathcal{G}_1(q)}{q^4}-\frac{\mathcal{G}_1(q')}{q'^4}\right] , \quad (3.36)$$

where $q=k-p$ and we introduce $q'=k-\mu$, where $\mu$ is a Euclidean four-vector at the renormalization scale $p^2=\mu^2$. We can add and subtract a term proportional to $\mathcal{G}_1(q)/q'^4$ to give

$$2\pi\int \sin^2\psi\, d\psi\, k^2dk^2\mathcal{G}_1(k)(-13-2\cos^2\psi) \times\left[\frac{\mathcal{G}_1(q)}{q^4q'^4}(q'^4-q^4)+\frac{1}{q'^4}[\mathcal{G}_1(q)-\mathcal{G}_1(q')]\right] . \quad (3.37)$$

In the first term the highest power of $k^2$ cancels in the $q'^4-q^4$ term, leaving us with a finite integral. The second term is still divergent by power counting. We integrate this term up to some large momentum $R$, with $R^2\gg\mu^2,p^2$. Above this momentum we can expand

$$\begin{aligned}\mathcal{G}_1(q)&=\mathcal{G}_1(k)+(q^2-k^2)\mathcal{G}_1'(k)+\tfrac{1}{2}(q^2-k^2)^2\mathcal{G}_1''(k)+O(1/k^3)\\ &=\mathcal{G}_1(k)+(p^2-2k\cdot p)\mathcal{G}_1'(k)+2(k\cdot p)^2\mathcal{G}_1''(k)+O(1/k^3) . \end{aligned}\quad (3.38)$$

A similar expansion for $\mathcal{G}_1(q')$ allows us to write $\mathcal{G}_1(q)-\mathcal{G}_1(q')$ as

$$(p^2-\mu^2+2k\cdot\mu-2k\cdot p)\mathcal{G}_1'(k) +2[(k\cdot p)^2-(k\cdot\mu)^2]\mathcal{G}_1''(k)+O(1/k^3) . \quad (3.39)$$

Since we will choose an explicit parametrization for $\mathcal{G}_1$, its derivatives can be calculated, and the extra powers of $k^2$ in the denominator coming from these make the integral finite. These powers can be explicitly extracted to enable us to perform the numerical integration. For $R^2$ sufficiently large, the errors introduced by ignoring higher terms in the expansion are small. We usually choose $R^2=10^3\max(p^2,\mu^2)$. The details of how the $C(k,p)$ and $D(k,p)$ terms are similarly integrated can be found in Ref. 9. All can be rendered suitable for numerical integration to an accuracy of 0.1%.

### D. The solution

Since Eq. (3.10) cannot be solved analytically, we choose a parametrization for $\mathcal{G}_R(p)$ consistent with the expected infrared enhancement, and the asymptotic behavior of Eq. (3.22). We introduce

$$\mathcal{G}_\infty(p)=\left[1+\frac{\beta_0\alpha_s(\mu)}{4\pi}\ln\left[1+\frac{p^2}{\mu^2}\right]\right]^{-\gamma_0/\beta_0} \quad (3.40)$$

which is well behaved at all values of $p^2$, where $\gamma_0/\beta_0=\frac{28}{53}$, and parametrize $\mathcal{G}_R(p)$ by

$$\mathcal{G}_R(p)=\frac{A\mu^2}{p^2}+\mathcal{G}_\infty(p)\sum_{n=1}^{N}a_n\left[\frac{p^2}{p^2+p_0^2}\right]^{nb} . \quad (3.41)$$

Here the parameters $A,p_0,a_n,b$ are to be determined by

TABLE I. Parameters for $\mathcal{G}_R(p)$, Eq. (3.41), for the solutions shown in Fig. 3.

| $\Lambda$ (MeV) | 200 | 500 |
|---|---|---|
| $\alpha_s(\mu)$ | 0.172 | 0.257 |
| $A$ | 0.019 34 | 0.073 63 |
| $b$ | 0.3 | 0.3 |
| $q_0^2$ (GeV$^2$) | 0.3 | 0.3 |
| $a_1$ | 2.637 | 1.469 |
| $a_2$ | 0.009 379 | −0.090 48 |
| $a_3$ | −0.440 7 | 0.254 7 |
| $a_4$ | −1.180 | −0.627 4 |

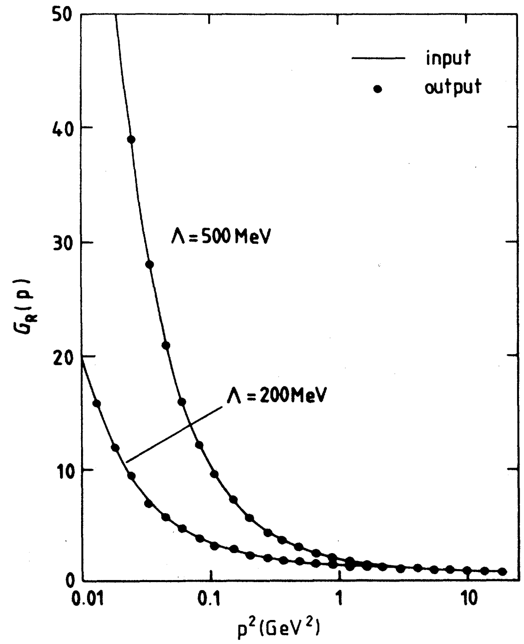


FIG. 3. Gluon renormalization function $\mathcal{G}_R(p)$ as a function of $p^2$ for $\Lambda = 200, 500$ MeV.

self-consistency of Eq. (3.10). We substitute this form into Eq. (3.10), treating the enhanced infrared term by our plus prescription, and performing the numerical integrations, the details of which are given in Sec. III C.

We then vary these parameters until good agreement is obtained over a range of values of $p^2$, between the right- and left-hand sides of Eq. (3.10). Because of the extreme length of time needed to perform these two-dimensional integrations numerically, we set "by hand" $p_0^2 = 0.3$ GeV$^2$ and $b = 0.3$, values for which we obtained approximate agreement. Since these are the only parameters on which the dependence of Eq. (3.41) is nonlinear, the integrations of Eq. (3.10) can be performed once, and the output stored. We then allow $A$ and the $a_n$ to vary within the CERN numerical minimization program MINUIT.

We choose our range of momenta to be $0.01 < p^2 < 100$ GeV$^2$, with the renormalization scale $\mu^2 = 10$ GeV$^2$, a scale we know from experiment to be in the perturbative regime. We set $\mathcal{G}_R(\mu) = 1$, and choose a value for $\alpha_s(\mu)$. With $N = 4$ in Eq. (3.41) we find this matching is achieved to impressive numerical accuracy, over the whole momentum range, from the deep infrared to the deep ultraviolet. Using the high-energy form of Eq. (3.20) for the running coupling, we detail the solutions obtained for $\Lambda = 200$ and 500 MeV, i.e., $\alpha_s(\mu) = 0.172$ and 0.257, respectively [see Eq. (3.21)]. The results are plotted in Fig. 3 and the parameters are detailed in Table I.

We find that the solution to this truncated Schwinger-Dyson equation for the gluon propagator does indeed possess a solution in which the function $\mathcal{G}_R(p) \sim 1/p^2$ as $p^2 \to 0$. Thus the propagator itself will behave like $1/p^4$ in this limit. Before we go on to discuss the importance of this result, and its relevance to confinement physics in Sec. IV, we first detail a far simpler approximation to this gluon equation, which remarkably possesses the same qualitative solutions as those we have obtained here using the full longitudinal part of the triple-gluon vertex.

## IV. THE MANDELSTAM APPROXIMATION

### A. Introduction

In this section we consider the Schwinger-Dyson equation for the inverse gluon propagator, but this time in a simpler approximation first proposed by Mandelstam.[4] We shall demonstrate, qualitatively at least, that the solutions using this approximation have the same features as those obtained in Sec. III, and discuss why this is so. Because of the simplicity of this so called "Mandelstam approximation" we have used it as a basis for the calculations where we study the interplay between quarks and gluons in Ref. 11. This section provides the justification for such a treatment.

In solving the Slavnov-Taylor identity for the triple-gluon vertex in terms of the gluon renormalization function $\mathcal{G}(p)$, Eq. (2.3), we see that the determined longitudinal part of the vertex, Eq. (2.4), always involves terms proportional to $1/\mathcal{G}$, with arguments $p,k,q$. The two-gluon propagators in the gluon loop of Fig. 2 give a contribution of $\mathcal{G}(k)\mathcal{G}(q)$, partially canceling some of these $1/\mathcal{G}$ terms. Mandelstam has suggested an approximation which assumes this cancellation to be complete, where we simply write the full triple-gluon vertex as $1/\mathcal{G}(q)$ times the bare vertex [cf. Eq. (2.5)].

Neglecting ghost contributions and the diagrams involving quartic-gluon couplings, as in Sec. III, allows the simplified gluon equation to be expressed diagrammatically by Fig. 4, giving us the following equation for the inverse gluon propagator:

$$\Pi_{ab}^{\mu\nu} = \Pi_{ab}^{\mu\nu} + \delta_{ab} \frac{C_A g_0^2}{32\pi^4} \int d^4k \, \Gamma_0^{\mu\alpha\delta}(-p,k,q)\Delta_{\alpha\beta}(k) \times \Delta_{\gamma\delta}(q) \frac{\Gamma_0^{\beta\nu\gamma}(-k,p,-q)}{\mathcal{G}(q)}, \tag{4.1}$$

where all the quantities are defined in Sec. III A, and again we work in the Landau gauge.

Projecting with $P^{\mu\nu}$ of Eq. (2.7) we obtain, on performing the angular integrations,

$$\frac{1}{\mathcal{G}(p)} = 1 + \frac{C_A g_0^2}{16\pi^2 p^2} \left[ \int_0^{p^2} dk^2 \mathcal{G}(k) \left[ \frac{7k^4}{6p^4} - \frac{17k^2}{6p^2} - \frac{3}{8} \right] + \int_{p^2}^{\kappa^2} dk^2 \mathcal{G}(k) \left[ -\frac{7p^2}{3k^2} + \frac{7p^4}{24k^4} \right] \right], \tag{4.2}$$

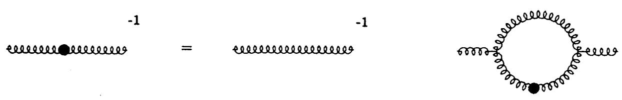


FIG. 4. The Mandelstam approximation to the Schwinger-Dyson equation for the inverse gluon propagator (cf. Fig. 1).

where we have anticipated that the $k$ integral is ultraviolet divergent by introducing a cutoff $\kappa^2$.

The beauty of this Mandelstam approximation is that it allows all the angular integrations to be performed analytically, leaving us with just the momentum integration to do. This is to be compared to the far more complicated situation in Sec. III. Thus Eq. (4.2) is a far easier equation to handle numerically, a feature which will be of essential importance when we consider the coupled gluon and quark system in Ref. 11.

Because of the presence of the ultraviolet cutoff, we have really only defined $\mathcal{G}(p,\kappa)$. As in Sec. III we define a renormalized gluon function by

$$Z_G(\kappa/\mu)\mathcal{G}_R(p)=\mathcal{G}(p,\kappa)\ . \tag{4.3}$$

To obtain an equation independent of the cutoff $\kappa^2$, we first evaluate Eq. (4.2) at $p^2=\mu^2$, where once again $\mu^2$ is some arbitrary momentum. We then subtract this from Eq. (4.2), and use the definition in Eq. (4.3) to obtain a renormalized equation:

$$\frac{1}{\mathcal{G}_R(p)} = \frac{1}{\mathcal{G}_R(\mu)}+\frac{C_A\alpha_1(\mu)}{4\pi}\int_0^{\kappa^2}dk^2[\mathcal{J}(k,p)-\mathcal{J}(k,\mu)]\mathcal{G}_R(k)\ , \tag{4.4}$$

where the kernel $\mathcal{J}(k,p)$ is simply read off from Eq. (4.2). Here the renormalized running coupling is given as

$$g^2(\mu)=Z_G(\kappa/\mu)^2 g_0^2 \tag{4.5}$$

and as usual $\alpha_1(\mu)=g^2(\mu)/4\pi$. The subscript 1 is to differentiate this coupling from that of Sec. III, Eq. (3.8). As in Sec. III we must also consider the gluon mass renormalization. A contribution violating in Eq. (3.3), viz., $p^2/\mathcal{G}(p)\to 0$ as $p^2\to 0$, will arise from terms where $\mathcal{G}(p)\sim A\mu^2/p^2$ on the right-hand side of Eq. (4.4). We can deal with this, as in Sec. III, by writing $\mathcal{G}(p)=A\mu^2/p^2+\mathcal{G}_1(p)$. The first term generates a gluon mass and must be subtracted. Since in the present approximation to the vertex the right-hand side of the gluon equation is linear in $\mathcal{G}(p)$, this subtraction means that only the $\mathcal{G}_1$ term appears under the integrals, greatly simplifying the evaluation as all integrals are now infrared finite.

### B. The solution

As in Sec. III, it has not proved possible to find an analytic solution to Eq. (4.4), and again we attempt a numerical solution. The analysis of the possible low-momentum behavior of $\mathcal{G}_R(p)$ is similar to the dimensional analysis of Sec. III B, so again we look for a solution with the enhanced infrared term, bearing in mind that the gluon mass renormalization above means that this term does not contribute to the integrations we must perform. As in Sec. III B we can also extract the asymptotic form of $\mathcal{G}_R(p)$ for large $p^2$. By expanding $\mathcal{G}_R=1+O(\alpha_1(\mu))$, we easily obtain, for the leading behavior,

$$\frac{1}{\mathcal{G}_R(p)}=\frac{1}{\mathcal{G}_R(\mu)}+\frac{\gamma_0'\alpha_1(\mu)}{4\pi}\ln\frac{p^2}{\mu^2}\ , \tag{4.6}$$

where $\gamma_0'=\frac{7}{3}C_A$. Using the relation of Eq. (3.15), we can recast our definition of the running coupling in the form

$$\alpha_1(p)=\alpha_1(\mu)\left[\frac{\mathcal{G}_R(p)}{\mathcal{G}_R(\mu)}\right]^2 . \tag{4.7}$$

Again expanding in powers of $\alpha_1(\mu)$, and using Eq. (4.6) we obtain

$$\frac{1}{\alpha_1(p)}=\frac{1}{\alpha_1(\mu)}+\frac{\beta_0'}{4\pi}\ln\frac{p^2}{\mu^2}\ , \tag{4.8}$$

where we easily see that $\beta_0'=2\gamma_0'=\frac{14}{3}C_A$ from the simple form of Eq. (4.4). The standard renormalization-group argument yields

$$\mathcal{G}_R(p)=\mathcal{G}_R(\mu)\left[\frac{\alpha_1(p)}{\alpha_1(\mu)}\right]^{\gamma_0'/\beta_0'} . \tag{4.9}$$

Since $\gamma_0'/\beta_0'=\frac{1}{2}$ [cf. $\frac{28}{53}$ in the approximation of Eq. (3.23)], we can use the usual one-loop asymptotics for the coupling arising from Eq. (4.8), to give us the asymptotic form for $\mathcal{G}_R(p)$ as

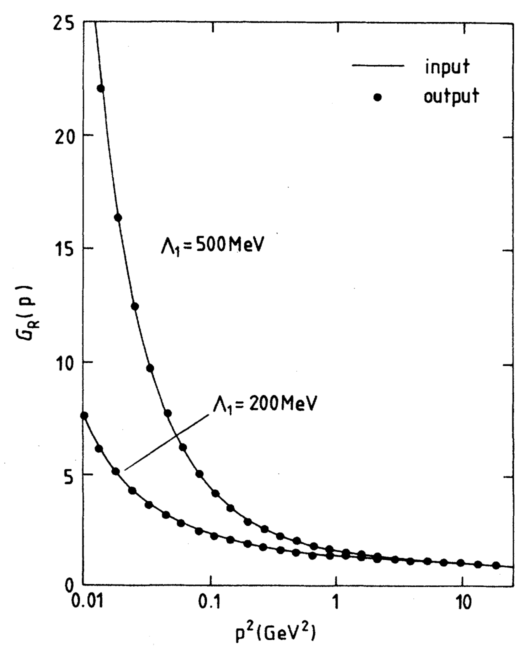


FIG. 5. Gluon renormalization function $\mathcal{G}_R(p)$ in the Mandelstam approximation, as a function of $p^2$ for $\Lambda_1=200,500$ MeV.

TABLE II. Parameters for $\mathcal{G}_R(p)$, Eq. (4.12), for the solutions shown in Fig. 5.

| $\Lambda_1$ (MeV) | 200 | 500 |
|---|---|---|
| $\alpha_s(\mu)$ | 0.163 | 0.243 |
| $A$ | 0.006 327 | 0.029 94 |
| $a_1$ | 0.932 0 | 0.968 8 |
| $p_0^2$ (GeV$^2$) | 0.044 95 | 0.198 7 |
| $b_1$ | 0.889 8 | 1.184 |
| $c_1$ | 0.133 5 | 0.114 6 |
| $q_0^2$ (GeV$^2$) | 0.028 95 | 0.139 8 |
| $d_1$ | 0.444 7 | 0.637 2 |

$$\mathcal{G}_R(p) \sim \left[\ln\frac{p^2}{\mu^2}\right]^{-1/2} . \tag{4.10}$$

Again we choose a parametrization for $\mathcal{G}_R(p)$ which reproduces this asymptotic behavior. We first introduce, as in Sec. III,

$$\mathcal{G}_\infty(p) = \left[1+\frac{\beta_0'\alpha_1(\mu)}{4\pi}\ln\left[1+\frac{p^2}{\mu^2}\right]\right]^{-\gamma_0'/\beta_0'} \tag{4.11}$$

allowing us to parametrize $\mathcal{G}_R(p)$ by

$$\mathcal{G}_R(p) = \frac{A\mu^2}{p^2} + \mathcal{G}_\infty(p)\left[\sum_{n=1}^{M} a_n\left[\frac{p^2}{p^2+p_n^2}\right]^{b_n} + \sum_{n=1}^{N} c_n\left[\frac{p^2\mu^2}{p^4+q_n^4}\right]^{d_n}\right] , \tag{4.12}$$

where the parameters $A$, $a_n$, $b_n$, $c_n$, $d_n$, $p_n$, and $q_n$ are permitted to vary. Since all the angular integrations have been performed, we can allow the dimensionful parameters $p_n, q_n$ to vary numerically. This contrasts with the situation for the full vertex, in Sec. III, where the corresponding dimensionful parameters were chosen by trial and error. This was because of the considerable computing time needed when the parameters are allowed to vary under the integral sign, where only one of the angular integrations can be performed numerically. Though we expect a nonzero value for the coefficient of the infrared-enhanced term $A$, again we do not prejudge this result, since we allow the possibility that this parameter can be zero.

As before, we set $\mathcal{G}_R(\mu)=1$ and choose $\mu^2=10$ GeV$^2$. We demand consistent numerical agreement between input and output for the range $0.01 < p^2 < 100$ GeV$^2$, with $\Lambda_1=200,500$ MeV (as in Sec. III) where with $\beta_0'=\frac{14}{3}C_A$, $\alpha(\mu)=0.163, 0.243$, respectively. The momentum scale of Eq. (4.4) can thus be expressed wholly in terms of the intrinsic parameter of the theory, viz., $\Lambda_1$. We find good agreement (to within 1%) with $N=M=1$ in Eq. (4.12). We plot our results in Fig. 5 and give the parameters in Table II.

Once again we obtain solutions which possess the infrared enhancement we saw in Sec. III. Note that the size of this enhanced term rises with $\Lambda_1$ exactly as we would expect if $\Lambda_1$ is the intrinsic momentum scale of the theory. We discuss this further in Sec. V, where we will also discuss the physical implications of this enhancement of the gluon propagator, and see how it is related to confinement. Before this we first discuss the relationship between the Mandelstam approximation and the full vertex approximation of Sec. III.

### C. How good is the Mandelstam approximation?

In this and the preceding section, we have used two different approximation schemes to solve the Schwinger-Dyson equation for the gluon propagator. In both cases we have obtained solutions for the gluon renormalization function $\mathcal{G}_R(p)$ which has an infrared-enhanced term $A\mu^2/p^2$, which we believe to be indicative of confinement. In Sec. III we solved the Slavnov-Taylor identity for the triple-gluon vertex in the absence of ghost contributions. This allowed us to write the longitudinal part of this vertex in terms of $\mathcal{G}$, where we drop the subscript $R$ for the discussion of this section. The solution for this longitudinal part is proportional to $1/\mathcal{G}$ with arguments $p$, $k$, and $q$. The propagators in the gluon loop of Fig. 2 contain a factor $\mathcal{G}(k)\mathcal{G}(q)$, and so we obtain three generic terms, proportional to

$$\frac{\mathcal{G}(k)\mathcal{G}(q)}{\mathcal{G}(p)}, \quad \mathcal{G}(k), \text{ and } \mathcal{G}(q) . \tag{4.13}$$

The last type can be made equal to the second by a change of variable in the loop integral. The first term we denoted by $\mathcal{L}(k,p)$, the second and third by $\mathcal{K}(k,p)$, see Eqs. (3.1) and (3.10). The Mandelstam approximation assumes that the cancellation which occurs for the second and third terms in Eq. (4.13), occurs for the entire vertex.

The gluon mass renormalization subtracts off those terms on the right-hand side of Eqs. (3.1) and (3.10), which do not vanish when we multiply by $p^2$ and let $p^2\to 0$. Such contributions only arise from the infrared-enhanced term, so we have written $\mathcal{G}(p)=A\mu^2/p^2+\mathcal{G}_1(p)$, and subtract from these equations the contribution where any $\mathcal{G}$ which appears in the numerator is put *equal* to its infrared-enhanced term only. This means that such singular terms do not contribute to the right-hand side of the Mandelstam approximation at all, and similarly not in the $\mathcal{K}(k,p)$ terms of the full vertex approximation. After this subtraction the only singular terms which contribute to the $\mathcal{L}(k,p)$ are those where one of the factors of $\mathcal{G}$ in the numerator is equal to $A\mu^2/k^2$, and the other is equal to $\mathcal{G}_1$: for example,

$$\frac{(A\mu^2/k^2)\mathcal{G}_1(q)}{\mathcal{G}(p)} . \tag{4.14}$$

In Sec. III C we defined the divergent contributions arising from such a term by means of a plus prescription and the technical details are described there. On dimensional grounds though, the integration must provide a dimensionful quantity to balance the factor of $\mu^2$ and since this can only be $p^2$, we have, after integrating such a quantity as Eq. (4.14),

$$\frac{A\mu^2/p^2}{\mathcal{G}(p)}W(p) , \tag{4.15}$$

where $W(p)$ is some function arising from the integration which is well behaved as $p^2 \to 0$. In this limit the factor in the denominator also behaves like $A\mu^2/p^2$ and cancels with the term in the numerator. Thus the cancellation which occurs completely in the Mandelstam equation, and partially for the $\mathcal{K}(k,p)$ term in the full vertex approximation *before* integration is seen, in a sense, to occur for the $\mathcal{L}(k,p)$ term *after* integration, at least in the limit $p^2 \to 0$. Since it is in this $p^2 \to 0$ limit in which using only the longitudinal part of the vertex, and neglecting the transverse part is valid, we might expect that the simpler Mandelstam equation will work. Of course, the arguments above concern only the qualitative analytic behavior, and there is no reason why the Mandelstam equation should not give quantitatively different results. By comparing Fig. 3 with Fig. 5 we see that this is indeed the case. However, as was mentioned in Ref. 3, although neglecting the transverse part of the triple-gluon vertex will not alter the existence of infrared enhanced solutions, it will affect the coefficient of such a term, and so we should not, therefore, worry about any further quantitative difference introduced by using the Mandelstam approximation. It is, nevertheless, amusing to note that representing the infrared enhancement in this approximation in terms of gluon condensates gives values for these in surprisingly good agreement with phenomenological expectations.[18]

Another difference between the two approximations lies in the different renormalizations and hence the different running couplings, Eqs. (3.16), (3.20), (4.7), and (4.8). The important fact though, is the dependence of the running coupling on $\mathcal{G}(p)$, and in both cases we have

$$\alpha(p) \sim \alpha(\mu) \left[ \frac{\mathcal{G}(p)}{\mathcal{G}(\mu)} \right]^n , \tag{4.16}$$

where $n=1$ for the full vertex approximation and $n=2$ for the Mandelstam approximation. Although different, both couplings become strongly enhanced at low $p^2$, because of the factor of $\mathcal{G}(p)$ in the numerator.[19] This is exactly what we would expect of the coupling in a confining theory, and in both cases this enhancement is by a power law.

In summary, the cancellation of factors of $\mathcal{G}$ used in the Mandelstam equation is seen to arise in the full vertex approximation, partially before integration, and partially after. The much simpler Mandelstam equation gives rise to qualitatively similar solutions, both for the gluon renormalization function $\mathcal{G}(p)$ and for the running coupling. Since the Mandelstam equation is far simpler to use, both analytically, and more importantly in this case, numerically, it is this approximation that we used in our extensive study of fermions in a non-Abelian gauge theory in Ref. 11. There the solution of the gluon equation in the Mandelstam approximation, Eqs. (4.2) and (4.4), is given for $\alpha_1(\mu)$ from 0.15 to 0.30 in 0.05 steps.

## V. CONFINEMENT AND A $1/p^4$ GLUON PROPAGATOR

### A. Introduction

In the studies of the Schwinger-Dyson equation for the gluon propagator reported here and in Ref. 8, we have demonstrated solutions for the gluon renormalization function $\mathcal{G}(p)$ which are as singular as $1/p^2$ as $p^2 \to 0$. This in turn means that the gluon propagator is as singular as $1/p^4$. This same behavior has been found by others[3-5,10,13] in various approximations and truncations of these equations. It seems likely then that this does reflect the true behavior of the gluon. Though the gluon propagator is neither an experimental observable nor even a gauge-invariant quantity, there is, nevertheless, good reason to suggest that this enhancement is physically meaningful and reflects the confinement mechanism. Here we shall briefly mention three ways in which to translate this behavior into statements about confinement.

In QED we can relate the long-range force between two static electric charges to the photon propagator. By virtue of gauge invariance this guarantees the Coulomb $1/r^2$ force law. The connection is based on considering a "one-photon-exchange" approximation, which for an Abelian theory correctly gives the large distance limit. For QCD, because of the self-coupling of the gluon, this approximation is no longer valid,[20] and we have to consider "multigluon" exchanges between static color charges as well. Nevertheless, it is well known that the Fourier transform of the time-time component of the propagator is related to the one-gluon-exchange contribution to this potential and a $1/p^4$ infrared behavior does generate a linearly confining potential at large distances. Of course, one-gluon exchange produces a vector contri-

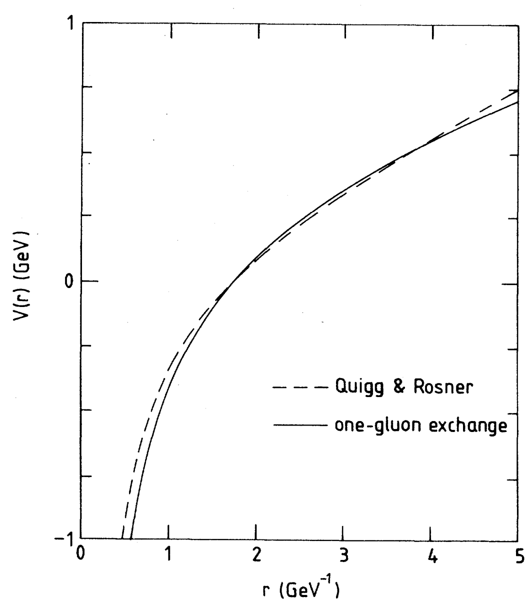


FIG. 6. One-gluon-exchange contribution to the static interquark potential $V(r)$ as a function of $r$ from our solution to the gluon propagator using the Mandelstam approximation for $\alpha(\mu)=0.25$. The renormalization constant is chosen (somewhat arbitrarily) to maximize agreement with the phenomenological potential of Quigg and Rosner (Ref. 21) in the region determined by $c\bar{c}$ and $b\bar{b}$ spectra. Their potential is shown for comparison.

bution to the potential. The multigluon exchanges we neglect will generate a scalar component and this is expected to play an important role.[20]

This one-gluon-exchange potential in the Mandelstam approximation with $\alpha_s(\mu)=0.25$ is shown in Fig. 6. For comparison we also plot the phenomenological potential of Quigg and Rosner,[21] which has been successful in describing the properties of heavy $q\bar{q}$ states.

### B. Wilson loop operator

As is well known, a gauge-invariant translation of the behavior of the gluon field is provided by the Wilson loop operator.[22] In a confining theory, this is expected to decay exponentially as the area enclosed by the loop increases.

West has proved an important result,[23] showing that if in *any* gauge the gluon propagator is as singular as $1/p^4$, then the Wilson loop operator does indeed satisfy an area law. This is a result of central importance to studies of the gluon propagator, as it allows us to relate the infrared behavior of the gauge-noninvariant propagator to a gauge-invariant signal for confinement. Note that although we have indeed found a $1/p^4$ behavior for the gluon propagator, we cannot really use this result of West's, as we have only studied the gluon in a truncation of the full theory. Nevertheless, our result is suggestive.

A comment is needed here on the results in axial gauges.[3] West has also shown[24] that the full axial gauge gluon propagator cannot be as singular as $1/p^4$ in the infrared limit. This result follows from a consideration of positivity of spectral functions in this physical gauge. This does not contradict the results in axial gauges obtained by Baker, Ball, and Zachariasen[3] since there the full propagator depends on two scalar functions and only one of these has been studied. Presumably the other is also singular, with the behavior canceling between the two terms. This does, however, call into question the approximation of only studying one of these functions, and further motivated our analysis in covariant gauges, where no such restrictions on the propagator apply. The conclusion of this "no-go" theorem of West's is that it will be impossible to prove confinement via the Wilson loop in axial gauges, which as we have stated, is the crucial link between our gauge-noninvariant propagator and gauge-invariant physics.

### C. The QCD vacuum and dual superconductivity

Here we briefly mention a suggestive analogy between a possible interpretation of our results and electromagnetic superconductivity. It is well known that it is possible to describe superconductivity by a mechanism whereby the photon acquires a mass $m_L$, and so only propagates over distances comparable to $1/m_L$, the so-called London length $\lambda_L$. This gives us the Meissner effect. This can also be thought of as the photon propagating in a medium with electrical permittivity $\epsilon(k^2)=k^2/m_L^2$ at low momenta. We can reexpress this in terms of "dual" potentials, these being related to the ordinary potentials by an electromagnetic duality transformation. Thus we can think of a "dual" photon having inverse propagator $\Pi_0^{\alpha\beta}\mu(k^2)$, where $\mu(k^2)$ is the magnetic permeability, where, as usual, we have $\epsilon(k^2)\mu(k^2)=1$, and $\Pi_0^{\alpha\beta}$ is the free inverse propagator. We can then deduce that this "dual" photon is as singular as $1/k^4$ as $k^2\to 0$.

The extension of such a formalism to non-Abelian theories is far from trivial,[25,26] yet a heuristic interpretation of our results would be that since the gluon is as singular as $1/k^4$ as $k^2\to 0$, the "dual" gluon will acquire a mass. We can, therefore, think of the QCD vacuum as an infinite "dual" color-electric superconductor.

This interpretation is given further impetus, when we consider the interplay between an ordinary electromagnetic superconductor and magnetic monopoles. Since the Meissner effect excludes magnetic fields from inside the superconducting medium, it would take infinite energy to place an isolated magnetic monopole inside the superconductor. With more than one monopole, however, we would have the possibility of forming monopole-antimonopole bound states, where the two particles would have a separation of $\lambda_L$ or less. In a type-I superconductor the magnetic flux would essentially be confined within a small volume, whereas for a type-II superconductor, the magnetic flux would be squeezed into a small tube, with the monopole and the antimonopole at the ends.

In a "dual" superconductor, it would not be magnetic charge which is so "confined," but electric charge. Thus we can formulate an intuitive picture of confinement, whereby color-*electric* monopoles, such as quarks, cannot exist singly, but must form bound states. Much work has been done on a "dual" formulation of QCD (Ref. 27). A possible signal for this simple model of confinement, might well be the $1/p^4$ behavior of the gluon propagator which we have found in our study. Moreover, the scale, the analogue of $m_L^2$, would be the coefficient of the enhanced term, viz., $A\mu^2$, in Eqs. (3.41) and (4.12). As shown in Ref. 11 this is roughly proportional to the $\Lambda^2$, the scale of QCD, and it is this, of course, that naturally sets the scale of the nonperturbative phenomena in strong interactions.

## VI. SUMMARY AND CONCLUSIONS

We have studied the Schwinger-Dyson equation for the gluon propagator and found solutions which indicate that the full gluon propagator is as singular as $1/p^4$ as $p^2\to 0$. The scale of this enhancement grows with increasing $\Lambda$, where $\Lambda$ is the intrinsic momentum scale of QCD. Indeed, in Ref. 11, we have shown that this enhancement is very nearly proportional to $\Lambda^2$, as we might expect on dimensional grounds, with only logarithmic deviations. Asymptotic freedom is a very important complementary aspect of QCD. The gluon renormalization functions illustrated in Figs. 3 and 5 have perturbative behavior for large momenta when $p^2 \gg \Lambda^2$. Our results have shown that this continues down to surprisingly low momenta. It is not until $p=O(\Lambda)$ that the nonperturbative confining effects enter and then very strongly.

These results have been deduced in two different approximations. In the first of these, studied in Sec. III, we used an *Ansatz* for the full triple-gluon vertex based on solving the Slavnov-Taylor identity. A partial cancellation between some of the terms in this vertex, and the propagators in the loop integral was observed. The simpler Mandelstam approximation, of Sec. IV, assumes this cancellation to be complete, yet gave similar quantitative solutions for the propagator. We discussed why this was so, despite the need to deal with infrared singularities which occurred in the first approximation, which we handled by means of a "plus" prescription. This comparison justifies the use of the Mandelstam approximation in the more complicated set of equations which arises when we include dynamical quarks.[11,12,28]

In Sec. V we discussed some of the implications this behavior may have for confinement, and saw that an infrared behavior of $1/p^4$ for the gluon propagator may well be an important signal that QCD does contain the dynamics to confine the colored quarks and gluons into the colorless hadrons we observe in experiment. This signal can be related[23] to the behavior of the Wilson loop operator, allowing us to extract gauge-invariant consequences of our study. We have also seen how such an infrared enhanced gluon propagator gives rise to a physically appealing model for confinement in terms of dual superconductivity.

Perhaps more importantly, this study lays the foundations for a nonperturbative study of QCD, our candidate theory for strong interactions, based on the Schwinger-Dyson equations. In QCD it is precisely the low-momentum behavior which cannot be determined by perturbation theory and it is in this region that the various approximations made here are most likely to be valid. While many problems remain, we believe that the physically realistic results obtained in this and other studies demonstrate the viability of the Schwinger-Dyson equations as a means of studying the full nonperturbative dynamics of a continuum non-Abelian gauge theory at all distance scales.

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[1]R. L. Stuller, Phys. Rev. D **13**, 513 (1976); H. Pagels, *ibid.* **14**, 2747 (1976); **15**, 2991 (1977).

[2]J. S. Ball and F. Zachariasen, Nucl. Phys. **B143**, 148 (1978); C. Nash and R. L. Stuller, Proc. R. Ir. Acad. **78**, 217 (1978); R. Anishetty, M. Baker, S. K. Kim, J. S. Ball, and F. Zachariasen, Phys. Lett. **86B**, 52 (1979); M. Baker, J. S. Ball, P. Lucht, and F. Zachariasen, *ibid.* **89B**, 211 (1980).

[3]M. Baker, J. S. Ball, and F. Zachariasen, Nucl. Phys. **B186**, 531 (1981); **B186**, 560 (1981).

[4]S. Mandelstam, Phys. Rev. D **20**, 3223 (1979).

[5]U. Bar-Gadda, Nucl. Phys. **B163**, 312 (1980).

[6]A. I. Alekseev, Yad. Fiz. **33**, 516 (1981) [Sov. J. Nucl. Phys. **33**, 269 (1981)]; W. J. Schoenmaker, Nucl. Phys. **B194**, 535 (1982); B. A. Arbuzov, Phys. Lett. **125B**, 497 (1983); E. J. Gardner, J. Phys. G **9**, 139 (1983); D. Atkinson, P. W. Johnson, W. J. Schoenmaker, and H. A. Slim, Nuovo Cimento A **77**, 197 (1983); J. L. Rodriquez Marrero and A. R. Swift, Phys. Rev. D **29**, 1823 (1984); **31**, 917 (1985); **32**, 476 (1985).

[7]J. S. Ball and F. Zachariasen, Phys. Lett. **106B**, 133 (1981).

[8]N. Brown and M. R. Pennington, Phys. Lett. B **202**, 257 (1988).

[9]N. Brown, Ph.D. thesis, University of Durham, 1988.

[10]A. D. Worrall, Ph.D. thesis, University of Durham, 1985; M. R. Pennington and A. D. Worrall (unpublished).

[11]N. Brown and M. R. Pennington, Phys. Rev. D **38**, 2266 (1988).

[12]M. R. Pennington, Phys. Rev. Lett. **60**, 267 (1988).

[13]D. Atkinson, J. K. Drohm, P. W. Johnson, and K. Stam, J. Math. Phys. **22**, 2704 (1981); D. Atkinson, P. W. Johnson, and K. Stam, *ibid.* **23**, 1917 (1982); D. Atkinson *et al.*, *ibid.* **25**, 2095 (1984); D. Atkinson and M. Koopmans, *ibid.* **28**, 717 (1987); A. Hädicke, Friedrich Schiller University Jena Report No. N/88/25, 1988 (unpublished).

[14]Of course, in axial gauges where ghosts do not enter, further assumptions are needed on how to treat the attendant $n \cdot p$ singularities, so nothing comes for free.

[15]W. Marciano and H. Pagels, Phys. Rep. **36C**, 137 (1983).

[16]S. K. Kim and M. Baker, Nucl. Phys. **B164**, 152 (1980).

[17]J. S. Ball and T. W. Chiu, Phys. Rev. D **22**, 2550 (1980).

[18]E. Bagan and M. R. Pennington, Phys. Lett. B (to be published).

[19]In Ref. 11 we discussed such couplings and those introduced by the inclusion of quarks.

[20]T. Appelquist, M. Dine, and I. J. Muzinich, Phys. Lett. **69B**, 231 (1977); F. Feinberg, Phys. Rev. Lett. **39**, 316 (1977); W. Fischler, Nucl. Phys. **B129**, 157 (1977).

[21]C. Quigg and J. L. Rosner, Phys. Rep. **56**, 167 (1980).

[22]K. G. Wilson, Phys. Rev. D **10**, 2445 (1974).

[23]G. B. West, Phys. Lett. **115B**, 468 (1982).

[24]G. B. West, Phys. Rev. D **27**, 1878 (1983).

[25]M. Creutz, Phys. Rev. D **10**, 2696 (1974); V. P. Nair and C. Rosenzweig, Phys. Lett. **135B**, 450 (1984).

[26]S. Mandelstam, Phys. Rev. D **19**, 2391 (1979).

[27]M. Baker, J. S. Ball, and F. Zachariasen, Phys. Rev. D **37**, 1036 (1988).

[28]The companion paper (Ref. 11) contains a number of misprints, which we correct here. In Eq. (3.3) $k'=p-k$. In Eq. (3.9) $I_2(k,p)=(k'^2+p^2)[k^2p^2-(k\cdot p)^2]$. In Eq. (3.10) $C_2(F)$ is $C_F$. In Eq. (3.11) in the form for $L$ the $(k'^2+p^2)$ in the denominator should be $(k'^2-p^2)$. In Eq. (3.14) the final term in the denominator should be $F_r(k'',\lambda)$ (where $k''=k-\mu$). In Eq. (4.1) the $k^2$, $k'^2$ in the denominator should be deleted—these factors are implicitly in $S_F(k)$, $S_F(k')$.