PHYSICAL REVIEW D VOLUME 39, NUMBER 8 15 APRIL 1989

# Nonunique solution to the Schwinger-Dyson equations

Carl M. Bender
*Department of Physics, Washington University, St. Louis, Missouri 63130*

Fred Cooper and L. M. Simmons, Jr.
*Theoretical Division, Los Alamos National Laboratory, Los Alamos, New Mexico 87545*
(Received 14 October 1988)

In principle, a path-integral representation for a quantum field theory uniquely determines all of the Green's functions of the theory. One possible way to calculate the Green's functions is to derive from the path-integral representation an infinite set of coupled partial differential equations for the Green's functions known as the Schwinger-Dyson equations. One might think that all nonperturbative information about the Green's functions is contained in the Schwinger-Dyson equations. However, we show that while the Schwinger-Dyson equations do determine the weak-coupling perturbation expansions of the Green's functions, the solution to the Schwinger-Dyson equations is not unique and therefore the nonperturbative content of the Green's functions remains undetermined. In particular, one cannot use the Schwinger-Dyson equations to compute high-temperature or strong-coupling expansions.

## I. INTRODUCTION

Recently there has been renewed interest in the Schwinger-Dyson equations for quantum field theory as a vehicle to regulate all field theories uniformly using stochastic quantization.[1] A new regulation scheme proposed by Bern *et al.*[1] involves smearing the stochastic noise term that drives the Langevin equation so that the source is no longer local. At large fictitious times the averaging over the noise is equivalent to averaging over the usual Euclidean action in a path integral. One can derive from the regulated Langevin equations a set of *regulated* Schwinger-Dyson equations where the regulation consists of smearing an external current which is coupled to the field theory in the usual fashion. Bern *et al.* show that this regulation scheme regulates all theories including gauge theories and gravity to all orders in weak-coupling perturbation theory. They also argue that this scheme does not lead to a regulated path integral and thus although all the Green's functions are regulated, the ground-state energy density is not regulated (is still infinite).

Several years ago we attempted to devise a continuum regulation scheme for quantum field theory in the strong-coupling approximation.[2] These attempts were only partially successful because the only continuum regulator that we were able to find that preserved Ward identities obeyed by the ground-state energy density involved step functions in momentum space and these were very difficult to use to obtain analytic results at high order. The new regulation scheme of Bern *et al.* is not associated with a path integral and does not regulate the ground-state energy density. One might hope, therefore, that regulated Schwinger-Dyson equations would be a good context in which to derive nonperturbative strong-coupling expansions. However, as our study here will show, deriving strong-coupling expansions starting from a path integral is easy, but the Schwinger-Dyson equations do not uniquely determine the strong-coupling series. Extra boundary conditions must be imposed to supplement the Schwinger-Dyson equations in the strong-coupling regime. It is impractical (if not impossible) to impose these conditions.

To present our arguments we consider in this paper the extremely simple case of a $\lambda\phi^4$ field theory in zero-dimensional space-time. Such a field theory is called ultralocal because there is no kinetic energy term in the field equations. (The ultralocal approximation is the starting point for deriving strong-coupling or high-temperature expansions.)

We will show in Sec. II that merely assuming that a weak-coupling expansion *exists* is sufficient for the Schwinger-Dyson equations to determine this expansion uniquely without our having to impose any supplementary conditions. However, if we wish to determine the strong-coupling expansion from the Schwinger-Dyson equations it is necessary to have additional information such as the value of the two-point Green's function. To leading order in the strong-coupling expansion we can obtain this information from boundary conditions on the generating functional $Z(J)$. However, if we wish to obtain higher orders in the kinetic-energy expansion it becomes extremely difficult to continue the calculation.

Having examined weak-coupling expansions of the Schwinger-Dyson equations (which are unique) and strong-coupling expansions (which are not unique) we examine in Sec. III the possibility of solving the Schwinger-Dyson equations by truncation. When these equations are truncated by simply ignoring the coupling to all Green's functions having more than $N$ legs, the resulting *finite* set of equations has a *unique* solution. We study the behavior of the solution to the truncated Schwinger-Dyson equations in the limit as $N\rightarrow\infty$ for both signs of the mass. We find the surprising result that as $N\rightarrow\infty$ the truncated solution approaches that solution to the Schwinger-Dyson equations having a weak-coupling expansion for that sign of the mass. This shows that truncation schemes are tied to weak-coupling expansions. Moreover, for $m^2=0$, the purely strong-coupling case, the truncated Schwinger-Dyson equations do not converge at all.

Finally, in Sec. IV we summarize briefly the work of Bern *et al.* on stochastic regulation and explain why regulated Schwinger-Dyson equations cannot be used in isolation to generate regulated strong-coupling expansions.

## II. WEAK-COUPLING AND STRONG-COUPLING EXPANSIONS OF THE PATH INTEGRAL

A quantum field theory is defined by giving its vacuum functional $Z(J)$ as a path integral in Euclidean space. For a $\lambda\phi^4$ field theory $Z(J)$ is given by

$$Z(J)=\int D\phi \exp\left\{-\int d^nx[\tfrac{1}{2}(\partial\phi)^2+\tfrac{1}{2}m^2\phi^2+\tfrac{1}{4}\lambda\phi^4-J\phi]\right\}. \quad (2.1)$$

Note that $Z(J)$ satisfies the functional differential equation

$$[(-\Box+m^2)\delta/\delta J(x)+\lambda\delta^3/\delta J(x)^3]Z(J)=J(x)Z(J). \quad (2.2)$$

To verify (2.2) we substitute (2.1) into (2.2) and use the identity

$$\int D\phi(\delta/\delta\phi)\exp\left\{-\int d^nx[\tfrac{1}{2}(\partial\phi)^2+\tfrac{1}{2}m^2\phi^2+\tfrac{1}{4}\lambda\phi^4-J\phi]\right\}=0 . \quad (2.3)$$

We can also obtain (2.2) directly from the field equation in the Heisenberg picture,

$$(-\Box+m^2)\phi(x)+\lambda\phi^3(x)=J(x) , \quad (2.4)$$

by taking its vacuum expectation value.

The path-integral representation for $Z(J)$ has both a strong- and a weak-coupling expansion. To obtain the weak-coupling expansion we rewrite $Z(J)$ as

$$\begin{aligned}Z(J)&=\exp\left[-\lambda\int d^nx\,\delta^4/\delta J^4(x)\right]\int D\phi\exp\left\{-\int d^nx[\tfrac{1}{2}(\partial\phi)^2+\tfrac{1}{2}m^2\phi^2-J\phi]\right\}\\ &\doteq\exp\left[-\lambda\int d^nx\,\delta^4/\delta J^4(x)\right]\exp\left[\tfrac{1}{2}\int d^nx\,d^ny\,J(x)G(x-y)J(y)\right] ,\end{aligned} \quad (2.5)$$

where $G^{-1}(x-y)=(-\Box+m^2)\delta(x-y)$. Expanding the first exponential in (2.5) in powers of $\lambda$ generates the weak-coupling perturbation series.

To obtain the strong-coupling perturbation series one writes instead

$$Z(J)=\exp\left\{\tfrac{1}{2}\epsilon\int d^nx\,d^ny[\delta/\delta J(x)]G_0^{-1}(x-y)[\delta/\delta J(y)]\right\}Z_0(J) , \quad (2.6)$$

where $G_0^{-1}(x-y)=-\Box\delta(x-y)$ and

$$Z_0(J)=\int D\phi\exp\left[-\int d^nx(\tfrac{1}{2}m^2\phi^2+\tfrac{1}{4}\lambda\phi^4-J\phi)\right] . \quad (2.7)$$

The parameter $\epsilon$ counts powers of the kinetic energy $p^2$ and this is equivalent to counting powers of $\lambda^{-1/2}$ if we further expand $Z_0(J)$ in a strong-coupling expansion. $Z_0(J)$ satisfies the functional differential equation

$$[m^2\delta/\delta J(x)+\lambda\delta^3/\delta J(x)^3]Z_0(J)=J(x)Z_0(J) . \quad (2.8)$$

A formal expression for $Z_0$ can be obtained by evaluating the path integral on the lattice and then taking the continuum limit. One discretizes $x$ by setting

$$\mathbf{x}=(n_x\mathbf{i}+n_y\mathbf{j}+n_z\mathbf{k})a ,$$

where $a$ is the lattice spacing. Symbolically we replace $J(x)$ by $j_i$ to denote this discretization. The path integral $\int D\phi$ becomes the product of ordinary integrals: $N\prod\int dx_i$. Normalizing $Z_0(J)$ so that $Z_0(0)=1$ we have

$$\begin{aligned}Z_0(J)&=\prod_i F(j_i)/F(0)\\ &=\exp\left[\sum_i \ln F(j_i)/F(0)\right]\\ &=\exp\left\{\delta(0)\int d^nx\,\ln[F(J(x))/F(0)]\right\} ,\end{aligned} \quad (2.9)$$

where

$$F(Y)=\int_{-\infty}^{\infty}dx\,\exp[-a^n(\tfrac{1}{2}m^2x^2+\tfrac{1}{4}\lambda x^4-xy)] . \quad (2.10)$$

One can verify that the formal expression for $Z_0$ in (2.9) exactly satisfies the functional differential equation

$$[m^2\delta/\delta J(x)+\lambda\delta^3/\delta J(x)^3]Z_0(J)=J(x)Z_0(J) \quad (2.11)$$

because $F(y)$ obeys the ordinary differential equation

$$m^2a^{-n}dF/Dy+\lambda a^{-3n}d^3F/dy^3=yF . \quad (2.12)$$

In this paper we are concerned with the weak-coupling and strong-coupling expansions of the integral for $F(y)$ in

(2.10) and to what extent these expansions can be recovered from the differential equation (2.12). Equation (2.12) is the functional equation which generates and is equivalent to the Schwinger-Dyson equations in zero-dimensional space-time.

In zero dimensions ($n=0$) we will call $H(y)$ the normalized generating functional

$$H(y)\equiv\frac{F(y)}{F(0)} =\frac{1}{F(0)}\int_{-\infty}^{\infty}dx\,\exp[-(\tfrac{1}{4}\lambda x^4+\tfrac{1}{2}m^2x^2-xy)]\ . \tag{2.13}$$

$H(y)$ obeys the differential equation

$$\lambda H'''+M^2H'=yH\ . \tag{2.14}$$

The differentiations are with respect to a constant external source $y$. The differential equation for $H$ is third order and requires three boundary conditions.

Clearly, one boundary condition is that $H(0)=1$ in order that probability be conserved for the vacuum persistence functional in the absence of sources. A second condition is that $H'(0)=0$ so that parity is conserved. Possible choices for the third boundary condition are to achieve the following. (i) Specify $H''(0)$. This amounts to specifying the two-point function. (ii) Specify $H(y)$ for large $y$. This involves specifying how the vacuum persists in the presence of strong external sources. (iii) Assume that $H$ has a weak-coupling expansion. The problem of how to specify the third boundary condition makes it difficult to obtain a strong-coupling expansion for $H$ from the differential equation.

From (2.14) we can see that there are three possible types of asymptotic behavior of $H(y)$ for large argument:

$$H(y)\sim\exp(\tfrac{3}{4}y^{4/3}\omega\lambda^{-1/3})\ , \tag{2.15}$$

where $\omega$ is one of the three roots of $\omega^3=1$:

$$\omega_1=1,\quad \omega_2=\exp(i2\pi/3),\quad \omega_3=\exp(i4\pi/3)\ . \tag{2.16}$$

There are three possible integral representations for the solution $H(y)$ that correspond to these three possible types of asymptotic behavior and these representations have quite different properties.

The usual Euclidean path-integral representation is

$$H_1(y)=N_1\int_{-\infty}^{\infty}dx\,\exp[-(\tfrac{1}{2}m^2x^2+\tfrac{1}{4}\lambda x^4-xy)]\ . \tag{2.17}$$

$H_1(y)$ has the asymptotic behavior in (2.15) corresponding to $\omega_1$ in (2.16). $H_1(y)$ has a weak-coupling expansion for $m^2>0$. Using $H_1$ one has the correspondence

$$x\Longleftrightarrow d/dy\ .$$

Thus

$$\langle x^2\rangle=H_1''(y=0)>0\ . \tag{2.18}$$

A second and third solution are linear combinations of

$$H_{2,3}(y)=N_{2,3}\int_{C_{2,3}}dx\,\exp(\tfrac{1}{2}m^2x^2-\tfrac{1}{4}\lambda x^4-ixy)\ , \tag{2.19}$$

where $C_{2,3}$ are complex contours. We will focus on $H_2$ here for which $C_2$ is the real axis. For $H_2$ one has the correspondence

$$x\Longleftrightarrow -i\,d/dy\ .$$

Thus,

$$\langle x^2\rangle=-H_2''(y=0)>0\ . \tag{2.20}$$

$H_2$ leads to different moments of $x$ than $H_1$ for the same value of the parameters $m$ and $\lambda$. $H_2$ has a weak-coupling expansion when $m^2<0$.

If one has a particular integral representation of the solution one can as easily make weak- or strong-coupling expansions of the integral. Consider $H_1(y)$. The Green's functions of the theory with no kinetic energy are obtained by expanding $H$ as a Taylor series in $y$:

$$H_1(y)=N_1\int_{-\infty}^{\infty}dx\,\exp(-\tfrac{1}{2}m^2x^2-\tfrac{1}{4}\lambda x^4+xy) =\sum G_{2n}y^{2n}/(2n)!\ , \tag{2.21}$$

where $G_{2n}$, the $2n$-point Green's function, is

$$G_{2n}=N_1\int_{-\infty}^{\infty}dx\,x^{2n}\exp(-\tfrac{1}{2}m^2x^2-\tfrac{1}{4}\lambda x^4),\ G_0=1\ , \tag{2.22}$$

where

$$N_1^{-1}=\int_{-\infty}^{\infty}dx\,\exp(-\tfrac{1}{2}m^2x^2-\tfrac{1}{4}\lambda x^4)\ .$$

The weak-coupling expansion of the $G_{2n}$ is obtained by using the Gaussian measure

$$G_{2n}=N_1\int_{-\infty}^{\infty}dx\,x^{2n}\exp(-\tfrac{1}{2}m^2x^2)\sum_{p=0}^{\infty}(-\tfrac{1}{4}\lambda x^4)^p/p!\sim N_1(2/m^2)^{n+1/2}\sum_{p=0}^{\infty}(-\lambda/m^4)^p\Gamma(2p+n+\tfrac{1}{2})/p!\ , \tag{2.23}$$

where $N_1$ must also be expanded as series in powers of $\lambda$.

The strong-coupling expansion, on the other hand, is obtained by using the quartic term for the measure and expanding the Gaussian term:

$$G_{2n}=N_1\int_{-\infty}^{\infty}dx\,x^{2n}\exp(-\tfrac{1}{4}\lambda^4x^4)\sum(-\tfrac{1}{2}m^2x^2)^p/p!=N_1\tfrac{1}{2}(4/\lambda)^{n+1/2}\sum(-m^2/\lambda^{1/2})^p\,\Gamma\left[\frac{p}{2}+\frac{k}{2}+\frac{1}{4}\right]\Big/p!\ , \tag{2.24}$$

where here $N_1$ must also be expanded as a series in powers of $\lambda^{-1/2}$.

Thus, the $2n$-point Green's functions for this theory (which contain disconnected as well as connected parts),

$$\frac{\langle x^{2k}\rangle}{\langle 1\rangle}=G_{2n} ,$$

can be expanded as a power series in $\lambda/m^4$ or as a power series in $m^2/\lambda^{1/2}$. The first series (weak coupling) is an asymptotic series, whereas the second series (strong coupling) is a convergent series.

Now let us consider how to determine these same series directly from the Schwinger-Dyson equations. The Schwinger-Dyson equations are derived from (2.14). They are an infinite set of coupled equations for the Green's functions $G_{2n}$:

$$\lambda G_n+m^2G_{n-2}=(n-3)G_{n-4},\quad G_0=1,\quad G_1=0 , \tag{2.25}$$

where

$$G_n\equiv\left[\frac{d}{dy}\right]^n H(y)\Big|_{y=0} .$$

If we assume that there *exists* a weak-coupling expansion (formal power series in $\lambda$) for $G_n$ then one quickly obtains a unique answer for this series.[3] In the differential equation (2.14) we set $\lambda=0$:

$$m^2H'(y)=yH(y) . \tag{2.26}$$

From the boundary condition $H(0)=1$. We immediately obtain, at $\lambda=0$,

$$H(y)=\exp\left[\frac{y^2}{2m^2}\right] . \tag{2.27}$$

Note that this function already satisfies the boundary condition $H'(0)=0$.

Equivalently we derive this result using the Schwinger-Dyson equations. Setting $\lambda=0$ in (2.24), we obtain using $H(0)=G_0=1, G_1=0$,

$$G_{2n}=\frac{(2n)!2^{-n}m^{-2n}}{n!},\quad G_{2n+1}=0 . \tag{2.28}$$

Thus, we recover the result

$$H(y)=\sum_{n=0}^{\infty} y^{2n}\frac{G_{2n}}{(2n)!}=\exp\left[\frac{y^2}{2m^2}\right] .$$

Having computed $H(y)$ at $\lambda=0$ we can iterate the Schwinger-Dyson equations to obtain a unique result to all orders in powers of $\lambda$. Formally, we can write the result of this iteration as

$$H(y)=\exp[-\lambda(d/dy)^4]\exp\left[\frac{y^2}{2m^2}\right] . \tag{2.29}$$

Then writing

$$\exp\left[\frac{y^2}{2m^2}\right]=N\int_{-\infty}^{\infty}dx\,\exp(-\tfrac{1}{2}m^2x^2+yx) \tag{2.30}$$

we recover the unique result

$$H_1(y)=N_1\int_{-\infty}^{\infty}dx\,\exp(-\tfrac{1}{2}m^2x^2-\tfrac{1}{4}\lambda x^4+xy) , \tag{2.31}$$

where

$$N_1=\int_{-\infty}^{\infty}dx\,\exp(-\tfrac{1}{2}m^2x^2-\tfrac{1}{4}\lambda x^4) . \tag{2.32}$$

What happens in the case of the strong-coupling expansion? First we must solve the theory with $m^2=0$. In this limit the differential equation (2.14) becomes

$$\lambda H'''(y)=yH . \tag{2.33}$$

As we noted before this is a third-order differential equation which requires three boundary conditions. One choice for the third boundary condition is to require that the solution have one of the three possible types of asymptotic behavior:

$$H(y)\sim\exp(\tfrac{3}{4}y^{4/3}\omega\lambda^{-1/3}) .$$

When $m^2=0$ the integral representations in (2.17) and (2.19) become

$$\begin{aligned}H_1(y)&=N_1\int_{-\infty}^{\infty}dx\,\exp(-\tfrac{1}{4}\lambda x^4+xy)\\&=\sum y^{2n}G_{2n}/(2n)! ,\end{aligned} \tag{2.34}$$

$$\begin{aligned}H_2(y)&=N_1\int_{-\infty}^{\infty}dx\,\exp(-\tfrac{1}{4}\lambda x^4+ixy)\\&=\sum(-1)^n y^{2n}G_{2n}/(2n)! ,\end{aligned} \tag{2.35}$$

where

$$G_{2n}=\left[\frac{4}{\lambda}\right]^{n/2}\frac{\Gamma(n/2+1/4)}{\Gamma(1/4)} .$$

Thus, $H_1$ and $H_2$ are quite distinguishable from their large-$y$ behavior.

What happens if we try to solve the Schwinger-Dyson equations (2.24) when $m^2=0$? Since the initial conditions $H(0)=1$, $H'(0)=0$ do not determine the value for the two-point function $G_2$ it must remain arbitrary: $G_2=\alpha$. The infinite system of coupled Schwinger-Dyson equations in the massless limit reads

$$\lambda G_n=(n-3)G_{n-4} . \tag{2.36}$$

Iterating this equation gives

$$\begin{aligned}G_{4n}&=(4/\lambda)^n\frac{\Gamma(n+\frac{1}{4})}{\Gamma(\frac{1}{4})} ,\\G_{4n+2}&=\alpha(4/\lambda)^n\frac{\Gamma(n+\frac{3}{4})}{\Gamma(\frac{3}{4})} .\end{aligned} \tag{2.37}$$

We do not know a simple way to determine $\alpha$ directly. One possible approach is to reconstruct the generating function $H(y)$ from (2.37). For arbitrary $\alpha$ we can sum the series if we make a particular assumption about the generating functional. For the correspondence $x\Longleftrightarrow d/dy$ we have

$$H_\alpha(y)=\sum_{n=0}^{\infty}G_{2n}(\alpha)y^{2n}/(2n)! . \tag{2.38}$$

We can rewrite this series putting a new parameter $\beta=(4/\lambda)^{1/2}\Gamma(\frac{3}{4})/\Gamma(\frac{1}{4})$:

$$H_\alpha(y)=\tfrac{1}{2}(1+\alpha/\beta)H_1(y)+\tfrac{1}{2}(1-\alpha/\beta)H_2(y) \,, \tag{2.39}$$

where $H_1(y)$ and $H_2(y)$ are given in (2.34) and (2.35). We now see that imposing the large-$y$ behavior of either $H_1$ or $H_2$ determines $\alpha$ to be either $\beta$ or $-\beta$. Unfortunately, at the next order in perturbation theory we have exactly the same problem all over again. We must assume again an unknown value of the two-point function valid to next order in $m^4/\lambda$ and sum the series. This is not very practical in general.

It is quite clear that in higher-dimensional space-time there is a kinetic energy term and the problem of determining the third boundary condition gets more difficult. For the case of the Langevin regulated Schwinger-Dyson equations, there is no formal path-integral solution and we do not know how to solve the problem of satisfying the third boundary condition.

## III. CONVERGENCE OF TRUNCATION SCHEMES

As an alternative to solving the Schwinger-Dyson equations by making weak-coupling or strong-coupling expansions we can use truncation methods. Here we decouple the first $N$ Green's functions from the rest and solve the closed system of equations that results.[4] One of the best-known truncation schemes, the Hartree-Fock (or large-$N$) approximation consists of solving the equations for the coupled one- and two-point functions. In zero dimensions truncating the equations at the $N$th Green's function gives a set of $N$ coupled algebraic equations which we solve explicitly. Therefore, we can study the limit $N\to\infty$ in detail.

The Schwinger-Dyson equations in (2.24) can be rewritten for the even Green's functions (the odd ones are zero) as

$$\lambda G_{2n+2}+m^2G_{2n}=(2n-1)G_{2n-2} \,, \tag{3.1}$$

with the boundary condition $G_0=1$.

Setting $G_{2n}=Q_n$ gives

$$m^2Q_1+\lambda Q_2=1 \tag{3.2}$$

and

$$m^2Q_n+\lambda Q_{n+1}-(2n-1)Q_{n-1}=0, \quad n\ge 1 \,. \tag{3.3}$$

We can think of $Q$ as a column vector so that this set of equations is equivalent to the matrix equation

$$MQ=I_1 \,. \tag{3.4}$$

If we restrict ourselves to finite truncations of $Q$ so that $Q$ is replaced by $Q^{[N]}$ then $M^{[N]}$ is the $N\times N$ matrix:

$$M^{[N]}=\begin{bmatrix} m^2 & \lambda & 0 & \cdots & 0 & 0 & 0 \\ -3 & m^2 & \lambda & & 0 & 0 & 0 \\ 0 & -5 & m^2 & & 0 & 0 & 0 \\ 0 & 0 & -7 & & \lambda & 0 & 0 \\ 0 & 0 & 0 & & m^2 & \lambda & 0 \\ 0 & 0 & 0 & & -(2N-3) & m^2 & \lambda \\ 0 & 0 & 0 & \cdots & 0 & -(2N-1) & m^2 \end{bmatrix} \tag{3.5}$$

and $I_1^{[N]}$ is the column vector with 1 in the first entry all other entries zero:

$$I_1^{[N]}=\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} . \tag{3.6}$$

Now that we have truncated the recursion relation (3.3) to a finite-dimensional matrix equation we can calculate all the Green's functions up to $Q_N$ by using Cramer's rule. Thus to obtain the two-point function $Q_1$ we first construct the matrix $M_1^{[N]}$, in which we replace the first column of $M^{[N]}$ by the column vector $I_1^{[N]}$. Then from Cramer's rule, the $N$th truncation approximation to $Q_1$ is

$$Q_1^{[N]}=\det M_1^{[N]}/\det M^{[N]}\equiv DM_1^{[N]}/DM^{[N]} \,. \tag{3.7}$$

The first few determinants are

$$DM_1^{[1]}=1, \quad DM_1^{[2]}=m^2, \quad DM_1^{[3]}=m^4+5\lambda,$$
$$DM^{[1]}=m^2, \quad DM^{[2]}=m^4+3\lambda, \quad DM^{[3]}=m^6+8\lambda m^2 \,. \tag{3.8}$$

Note that the ratio of determinants can be reexpanded in terms of an infinite Taylor series in $\lambda/m^4$ whose leading term is $1/m^2$. This gives the exact weak-coupling expansion out to order $N$.

Recall that the integral representation for $H_1(y)$ in (2.17) has a weak-coupling expansion for $m^2>0$. From $H_1(y)$ we obtain the correspondence

$$x \Longleftrightarrow d/dy \,.$$

Thus

$$\frac{\langle x^2\rangle}{\langle 1\rangle}=H_1''(y=0)>0 \,.$$

Recall, also, that $H_2(y)$ in (2.19) has a weak-coupling expansion for $m^2<0$. For $H_2$ we have the correspondence

$$x \Longleftrightarrow -i\, d/dy \ .$$

Thus

$$\frac{\langle x^2\rangle}{\langle 1\rangle}=-H_2''(y=0)>0 \ .$$

Apparently, our truncation of the Schwinger-Dyson equations has the property that in leading order

$$H''(y=0)=1/m^2 \ .$$

Thus for $m^2>0$ it must be approximating the Green's functions of $H_1$, and for $m^2<0$ it must be approximating the Green's functions of $H_2$. Hence, for $m^2>0$ we must compare $Q_1$ with

$$\frac{\langle x^2\rangle}{\langle 1\rangle}=\frac{\int_{-\infty}^{\infty}dx\, x^2\exp(-\frac{1}{4}\lambda x^4-\frac{1}{2}m^2x^2)}{\int_{-\infty}^{\infty}dx\, \exp(-\frac{1}{4}\lambda x^4-\frac{1}{2}m^2x^2)} \tag{3.9}$$

and for $m^2<0$ we need to compare $Q_1$ with $-\langle x^2\rangle$ , where

$$\frac{\langle x^2\rangle}{\langle 1\rangle}=\frac{\int_{-\infty}^{\infty}dx\, x^2\exp(-\frac{1}{4}\lambda x^4+\frac{1}{2}m^2x^2)}{\int_{-\infty}^{\infty}dx\, \exp(-\frac{1}{4}\lambda x^4+\frac{1}{2}m^2x^2)}. \tag{3.10}$$

It is very easy to obtain the two-point function $G_2=Q_1$ at order $N$ in the truncation scheme because the determinants $DM^{[N]}$ and $DM_1^{[N]}$ both satisfy the same recursion relation

$$DA^{[N]}=m^2DA^{[N-1]}+(2N-1)\lambda DA^{[N-2]} \tag{3.11}$$

with different initial conditions [see (3.8)].

Numerically we find that for $\lambda/m^4\le 1$ the convergence of the truncation scheme to the exact answer is quite rapid. For larger values of $\lambda/m^4$ the convergence rate becomes worse until finally as $\lambda/m^4\to\infty$ it does not converge at all. The ratio in (3.7) merely oscillates between positive and negative values. Thus, the truncation scheme does not give accurate results in the strong-coupling regime and in the strong-coupling limit there is no answer at all. Furthermore, the truncation scheme converges to that particular path-integral solution of the Schwinger-Dyson equations that has a weak-coupling expansion for the particular choice of sign for $m^2$. This shows that truncation schemes are indirectly tied to weak-coupling expansions.

## IV. STOCHASTIC REGULARIZATION

In this section we summarize the work of Bern *et al.* as it pertains to $\lambda\phi^4$ field theory and show that it is unlikely that we can reconstruct the field theory in the strong-coupling regime from the regulated Schwinger-Dyson equations. The starting point for the continuum regulation scheme is the regulation of the Langevin equations involved in stochastic quantization.[5] Introducing the Langevin fictitious time $\tau$ we have

$$\dot\phi(x,\tau)=-\delta S(x,\tau)/\delta\phi+\int dy\, R_{xy}(\Box)\eta(y,\tau) \ , \tag{4.1}$$

where the noise term $\eta$ satisfies

$$\langle \eta(x,\tau)\eta(y,\tau')\rangle=2\delta(\tau-\tau')\delta^n(x-y) \ . \tag{4.2}$$

In (4.1) $R$ is a regulator. For example, in heat-kernel regulation we take

$$R_{xy}(\Box)=[\exp(\Box/\Lambda^2)]_{xy} \ , \tag{4.3}$$

where $\Box$ is the Laplacian.

It has been shown[5] that the fixed-$\tau$ correlation functions obtained by averaging over the noise approach the field-theoretic Green's functions as $\tau\to\infty$. If $F[\phi]$ is any functional of the field $\phi$ then we have the $\tau$ evolution equation

$$\frac{d}{d\tau}\langle F[\phi]\rangle_\eta=\left\langle \int dx\left[-\delta S/\delta\phi+\int dy\, R_{xy}^2\delta/\delta\phi(y)\right]\delta F/\delta\phi(x)\right\rangle_\eta \ . \tag{4.4}$$

At long times $\tau$ an equilibrium state is reached and choosing $F=\exp[\int dx J(x)\phi(x)]$, and $Z[j]=\langle\exp[\int dx\, J(x)\phi(x)]\rangle$ we obtain the regulated functional differential equation

$$0=\int dx\, J(x)\left[-\delta S/\delta\phi+\int dy\, R_{xy}^2J(y)\right]Z[J] \ , \tag{4.5}$$

where $\phi(x)=\delta/\delta J(x)$ in $\delta S/\delta\phi$. Equation (4.5) is equivalent to an infinite set of regulated Schwinger-Dyson equations.

Let us look at the one-dimensional version of these equations (quantum mechanics). The regulated functional equation for the anharmonic oscillator is

$$\int dt\, J(t)[(-d^2/dt^2+m^2)\delta Z/\delta J+\lambda\delta^3Z/\delta J^3]=\int\int dt\, dt' J(t)R_{tt'}^2J(t')Z[J] \ . \tag{4.6}$$

This equation is equivalent to an infinite set of coupled ordinary differential equations for the Green's functions. The first of these is

$$(-d^2/dt_2^2+m^2)W_2(t_2,t_1)+(-d^2/dt_1^2+m^2)W_2(t_1,t_2)+\lambda W_4(t_1,t_1,t_1,t_2)+\lambda W_4(t_2,t_2,t_2,t_1)$$
$$=R^2(t_1,t_2)+R^2(t_2,t_1) \ . \tag{4.7}$$

Note that instead of having a delta function $\delta(t_1-t_2)$ on the right-hand side of (4.7) as in the conventional Schwinger-Dyson equation, the square of the regulator function appears. From (4.7) it is clear that when $\lambda=0$ one has a unique starting point for the weak-coupling expansion

$$W_2(p^2)=\tilde{R}^2(p^2/\Lambda^2)/(p^2+m^2) , \tag{4.8}$$

where $\tilde{R}$ is the Fourier transform of $R$. Evidently, (4.8) replaces $(p^2+m^2)^{-1}$ as the *regulated* propagator in the weak-coupling expansion.

Unfortunately, in the strong-coupling regime we have several problems. We not only have the problem of specifying $W_2$ as we had in the unregulated case, but now, because this is not the ultralocal approximation where the $N$-point functions are all delta functions, we also have to know how each leg is regulated. In the strong-coupling limit the first Schwinger-Dyson equation relates $W_4(t_1,t_1,t_1,t_2)$ to $R^2(t_1,t_2)$, the second Schwinger-Dyson equation relates $W_6(t_1,t_1,t_1,t_2,t_3,t_4)$ to $W_4(t_1,t_2,t_3,t_4)$ and $W_2(t_1,t_2)$, and so on. There is no equation for $W_4(t_1,t_2,t_3,t_4)$. We only obtain information about $W_4$ when three of the legs are tied together; from (4.7) we have, for large $\lambda$,

$$W_4(t_1,t_1,t_1,t_2)=\lambda^{-1}R^2(t_1,t_2) . \tag{4.9}$$

This is not sufficient information to allow us to reconstruct the theory in the strong-coupling domain from the regulated Schwinger-Dyson equations. Thus, we believe that the regulated Schwinger-Dyson equations found by Bern *et al.*, although quite adequate in reconstructing a regulated theory in the weak-coupling domain are not sufficient to reconstruct the theory in the strong-coupling domain.

## ACKNOWLEDGMENTS

We thank the U. S. Department of Energy for partial financial support.

[1]Some useful references are A. J. Niemi and L. C. R. Wijewardhana, Ann. Phys. (N. Y.) **140**, 247 (1982): J. D. Breit, S. Gupta, and A. Zaks, Nucl. Phys. **B233**, 61 (1984); Z. Bern, M. B. Halpern, L. Sadun, and C. Taubes, Phys. Lett. **165B**, 151 (1985); Nucl. Phys. B **284**, 1 (1987); **B284**, 35 (1987); Z. Bern, M. B. Halpern, and L. Sadun, *ibid.* **B284**, 92 (1987); Z. Bern, M. B. Halpern, and N. G. Kalivas, Phys. Rev. **D 35**, 753 (1987).

[2]F. Cooper and R. Kenway, Phys. Rev. D **24**, 2706 (1981); C. M. Bender, F. Cooper, R. Kenway, and L. M. Simmons, Jr., *ibid.* **24**, 2693 (1981).

[3]One can find simple analogs in ordinary differential equations. The first-order equation $xy'+y=e^x$ has an infinite number of solutions $y=(e^x+C)/x$ parametrized by $C$. All of these solutions are singular at $x=0$ except when $C=-1$. Thus, requiring that the solution have a Taylor expansion at $x=0$ (this is like requiring that the Green's functions have weak-coupling expansions) *uniquely* determines the solution.

[4]See, for example, C. M. Bender, G. S. Guralnik, R. W. Keener, and K. Olaussen, Phys. Rev. **D 14**, 2590 (1976).

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