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Covariant Gaussian approximation. I. Formalism

A. Kovner
High Energy Physics Department, School of Physics and Astronomy, Tel Aviv University, Ramat Aviv, Tel Aviv 69978, Israel

B. Rosenstein
Theory Group, Department of Physics, University of Texas, Austin, Texas 78712
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The variational Gaussian approximation is generalized to the time-dependent approach capable of giving time-dependent Green's functions. This covariant Gaussian approximation is represented, in full analogy with the classical approximation, as an initial truncation of the Dyson-Schwinger equations followed by functional differentiation of the effective action. Intuitively simple Schrödinger and Heisenberg pictures of the approximation are also discussed.

I. INTRODUCTION

Perturbation theory remains to this day the main analytical tool in investigating continuum quantum field theory (QFT). Its wide application is due to the fact that it is the simplest expansion scheme for calculating Green's functions and to its triumphs in the case of QED.

It is clear however that perturbation theory is unable to uncover the whole rich structure of QFT. The simplest example of its failure to predict qualitatively correct results is the spontaneously broken scalar theory. While the phenomenon of spontaneous symmetry breakdown is easily discovered in the classical approximation, it can never be seen in ordinary "naive" perturbation theory. To accommodate this phenomenon the framework of perturbation theory must be changed, for example, via the introduction of "phenomenological" quantities corresponding to condensates of local operators in the manner of Shifman, Vainshtein, and Zakharov.[1,2] Those quantities themselves cannot be calculated in perturbation theory and must be supplied by some other, nonperturbative method.

The simplest method that was employed to supply this nonperturbative information is the classical approximation. With its help in scalar theory one can calculate the lowest condensate $\langle\Phi\rangle$. However it is clear that the classical approximation must be inadequate in many cases. Apart from the fact that it completely ignores quantum corrections in some cases it is *a priori* clear that it must be useless altogether. For example, in Yang-Mills theory, where it is believed that the local condensate $\langle F^{a\mu\nu}F^{a}{}_{\mu\nu}\rangle$ plays a crucial role at low energies,[1] the classical approximation cannot help. The reason is that in this approximation the following relation between the expectation values holds:

$$\langle F^{a\mu\nu}F^{a}{}_{\mu\nu}\rangle=\langle F^{a\mu\nu}\rangle\langle F^{a}{}_{\mu\nu}\rangle\ . \tag{1.1}$$

However, $F^{a\mu\nu}$ carries Lorentz and color indices and if either of these symmetries is not broken it cannot acquire a nonzero vacuum expectation value.

Thus it is desirable to have some method which on one hand must go beyond both perturbation theory and the classical approximation in the points where those fail and on the other hand be sufficiently simple so analytical calculations could be performed in its framework.

The Gaussian approximation is a natural candidate for such a method. The essence of the method is that, by minimizing the expectation value of the Hamiltonian on the set of Gaussian states,

$$\Psi[\Phi(x)]=N\exp\{-\tfrac{1}{2}[\Phi(x)-\phi(x)]\times G(x-y)[\Phi(y)-\phi(y)]\} \tag{1.2}$$

one finds such a state with minimum energy. This state is then used to approximate the vacuum of the system. This approximation is widely used in nonrelativistic many-body theory, where it was very successful in the Bardeen-Cooper-Schrieffer (BCS) theory of superconductivity.[3] In quantum mechanics it is in excellent agreement with numerical results for ground-state and first-excited-state energies for a wide range of potentials.[4] This method is sometimes called the Hartree approximation and under this name it has a rather long history of applications also in relativistic bosonic QFT (Ref. 5).

Recently interest in this approximation has been revived.[6–14] The method was applied to renormalizable scalar theories.[8–11] Several attempts were also recently made to apply it to pure Yang-Mills theory.[12–14]

The method is, however, somewhat inconvenient in application to relativistically invariant QFT since it is not explicitly covariant. In fact it is represented in Hamiltonian formalism. While this is of less importance when scalars are considered, a covariant formulation becomes essential in application to theories containing fields in nontrivial representations of the Lorentz group (for example, vectors).

Another related feature of this approach is that its application as a time-independent variational method provides only the possibility of calculating the effective potential rather than effective action. Thus one can obtain only Green's functions at zero momenta. In renormalizable theory this is a serious drawback, since it is impossi-

ble to impose one of the necessary normalization conditions: finiteness of the residue of the propagator. As a result the renormalization becomes partly guesswork and, for example, several different versions of the renormalization of ϕ^4 theory exist.[6,9,15]

In this work we discuss the time-dependent generalization of the Gaussian (Hartree) variational method.[5] We show that it can be formulated Lorentz covariantly in terms of truncation of the Dyson-Schwinger equations (DSE's). It permits one to calculate effective action and Green's functions at nonzero momentum in an explicitly covariant way and sets a convenient framework for discussion of renormalization.

In Sec. II we present a simple intuitive picture of the time-dependent Gaussian approximation in the Schrödinger picture. In Sec. III we formulate the same approximation in terms of truncation of Dyson-Schwinger equations (DSE's). In Sec. IV the Heisenberg picture representation of the approximation is discussed. For the sake of simplicity the discussion in these sections is confined to quantum mechanics. Section V contains the generalization of the method to QFT. In Sec. VI we discuss our results and indicate how this approach can be incorporated in a more general approximation scheme. The Appendix contains a short discussion of the dependence of the approximation on the basis in the Hilbert space.

II. SCHRÖDINGER PICTURE

Our aim in this section is to obtain the approximate effective action of a quantum theory in the Gaussian approximation.

The effective action of a quantum system is the Legendre transform of a functional $W[J]$ (Ref. 16) defined by

$$\exp(iW[J])=\langle 0_-|0_+\rangle_J \ . \tag{2.1}$$

It is worthwhile to discuss the meaning of this definition. One starts with the asymptotic state $|0_-\rangle$ that is a vacuum of the system's Hamiltonian $H(\hat{\Phi},\hat{\Pi})$ which is independent of J. Then a time-dependent external source $J(t)$ is adiabatically turned on and the initial state propagates under the Hamiltonian $H_J=H-J(t)\hat{\Phi}$. Gradually the source is turned off and the system again is governed by H. Under these conditions after a sufficiently long (infinite) time the system falls back into a vacuum of H—the state $|0_+\rangle$. This state however can differ from $|0_-\rangle$ by a phase, which is equal to $W[J]$.

In this section we present an approach in which $W[J]$ is calculated by direct approximation of quantum evolution of the initial state rather than by expansion in some parameter. We start with the classical approximation as the simplest example of an approximation of this kind.

A. Classical approximation

For the sake of definiteness we shall discuss the quantum system governed by the Hamiltonian of the form

$$H=\tfrac{1}{2}\hat{\Pi}^2+V(\hat{\Phi}) \ . \tag{2.2}$$

A ground state of Hamiltonian H in the classical approximation is taken to be a coherent state[17,18]

$$|0_-\rangle=|\phi_0,\pi_0\rangle = \left[\frac{\omega}{\pi}\right]^{1/4}\exp\left[-\frac{\omega}{2}(\Phi-\phi_0)^2+i\pi_0(\Phi-\phi_0)\right] , \tag{2.3}$$

where ω is an arbitrary real parameter. The distinguishing property of these states is their minimal indeterminacy

$$\Delta\Phi\,\Delta\Pi=\tfrac{1}{2}\hbar \ . \tag{2.4}$$

When the source is turned on this state evolves into $\Psi_t(\Phi)$. The state $\Psi_t(\Phi)$ is also approximated by the optimal coherent state

$$|\phi(t),\pi(t)\rangle=\left[\frac{\omega}{\pi}\right]^{1/4}\exp\left[-\frac{\omega}{2}[\Phi-\phi(t)]^2+i\pi(t)[\Phi-\phi(t)]+iA(t)\right] , \tag{2.5}$$

where $\phi(t)$ and $\pi(t)$ are time dependent. For this state

$$\langle\hat{\Phi}\rangle=\phi \ , \tag{2.6}$$

$$\langle\hat{\Pi}\rangle=\pi \ . \tag{2.7}$$

We look for this optimal coherent state along the lines of Ref. 18.

The initial state propagates according to the Schrödinger equation

$$i\frac{d}{dt}\Psi=H_J\Psi \ . \tag{2.8}$$

The coherent state $|\phi(t),\pi(t)\rangle$ cannot satisfy this equation. The set of coherent states is a three-parameter subset of the infinite-dimensional Hilbert space. The best one can do is to adjust these three parameters: the phase and the expectation values of Φ and Π. We choose a coherent state which is closest to satisfying the following three conditions, which would have been exact if it were a solution of Eq. (2.8):

$$\langle\Psi_t|\phi(t),\pi(t)\rangle=1 \ , \tag{2.9}$$

$$\langle\phi(t),\pi(t)|\hat{\Phi}|\phi(t),\pi(t)\rangle=\langle\Psi_t|\hat{\Phi}|\Psi_t\rangle \ , \tag{2.10}$$

$$\langle\phi(t),\pi(t)|\hat{\Pi}|\phi(t),\pi(t)\rangle=\langle\Psi_t|\hat{\Pi}|\Psi_t\rangle \ . \tag{2.11}$$

Equation (2.9) means that we want the overlap between the exact solution $|\Psi_t\rangle$ and the optimal coherent state to be maximal. Equations (2.10) and (2.11) state that the expectation values of operators $\hat{\Phi}$ and $\hat{\Pi}$ in the approximate solution are the same as in the exact solution (Fig. 1).

The differential form of Eqs. (2.9)–(2.11) is

$$\left\langle\phi(t),\pi(t)\left|\frac{d}{dt}\right|\phi(t),\pi(t)\right\rangle = -i\langle\phi(t),\pi(t)|H_J|\phi(t),\pi(t)\rangle \ , \tag{2.12}$$

FIG. 1. Solution of the Schrödinger equation $\psi(t)$ is approximated at all times by a coherent state $|\phi(t),\pi(t)\rangle$. The width of the Gaussian is kept constant; the coordinates of the center are optimized.

$$\left\langle \phi(t),\pi(t)\left|\hat{\Phi}\frac{d}{dt}\right|\phi(t),\pi(t)\right\rangle + \text{c.c.} = -i\langle \phi(t),\pi(t)|[\hat{\Phi},H_J]|\phi(t),\pi(t)\rangle \ , \quad (2.13)$$

$$\left\langle \phi(t),\pi(t)\left|\hat{\Pi}\frac{d}{dt}\right|\phi(t),\pi(t)\right\rangle + \text{c.c.} = -i\langle \phi(t),\pi(t)|[\hat{\Pi},H_J]|\phi(t),\pi(t)\rangle \ . \quad (2.14)$$

These are the equations giving the evolution of the optimal coherent state in the classical approximation.

Using the explicit form of $|\phi(t),\pi(t)\rangle$ we obtain differential equations for $A(t)$, $\phi(t)$, and $\pi(t)$:

$$\dot{A}=\pi\dot{\phi}-\langle \phi(t),\pi(t)|H|\phi(t),\pi(t)\rangle + J(t) \ , \quad (2.15)$$

$$\dot{\phi}=\pi \ , \quad (2.16)$$

$$\dot{\pi}=-\frac{d}{d\phi}\langle \phi(t),\pi(t)|V|\phi(t),\pi(t)\rangle + J(t) \ . \quad (2.17)$$

Thus

$$A(t)=\int_{-\infty}^{t} dt[\pi\dot{\phi}-E(\phi(t),\pi(t))+J(t)\phi(t)] \ , \quad (2.18)$$

where

$$E(\phi(t),\pi(t))=\langle \phi(t),\pi(t)|H|\phi(t),\pi(t)\rangle \quad (2.19)$$

and $\phi(t)$ and $\pi(t)$ obey Eqs. (2.16) and (2.17). If $J=0$ the static solution of Eqs. (2.16) and (2.17) gives $|\phi_0,\ \pi_0\rangle$ which is the approximate vacuum in classical approximation. It is equivalent to the variational principle

$$\frac{\partial}{\partial\phi}\langle \phi,\pi|H|\phi,\pi\rangle = 0 \ , \quad (2.20)$$

$$\frac{\partial}{\partial\pi}\langle \phi,\pi|H|\phi,\pi\rangle = 0 \ ; \quad (2.21)$$

that is, minimization of the expectation value of Hamiltonian on the set of coherent states.

By definition [Eq. (2.1)]

$$W[J]=A(\infty) \ . \quad (2.22)$$

Differentiating $W[J]$ with respect to $J(t)$ and using Eqs. (2.16) and (2.17) we obtain

$$\frac{\delta}{\delta J(t)}W[J]=\phi(t)+\int dt'\frac{\delta A}{\delta\phi(t')}\frac{\delta\phi(t')}{\delta J(t)}=\phi(t) \ . \quad (2.23)$$

The effective action is defined as a Legendre transform of $W[J]$. We therefore obtain for the effective action in classical approximation:

$$S[\phi(t)]=\int dt\ \tfrac{1}{2}\dot{\phi}^2-\langle \phi(t),\pi(t)|V|\phi(t),\pi(t)\rangle \ . \quad (2.24)$$

We remark that $S[\phi(t)]$ is equal to the classical action if the Hamiltonian H is normal ordered with respect to the same parameter ω that enters the definition of the coherent state equations (2.3) and (2.5). In this case Eqs. (2.16) and (2.17) are the classical Hamilton equations. For details see Refs. 17–19.

Proper Green's functions (PGF's) are obtained from the effective action Eq. (2.24) by functional differentiation with respect to $\phi(t)$ at $\phi(t)=\phi_0$. These are the usual tree-level Green's functions.

B. Gaussian approximation

The classical approximation is of course inadequate for description of quantum effects. The deficiency of this approximation is that it does not allow for states without minimal quantum indeterminacy and time-dependent width. The easiest way to improve on it is to consider a wider set of Gaussian states:[19]

$$|\phi,\pi,G,I\rangle = \left[\frac{1}{G\pi}\right]^{1/4}\exp\{-\tfrac{1}{2}[\Phi-\phi(t)] \times[G^{-1}(t)+iI(t)][\Phi-\phi(t)] +i\pi(t)[\Phi-\phi(t)]+iA(t)\} . \quad (2.25)$$

In these states

$$\langle\hat{\Phi}\rangle=\phi \ , \quad (2.26)$$

$$\langle\hat{\Pi}\rangle=\pi \ , \quad (2.27)$$

$$\langle\Phi^2\rangle-\phi^2=\tfrac{1}{2}G \ , \quad (2.28)$$

$$\langle\Pi^2\rangle-\pi^2=\tfrac{1}{2}(G^{-1}+IGI) \ . \quad (2.29)$$

We shall follow now the approximation procedure analogous to the classical one. The ground state is approximated by a Gaussian

$$\Psi_0(\Phi)=|\phi_0,\pi_0;G_0,I_0\rangle \ . \quad (2.30)$$

Under the influence of time-dependent source $J(t)$ this state evolves into a state $\Psi_t(\Phi)$ which is approximated by another Gaussian $|\phi(t),\pi(t);G(t),I(t)\rangle$. In addition to the optimization requirements, Eqs. (2.9)–(2.11), we optimize now also the expectation values of quadratic operators:

$$\langle I(t),G(t);\phi(t),\pi(t)|\hat{\Phi}^2|\phi(t),\pi(t);G(t),I(t)\rangle=\langle\Psi_t|\hat{\Phi}^2|\Psi_t\rangle \ , \quad (2.31)$$

$$\langle I(t),G(t);\phi(t),\pi(t)|\hat{\Pi}^2|\phi(t),\pi(t);G(t),I(t)\rangle=\langle\Psi_t|\hat{\Pi}^2|\Psi_t\rangle \ , \quad (2.32)$$

$$\langle I(t),G(t);\phi(t),\pi(t)|\hat{\Phi}\hat{\Pi}+\hat{\Pi}\hat{\Phi}|\phi(t),\pi(t);G(t),I(t)\rangle=\langle\Psi_t|\hat{\Phi}\hat{\Pi}+\hat{\Pi}\hat{\Phi}|\Psi_t\rangle \ . \quad (2.33)$$

Physically this adjusts the width of a Gaussian to the exact solution, in addition to its phase and position; see Fig. 2.

The infinitesimal form of the full set of optimization conditions is

$$\left\langle \frac{d}{dt} \right\rangle + \text{c.c.} = -i\langle H \rangle \ , \tag{2.34}$$

$$\frac{d}{dt}\langle \hat{\Phi} \rangle = -i\langle [\hat{\Phi}, H] \rangle \ , \tag{2.35}$$

$$\frac{d}{dt}\langle \hat{\Pi} \rangle = -i\langle [\hat{\Pi}, H] \rangle \ , \tag{2.36}$$

$$\frac{d}{dt}\langle \hat{\Phi}^2 \rangle = -i\langle [\hat{\Phi}^2, H] \rangle \ , \tag{2.37}$$

$$\frac{d}{dt}\langle \hat{\Pi}^2 \rangle + -i\langle [\hat{\Pi}^2, H] \rangle \ , \tag{2.38}$$

$$\frac{d}{dt}\langle \hat{\Phi}\hat{\Pi} + \hat{\Pi}\hat{\Phi} \rangle = -i\langle [\hat{\Phi}\hat{\Pi} + \hat{\Pi}\hat{\Phi}, H] \rangle \ . \tag{2.39}$$

We have six equations for five parameters: ϕ, π, G, I, A. However, one of the equations (2.34)–(2.39) is trivially satisfied. This can be seen in the following way. The Gaussian state [Eq. (2.25)] is annihilated by the operator

$$\hat{a} = e^{-i\chi/2} \left[\frac{\omega}{2\cos\chi} \right]^{1/2} \hat{\Phi} - ie^{i\chi/2} \left[\frac{1}{2\omega\cos\chi} \right]^{1/2} \hat{\Pi} \ , \tag{2.40}$$

where

$$\omega e^{-i\chi} = G^{-1} + iI \ .$$

Consequently the quadratic operator $\hat{a}^\dagger \hat{a}$ trivially satisfies the equation

$$\frac{d}{dt}\langle \hat{a}^\dagger \hat{a} \rangle = -i\langle [\hat{a}^\dagger \hat{a}, H] \rangle \ , \tag{2.41}$$

which is a linear combination of Eqs. (2.34)–(2.39).

Using the explicit form of a Gaussian equation, (2.25), the optimization conditions lead to

$$\dot{A} = \dot{\phi}\pi + \tfrac{1}{4} I\dot{G} - \langle H \rangle \ , \tag{2.42}$$

$$\dot{\phi} = \pi \ , \tag{2.43}$$

$$\dot{\pi} = -\frac{\partial}{\partial \phi}\langle V \rangle + J \ , \tag{2.44}$$

$$\dot{G} = 2IG \ , \tag{2.45}$$

$$\dot{I} = G^{-2} - I^2 - 4\frac{\partial}{\partial G}\langle V \rangle \ . \tag{2.46}$$

Equations (2.42)–(2.46) are equivalent to two second-order equations

$$\ddot{\phi} + \frac{\partial}{\partial \phi}\langle V \rangle = J \ , \tag{2.47}$$

$$\ddot{G} - \frac{1}{2}\frac{\dot{G}^2}{G} - 2G^{-1} + 8G\frac{\partial}{\partial G}\langle V \rangle = 0 \ . \tag{2.48}$$

Equation (2.47) will be called in the following the shift equation, while Eq. (2.48) will be called the gap equation.

The generating functional of connected Green's functions, $W[J]$, is equal to $A(t=\infty)$ for ϕ and G that solve Eqs. (2.47) and (2.48). The initial conditions must be taken in accordance with definition (2.1), that is, for $t=-\infty$ we should start from the approximate vacuum. This approximate vacuum is determined by minimizing the energy expectation value on the set of Gaussians.[4] It is easy to see that the minimization equations coincide with the static version of Eqs. (2.47) and (2.48) for zero external current J:

$$\frac{\partial}{\partial \phi}\langle H \rangle = 0, \quad \frac{\partial}{\partial G}\langle H \rangle = 0 \ . \tag{2.49}$$

Thus $\phi(-\infty)$ and $G(-\infty)$ are determined by solving Eq. (2.49), and $\dot{G}(-\infty)=\dot{\phi}(-\infty)=0$. As in the classical approximation $A(\infty) - \int J\phi$ is an approximate effective action. It can be conveniently expressed as a functional of $\phi(t)$ and $G(t)$.

$$S_{\text{eff}}[\phi(t), G(t)] = \int dt \left[\frac{1}{2}\dot{\phi}^2 + \frac{1}{16}\frac{\dot{G}^2}{G} - \frac{1}{4}G^{-1} - \langle V(\phi) \rangle \right] . \tag{2.50}$$

The Euler-Lagrange equations for this action coincide with Eqs. (2.47) and (2.48).

In the next section we shall reformulate the time-dependent Gaussian approximation in the language of Dyson-Schwinger equations (DSE's) which will be useful in application to relativistic QFT.

III. DYSON-SCHWINGER APPROACH TO GAUSSIAN APPROXIMATION

The time-dependent Gaussian approximation presented above can be applied to QFT as well as to quantum mechanics. The quantities of primary interest in QFT are Green's functions.

Proper Green's functions (PGF's) are defined as functional derivatives of the effective action $S[\phi(t)]$:

$$\Gamma^n(t_1, \ldots, t_n) = \frac{\delta^n S[\phi]}{\delta\phi(t_1)\cdots\delta\phi(t_n)} \ . \tag{3.1}$$

Any quantum theory can be defined in terms of PGF's only. In this formulation the dynamics is given by a (infinite) set of intertwining integro-differential Dyson-Schwinger equations[16,20] (DSE's), which replace the Schrödinger equation for the wave function.

For example, for the anharmonic oscillator

$$H = \tfrac{1}{2}\hat{\Pi}^2 + \tfrac{1}{2}m^2\hat{\Phi}^2 + \frac{\lambda}{4!}\hat{\Phi}^4 \ , \tag{3.2}$$

the first two DSE's are

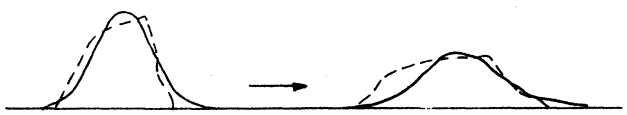
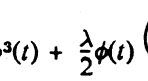


FIG. 2. Solution of the Schrödinger equation $\psi(t)$ is approximated at all times by a Gaussian state. Width of the Gaussian as well as the coordinates of its center are optimized.

$$-D^{-1}\phi(t') + \frac{\lambda}{3!}\phi^3(t) + \frac{\lambda}{2}\phi(t)$$  $$+ \frac{i\lambda}{3!}$$  $$= J(t) , \tag{3.3}$$

$$\Gamma^2(t,t') = D^{-1}(t,t') - \frac{\lambda}{2}\left[i\phi^2(t) + \right.$$  $$+ i\phi(t)$$  $$-$$  $$+ \frac{i}{3}$$  $$\left.\right]\delta(t-t') . \tag{3.4}$$

Here lines represent propagators, full circles represent PGF's, and

$$D^{-1} = (-\partial_t^2 - m^2)\delta(t-t') . \tag{3.5}$$

There is a close connection between the DSE formulation and the Heisenberg picture of quantum dynamics. Indeed Eq. (3.3) is obtained by taking the expectation value of the operator Heisenberg equation for operator $\hat{\Phi}(t)$ in a vacuum:

$$\frac{\partial^2}{\partial t^2}\langle\hat{\Phi}\rangle + m^2\langle\hat{\Phi}\rangle + \frac{\lambda}{3!}\langle\hat{\Phi}^3\rangle = J(t) . \tag{3.6}$$

Since PGF's defined by Eq. (3.1) are simply connected with expectation values of products of fields, e.g.,

$$G(t_1,l)[-i\Gamma^2(l,m)]G(m,t_2) = \langle T\hat{\Phi}(t_1)\hat{\Phi}(t_2)\rangle - \phi(t_1)\phi(t_2) , \tag{3.7}$$

$$G(t_1,l)G(t_2,m)G(t_3,n)[i\Gamma^3(l,m,n)] = \langle T\hat{\Phi}(t_1)\hat{\Phi}(t_2)\hat{\Phi}(t_3)\rangle - [\langle T\hat{\Phi}(t_1)\hat{\Phi}(t_2)\rangle\phi(t_3) + \text{perm.}] - \phi(t_1)\phi(t_2)\phi(t_3) , \tag{3.8}$$

Eq. (3.3) is just the diagrammatic representation of Eq. (3.6). Here $G(t_1,t_2)$ denotes the connected propagator and $\phi(t) = \langle\hat{\Phi}(t)\rangle$. We use here matrix notation, i.e., integration is implied over the time variable appearing twice except for $G(t,t)$ which will denote the equal-time propagator.

Multiplying the Heisenberg equation by $\hat{\Phi}(t')$, performing time ordering, and taking its vacuum expectation value (VEV), we obtain

$$\phi(t')\partial_t^2\phi(t) + m^2\phi(t')\phi(t) + \frac{\lambda}{3!}\phi(t')\langle\hat{\Phi}^3\rangle - D^{-1}(t,l)G(l,t') + \frac{\lambda}{3!}\{3G(t,t')G(t,t) + 3G(t',l)G(t,m)G(t,n)[i\Gamma^3(l,m,n)]\phi(t) + G(t',l)G(t,m)G(t,n)G(t,k)[i\Gamma^3(m,n,p)][i\Gamma^3(k,p,l)] + G(t',l)G(t,m)G(t,n)G(t,k)[i\Gamma^4(l,m,n,k)]\} = \delta(t-t') . \tag{3.9}$$

Equation (3.4) is its proper part. Higher DSE's can be obtained by successive multiplications of the Heisenberg equations by field operators.

We shall now represent the time-dependent Gaussian approximation directly in terms of PGF's and show that the optimization conditions (2.35)–(2.39) are equivalent to a certain truncation of the first two DSE's.

In the Gaussian approximation $S[\phi]$ is given by Eq. (2.50) where G is a functional of $\phi(t)$ found from the gap equation (2.48). In order to recast the effective action as well as the gap equation (2.48) into a form more suitable for QFT we introduce the auxiliary quantity, which will be called the truncated propagator

$$iG_{\text{tr}}^{-1} = \Gamma^2_{\text{tr}} = \frac{\partial^2 S[\phi,G]}{\partial\phi(t)\partial\phi(t')} . \tag{3.10}$$

Note that the derivatives in Eq. (3.10) are partial functional derivatives (G is not differentiated) in contrast with Eq. (3.1) and thus the propagator is not equal to the full Gaussian two-point Green's function (GF).

Differentiating the explicit form of $S[\phi,G]$, Eq. (2.50), we obtain

$$\Gamma^2_{\text{tr}} = -\left[\partial_t^2 + \frac{\partial^2}{\partial\phi^2}\langle V\rangle\right]\delta(t-t') . \tag{3.11}$$

Using Eq. (2.48) and the property of Gaussian states

$$\frac{\partial^2}{\partial\phi^2}\langle V(\Phi)\rangle = 4\frac{\partial}{\partial G}\langle V(\Phi)\rangle ,$$

this is transformed into

$$\Gamma^2_{\text{tr}} = \left[-\partial_t^2 + \frac{1}{2G}\left[-\partial_t^2 G + \frac{1}{2}\frac{(\partial_t G)^2}{G} + 2G^{-3}\right]\right]\delta(t-t') . \tag{3.12}$$

Thus the truncated propagator is a Green's function of the differential equation

$$\left[\partial_t^2 - \frac{1}{2G}\left[-\partial_t^2 G + \frac{1}{2}\frac{(\partial_t G)^2}{G} + 2G^{-3}\right]\right]G_{\text{tr}}(t,t') = i\delta(t-t') . \tag{3.13}$$

This Green's function is

$$G_{\rm tr}(t,t')=\left[\theta(t-t')\exp\left[i\int_{t'}^{t}ds\,G^{-1}(s)\right]+\theta(t'-t)\exp\left[-i\int_{t'}^{t}ds\,G^{-1}(s)\right]\right]\times\frac{\sqrt{G(t)G(t')}}{2}\ . \tag{3.14}$$

In particular

$$G_{\rm tr}(t,t)=\frac{G(t)}{2}\ . \tag{3.15}$$

With this information at hand we rewrite the shift and gap equations in terms of $\phi(t)$ and $G_{\rm tr}(t,t')$:

$$\partial_t^2\phi+\frac{\partial}{\partial\phi}\langle V(\phi,G_{\rm tr}(t,t))\rangle=J\ , \tag{3.16}$$

$$\Gamma^2_{\rm tr}=\left[-\partial_t^2-\frac{\partial^2}{\partial\phi^2}\langle V(\phi(t),G_{\rm tr}(t,t))\rangle\right]\delta(t-t')\ . \tag{3.17}$$

This is a set of equations for $\phi(t)$ and $G_{\rm tr}(t,t')$ since $\Gamma^2_{\rm tr}$ is connected with $G_{\rm tr}$ via Eq. (3.10).

The effective action is

$$S[\phi(t),G_{\rm tr}(t,t')]=\int dt\,\tfrac{1}{2}(\partial_t\phi)^2-\langle V\rangle-\tfrac{1}{2}\partial_t^2G_{\rm tr}(t,t')_{t=t'}-\frac{i}{2}[\ln(G_{\rm tr}(t,t'))]_{t=t'}\ . \tag{3.18}$$

The vacuum energy is equal to $-\dot{S}$ for the translationally invariant configuration $\phi(t)=\phi$, $G_{\rm tr}(t,t')=G_{\rm tr}(t-t')$. It can be conveniently expressed as

$$E=\tfrac{1}{2}\int dk\,k^2G_{\rm tr}(k)+\frac{i}{2}\int dk\,\ln G_{\rm tr}(k)+\langle V\rangle\ . \tag{3.19}$$

When written in this form Eqs. (3.16) and (3.17) look very suggestive.

As an illustration we consider the anharmonic oscillator equation (3.2). In this case Eqs. (3.16) and (3.17) can be represented diagrammatically as

$$-D^{-1}\phi(t)+\frac{\lambda}{3!}\phi^3(t)+\frac{\lambda}{2}\phi(t)\ \bigcirc\ =J\ , \tag{3.20}$$

$$\Gamma^2_{\rm tr}(t,t')=D^{-1}(t-t')-\frac{\lambda}{2}\left[\phi^2(t)+\ \bigcirc\ \right]\delta(t-t') \tag{3.21}$$

with lines representing $G_{\rm tr}(t,t')$.

We immediately observe that they are the same as Eqs. (3.3) and (3.4) with terms containing three- and higher-point GF's omitted. Because of the factorization property of Gaussian states,[18]

$$\langle\hat{\Phi}^n\rangle=\phi^n+\frac{n(n-1)}{2}\phi^{n-2}\langle\hat{\Phi}^2\rangle+\frac{n(n-1)(n-2)(n-3)}{8}\phi^{n-4}\langle\hat{\Phi}^2\rangle^2+\cdots\ , \tag{3.22}$$

this feature does not depend on specific form of potential $V(\hat{\Phi})$. Thus we arrive at the conclusion that the optimization conditions (2.35)–(2.39) are equivalent to the truncation of the first two DSE's in which three- and higher-point PGF's are omitted.

It is interesting to remark that the same framework can also accommodate the classical approximation. In this case (for a normal-ordered Hamiltonian) the optimization conditions (2.13) and (2.14) are equivalent to the truncation of the first DS equation with terms containing two- and higher-point PGF's omitted.

Truncated DSE's [Eqs. (3.20) and (3.21)] enable us to express in principle the truncated propagator $G_{\rm tr}(t,t')$ as a functional of $\phi(t)$. This expression should be inserted into the Gaussian effective action Eq. (3.18) and the Gaussian GF's are obtained from it by functional differentiation. Actually, in order to calculate GF's we do not need the explicit expression for $S[\phi(t)]$. Indeed the left-hand side (LHS) of the shift equation is precisely the first functional derivative of $S[\phi(t)]$. Thus PGF's in Gaussian approximation are obtained by successive differentiation of the LHS of the shift equation.

IV. THE APPROXIMATION IN THE HEISENBERG PICTURE

In this section we describe the representation of this approximation in the Heisenberg picture of quantum mechanics. In this picture dynamics is given by the time dependence of the Heisenberg operators $\Phi(t)$ and $\Pi(t)$ which satisfy the Heisenberg equations

$$\dot{\Phi}=-i[\Phi,H]\ , \tag{4.1}$$

$$\dot{\Pi}=-i[\Pi,H]\ . \tag{4.2}$$

Let us choose in the Hilbert space a harmonic oscillator basis generated by the (time-independent) creation operator $a^\dagger$:

$$|n\rangle=\frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle\ ,\qquad a|0\rangle=0\ . \tag{4.3}$$

We approximate $\Phi(t)$ and $\Pi(t)$ by operators linear in a and $a^\dagger$ with time-dependent coefficients

$$\Phi_{\rm lin}(t)=\phi(t)+\frac{1}{\sqrt{2}}[v^*(t)a+v(t)a^\dagger]\ , \tag{4.4}$$

$$\Pi_{\rm lin}(t)=\pi(t)+\frac{i}{\sqrt{2}}[u^*(t)a-u(t)a^\dagger]\ . \tag{4.5}$$

Here $\phi(t)$ and $\pi(t)$ are real and $v(t)$ and $u(t)$ complex functions. In order for $\Phi_{\rm lin}(t)$ and $\Pi_{\rm lin}(t)$ to satisfy canonical commutation relations at all times, the following relation between the coefficients must hold:

$$\tfrac{1}{2}(uv^*+vu^*)=1\ . \tag{4.6}$$

Equations (4.4)–(4.6) are equivalent to the statement that the time evolution is approximated by a Bogoliubov-Valatin transformation:

$$\Phi(t)=U\Phi(0)U^{-1}\ ,$$
$$\Pi(t)=U\Pi(0)U^{-1}\ , \tag{4.7}$$
$$U=\exp[iQ(a,a^{\dagger})]\ ,$$

where Q is a general time-dependent Hermitian operator quadratic in a and $a^{\dagger}$.

To obtain equations for coefficients u, v, ϕ, and π we proceed by substituting Eqs. (4.4) and (4.5) into the Heisenberg operator equations (4.1) and (4.2) and retaining in their RHS only terms linear in a and $a^{\dagger}$ in the normal-ordered expression. As an example consider an anharmonic oscillator Eq. (3.2). In this case the approximate equations of motion for ϕ, π, u, and v are

$$\dot{\phi}=\pi\ , \tag{4.8}$$
$$\dot{\pi}=-m^2\phi-\frac{\lambda}{3!}\phi^3-\frac{\lambda}{4}\phi v^* v\ , \tag{4.9}$$
$$\dot{u}=i\left[m^2 v+\frac{\lambda}{2}\phi^2 v+\frac{\lambda}{4}v^* v^2\right]\ , \tag{4.10}$$
$$\dot{v}=iu\ . \tag{4.11}$$

Let us discuss now the connection of this procedure with the time-dependent Gaussian approximation in the Schrödinger picture. The correspondence between the two representations is given by

$$\langle\psi(t)|O|\psi(t)\rangle=\langle\psi(0)|O(t)|\psi(0)\rangle\ , \tag{4.12}$$

where O is an arbitrary operator and $|\psi(t)\rangle=U(t)|\psi(0)\rangle$.

The unitary operator U [Eq. (4.7)] transforms a Gaussian state into another Gaussian state. For a Gaussian state, Eq. (2.25), expectation values of the linear and quadratic operators are given by Eqs. (2.26)–(2.29). On the other hand taking $|\psi_0\rangle$ to be an eigenstate of the operator a and calculating the same expectation values for the linearized operators $\Phi_{\rm lin}(t)$, $\Pi_{\rm lin}(t)$ Eqs. (4.4) and (4.5) we arrive at the following relations between u, v, and G,I:

$$vv^*=G\ ,$$
$$u^*u=G^{-1}+IGI\ . \tag{4.13}$$

One can easily verify that with these relations Eqs. (4.8)–(4.11) are equivalent to Eqs. (2.43)–(2.46) for the anharmonic oscillator. Moreover the two approaches are generally equivalent because of the following. Satisfying the Heisenberg equations up to terms linear in a and $a^{\dagger}$ is equivalent to equating to zero their expectation values in $|0\rangle$ state and their matrix elements between $|0\rangle$ and $|1\rangle$. However, taking the matrix element of an operator between $|0\rangle$ and $|1\rangle$ is equivalent to the multiplication of an operator by Φ or Π and taking its expectation value in $|0\rangle$. Therefore the linearized Heisenberg equations are completely equivalent to Eqs. (2.35)–(2.38).

To pass to the DSE representation let us introduce an auxiliary function which we shall call the truncated propagator

$$G^{\rm tr}(t,t')\equiv\langle 0|T\Phi_{\rm lin}(t)\Phi_{\rm lin}(t')|0\rangle$$
$$=\tfrac{1}{2}[\theta(t-t')v(t)v^*(t')+\theta'(t'-t)v(t')v^*(t)]\ , \tag{4.14}$$

where $\Phi_{\rm lin}$ is defined by Eq. (4.4). It can be easily seen that because of the constraint (4.6) and equation of motion (4.11) $G_{\rm tr}$ can be rewritten in the form of Eq. (3.14). We shall replace now the first-order Eqs. (4.8)–(4.11) by two second-order equations for $\phi(t)$ and $G^{\rm tr}(t,t')$. To this end we have to obtain the linearized second-order equation for $\Phi_{\rm lin}(t)$. For the Hamiltonian Eq. (3.2) we find

$$\ddot{\phi}+\frac{1}{\sqrt{2}}(\ddot{v}a+\ddot{v}^*a^{\dagger})$$
$$=-m^2\phi-\frac{\lambda}{3!}\phi^3-\frac{\lambda}{4}\phi vv^*$$
$$-\frac{1}{\sqrt{2}}\left[\left[m^2 v+\frac{\lambda}{2}\phi^2 v+\frac{\lambda}{4}v^2 v^*\right]a+\text{H.c.}\right]\ . \tag{4.15}$$

We sandwich Eq. (4.15) between two $|0\rangle$ states, then multiply Eq. (4.15) by $\Phi_{\rm lin}(t')$, take the time-ordered product and again sandwich it between $|0\rangle$ states. The resulting equations coincide with the truncated DSE's (3.20) and (3.21).

V. GENERALIZATION TO FIELD THEORY

The method can be easily extended to QFT. One should replace the operators $\hat{\Phi}(t)$ and $\hat{\Pi}(t)$ by an infinite set of operators $\hat{\Phi}(\mathbf{x},t)$ and $\hat{\Pi}(\mathbf{x},t)$. The Gaussian state in the field basis is described by the wave functional

$$\Psi[\Phi(\mathbf{x})]=N\exp(-\tfrac{1}{2}\{[\Phi(\mathbf{x})-\phi(\mathbf{x})]\Omega(\mathbf{x},\mathbf{y})[\Phi(\mathbf{y})-\phi(\mathbf{y})]$$
$$-i\pi(\mathbf{x})[\Phi(\mathbf{x})-\phi(\mathbf{x})]+iA(\mathbf{x})\})\ , \tag{5.1}$$

where N is a normalization factor and $\Omega(\mathbf{x},\mathbf{y})$ is a complex function.

The Schrödinger and Heisenberg picture approach can be transferred verbatim to QFT; however, in relativistic QFT the DSE approach has a decisive advantage. Since PGF's are Lorentz covariant objects it is preferable to represent the method by a set of covariant equations. It seems proper to call the generalization of the Gaussian approximation based on truncated covariant DSE's the covariant Gaussian approximation (CGA).

We shall expose the general procedure with the help of an example of the simplest field theory: self-interacting scalar field

$$L=\tfrac{1}{2}(\partial_\mu\Phi)^2-\tfrac{1}{2}m^2\Phi^2-\frac{\lambda}{4!}\Phi^4\ . \tag{5.2}$$

The first two DSE's are the same as given in Eqs. (3.3) and (3.4) (the dimensionality is not important) where now

$$D^{-1}(x,y)=-(\partial_\mu\partial_\mu+m^2)\delta^4(x-y)\ . \tag{5.3}$$

The truncated shift and gap equations are analogously

given by Eqs. (3.20) and (3.21). It is not an easy task to solve the gap equation for $G_{\rm tr}(x,y)$ as function of an arbitrary $\phi(x)$. However to calculate the n-point GF only $G_{\rm tr}$ and its finite-order derivatives evaluated in a vacuum [that is, for $\phi(x)=\phi$] are required.

The truncated propagator in a vacuum is easily found by solving the gap equation for constant ϕ. (In translationally invariant field theory $\langle\Phi\rangle$ must be position independent.) The derivatives of $G_{\rm tr}$ are found by the differentiation of the LHS of the gap equation.

We remark that since the J appears in the RHS of the shift equation the Gaussian GF can be formally represented as

$$\Gamma^n=-\frac{\delta^{n-1}J}{\delta\phi_1\cdots\delta\phi_{n-1}}\ , \tag{5.4}$$

an expression familiar from the functional integral approach to QFT.

We shall sketch now the calculation of a two-point PGF. The gap equation in the translationally invariant case [$\phi(x)=\phi$, $G_{\rm tr}(x,y)=G_{\rm tr}(x-y)$] becomes very simple. In Fourier space it reads

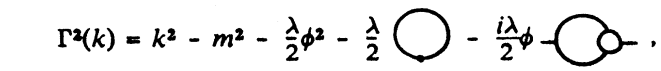
$$\Gamma^2_{\rm tr}(k)=k^2-m^2-\frac{\lambda}{2}\phi^2-\frac{\lambda}{2}\int\frac{d^4k}{(2\pi)^4}G_{\rm tr}(k)\ . \tag{5.5}$$

The solution obviously is of the form[21]

$$\Gamma^2_{\rm tr}(k)=k^2-M^2\ . \tag{5.6}$$

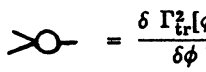
Thus the problem of solving a complicated equation for $G_{\rm tr}[\phi(x)]$ is reduced to a simple algebraic equation for M^2. Since in this paper we are interested only in presenting the general procedure rather than in calculational details we shall not solve explicitly this algebraic equation. This calculation is performed in detail in the companion paper.[22]

The two-point PGF is given by the functional derivative of the LHS of Eq. (3.20):



(5.7)

where lines denote the truncated propagator $G_{\rm tr}$ and empty circles denote the functional derivative of $\Gamma^2_{\rm tr}[\phi]$:

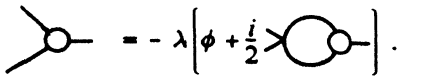


(5.8)

The diagrammatic rules of differentiation are very simple. Since

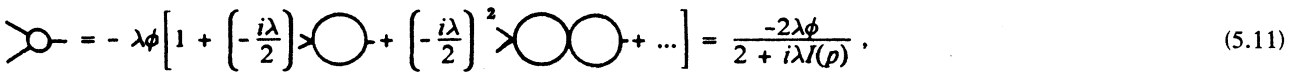
$$\frac{\partial G(x,y)}{\partial\phi(z)}=G(x,u)\left[-\frac{\partial G^{-1}(u,v)}{\partial\phi(z)}\right]G(v,y)=G(x,u)\left[\frac{i\,\partial\Gamma^2(u,v)}{\partial\phi(z)}\right]G(v,y) \tag{5.9}$$

we can represent it diagrammatically as shown on Fig. 3. This auxiliary function is found by differentiating the gap equation (3.21) according to these rules:



(5.10)

By the repeated use of Eq. (5.9) one obtains a geometric series which can be summed:

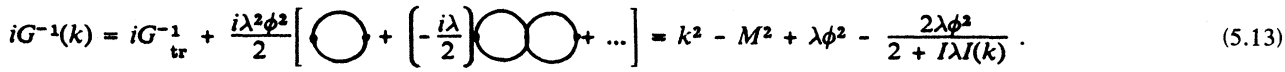


(5.11)

where

$$I(p)=-\int\frac{d^4k}{(2\pi)^4}\frac{1}{(k^2-M^2)[(k+p)^2-M^2]}\ . \tag{5.12}$$

Inserting this into Eq. (5.7) we obtain the diagrammatic representation as well as the analytic expression for the full Gaussian inverse propagator:



(5.13)

The value of ϕ should be found by solution of the (translationally invariant) shift equation (3.21) (Ref. 22).

Thus the propagator has a simple diagrammatic representation in the CGA. It is shown in Ref. 22 that the same is true for any GF.

To conclude this section we summarize the algorithm for calculation of GF's in the covariant Gaussian approximation: (1) Draw first two DSE's with terms containing

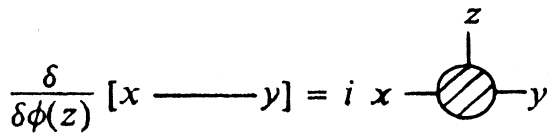


FIG. 3. Diagrammatic differentiation rules for Green's functions.

three- and higher-point PGF's omitted; (2) solve those equations for $G_{\rm tr}$ and ϕ in the translationally invariant case (when they become algebraic equations); (3) differentiate the LHS of the shift equation in order to obtain Γ^n; (4) differentiate the LHS of the gap equation; (5) find functional derivatives of $G_{\rm tr}$ by summing geometric series; (6) insert the derivatives of $G_{\rm tr}$ found in (5) into Γ^n.

VI. DISCUSSION

In this work we have presented the time-dependent generalization of the Gaussian approximation. It was shown that in the Schrödinger picture it has a simple intuitive meaning. It approximates a vacuum of Hamiltonian H evolving in time under the influence of external source J by Gaussian wave packet at all times. In the Heisenberg picture we showed that the approximation is the time-dependent generalization of the Bogoliubov-Valatin method.[3] The evolution operator is approximated by the exponent of time-dependent form quadratic in canonical creation and annihilation operators.

In the framework of the DSE formulation of quantum theory, more convenient in QFT, the approximation was shown to be equivalent to initial truncation of first two DSE's with subsequent differentiation of the effective action in order to find the PGF. The effective action in this approximation is a nonlocal functional of the expectation value of the field $\phi(t)$. The exact effective action of the quantum theory is also a nonlocal functional and thus this is a distinct advance over the classical approximation.

It is also possible to consider the CGA from the functional integral formulation point of view. In this framework there exists an expression for the effective action as a functional of ϕ and G (Ref. 23). The Gaussian effective action of Eq. (3.18) is the approximation to this Cornwall, Jackiw, Tomboulis effective action in which only the nonoverlapping diagrams are retained.

It is not an easy task to find the effective action explicitly for any value of $\phi(t)$ since this involves solution of a partial differential equation with an arbitrary function as a coefficient. However, it turns out that in order to calculate the PGF there is no need in the explicit form of effective action. Hence only a simple algebraic equation must be solved. We presented an example of a calculation of this sort. It involves the summation of the infinite sum of perturbation-theory-like graphs. We have seen that to calculate Γ^2 in the ϕ^4 theory those graphs can be easily summed. This is due to the fact that they contain only nonoverlapping diagrams. This turns out to be the general feature of the approximation. It is shown in the companion paper[22] by applying the algorithm formulated at the end of Sec. IV that any PGF is given by the sum of all nonoverlapping diagrams drawn in accord with Feynmann rules.

We should like to stress that although the CGA has a simple diagrammatic representation, it is essentially a nonperturbative method. There are two nonperturbative aspects in it. First, it performs a partial summation of an infinite number of diagrams. The second and less trivial aspect is that the diagrams are not exactly those of naive perturbation theory: the propagator running in them is not the bare propagator of the quadratic part of the initial Lagrangian, but $G_{\rm tr}$, which is a solution of the gap equation. Thus here the essential nonperturbative step precedes the diagrammatic calculation.

It is interesting to note that in some sense the CGA is a natural extension of the classical approximation. The classical approximation corresponds to the simplest possible truncation of the DSE—only the first DSE is considered and in it all terms containing two- and higher-point functions are omitted. The CGA considers first two DSE's leaving out Γ^n for $n>2$. This is suggestive of the possibility of generalizing the approach. As the next step one could try to consider first three DSE's with ϕ, $G_{\rm tr}$, and $\Gamma^3_{\rm tr}$ as unknowns. This would correspond to a variational principle on the states with different factorization properties than Gaussians. For those states $\langle\Phi^n\rangle$ can be expressed via three independent quantities: $\langle\Phi\rangle$, $\langle\Phi^2\rangle$, and $\langle\Phi^3\rangle$. Such calculation can be considered as the first post-Gaussian correction and it can give the evaluation of the precision of the Gaussian approximation. It seems, however, that this step is far from being simple and it would necessitate solution of a genuine integral equation.

Note added. After this work was completed we became aware of the following paper: R. Jackiw and A. Kerman, Phys. Lett. **71A**, 158 (1979). It is shown there that the effective action of a quantum theory can be represented [for slowly varying $J(t)$] as

$$S_{\rm eff}[\phi(t)]=\int dt\langle i\partial_t - H\rangle ,$$

where the expectation value on the RHS is taken in the normalized time-dependent states for which $\langle\hat{\Phi}\rangle=\phi(t)$ and all other parameters are chosen so as to extremize the RHS for a given $\phi(t)$.

One can show that the Gaussian effective action equation (2.50) is an approximation to this expression where the extremization is performed not on the whole Hilbert space but on the states that have a form of a Gaussian wave packet, Eq. (2.25).

We are grateful to Professor R. Jackiw for bringing this reference to our attention.

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APPENDIX: THE DEPENDENCE OF THE APPROXIMATION ON THE CHOICE OF BASIS

We should like to comment here on the dependence of the Gaussian approximation on the basis in Hilbert space in which one is working. It is clear that such a dependence exists. The Gaussian approximation searches for the approximate vacuum among the states connected by a most general Bogoliubov-Valatin transformation, which is linear in terms of Φ and Π. Suppose one chooses to work in a different basis:

$$\Phi' = U\Phi U^{-1} = f(\Phi,\Pi) , \tag{A1}$$

$$\Pi' = U\Pi U^{-1} = g(\Phi,\Pi) , \tag{A2}$$

where the functions f and g are not linear. The states that are Gaussian wave packets in the basis Φ,Π will not be Gaussians in the basis Φ',Π'. As a consequence Gaussian approximations in those two bases are certainly not equivalent. In the framework of the DSE approach this means that one obtains different truncated DSE. As an example an anharmonic-oscillator Hamiltonian in different bases:

$$H = \tfrac{1}{2}\Pi^2 + \frac{\alpha}{2}(\Pi\Phi^2 + \Phi^2\Pi) + \tfrac{1}{2}m^2\Phi^2 + \frac{\lambda}{4!}\Phi^4 . \tag{A3}$$

The transformation changing this Hamiltonian into the Hamiltonian of Eq. (3.2) is

$$\Phi' = U\Phi U^{-1} = \Phi , \tag{A4}$$

$$\Pi' = U\Pi U^{-1} = \Pi + \alpha\Phi^2 , \tag{A5}$$

where

$$U = \exp(i\alpha\Phi^3) . \tag{A6}$$

This is a simple transformation which changes only the phase of a wave function in the Φ representation. It is, however, nonlinear. We shall now obtain the truncated DSE in this basis. The Heisenberg operator equations are

$$\dot{\Phi} = \Pi + \alpha\Phi^2 , \tag{A7}$$

$$\dot{\Pi} = -\alpha\{\Pi,\Phi\} - m^2\Phi - \frac{\lambda}{3!}\Phi^3 . \tag{A8}$$

It should be noted that now the relation between Π and $\dot{\Phi}$ is not linear. This means that in going to the second-order equations, Eq. (A7) is not completely untouched but only its linearized version is preserved. Performing again the steps that lead to the DSE we obtain

$$\ddot{\phi}(t) + m^2\phi(t) + \frac{\lambda - 12\alpha^2}{3!}\phi^3(t) + \frac{\lambda - 12\alpha^2}{2}\phi(t) \bigcirc = J(t), \tag{A9}$$

$$iG^{-1}_{\mathrm{tr}}(t,t') = \left[-\partial^2_t - m^2 - \frac{\lambda - 12\alpha^2}{2}\phi^2(t) + \frac{\lambda - 4\alpha^2}{2} \bigcirc\right]\delta(t-t') . \tag{A10}$$

The equations explicitly depend upon the parameter of the transformation α. Here two comments are in order. First one can enlarge the set of trial states for a variational calculation by considering α as an additional variational parameter and minimizing energy also with respect to it. For the Hamiltonian of the anharmonic oscillator this transformation cannot improve the vacuum state. The reason is that if the dependence on the canonical momentum in the Hamiltonian is just of the form Π^2, addition of phase to the trial ground state can only increase its energy. However, for other Hamiltonians the situation is different. For example, interaction of a particle with a magnetic field is linear in Π. The same is true for QFT of gauge fields interacting with scalars. In these cases transformations of this kind have a chance to improve the Gaussian variational calculation. Similar transformations were recently considered in the context of variational calculations in Yang-Mills theory.[24] Second, this stresses the fact that the CGA is defined in Sec. III is equivalent to time-dependent Gaussian approximation only in a certain basis. In the case of a field theory, the Gaussian variational approach in a different basis which does not coincide with the CGA may lead, for instance, to noncovariant equations for the condensates. This must be kept in mind when one tries to improve on the CGA by transformations of the type of Eq. (A6).

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