Adiabatic holonomy and evolution of fermionic coherent state

Sumiyoshi Abe
Department of Physics and Atomic Energy Research Institute, College of Science and Technology, Nihon University, Kanda-Surugadai, Chiyoda-ku, Tokyo 101, Japan
(Received 9 August 1988)

The adiabatic evolution of Pauli spin in a slowly varying external magnetic field is studied by using the fermionic coherent state defined by the Grassmann-variant unitary displacement from the stable adiabatic vacuum. This manifests the interrelation between Berry's phase and Hannay's angle in a peculiar manner. The gauge field strength giving rise to the geometrical phase is found to be that of a monopole of a Grassmann charge. We quantize this charge based on the Bohr-Sommerfeld-type rule for Grassmannian systems and then correctly obtain Berry's phase.

I. INTRODUCTION

An important modification in the quantum adiabatic theorem[1] has been recently made by Berry[2] by grasping geometrically the space R that consists of a set of external adiabatic parameters. He showed that, in certain systems, the usual dynamical phase factor $\exp[-i\int_0^T dt\, E_n(t)]$ in the energy eigenstate must be accompanied by an additional geometrical phase factor $\exp[i\gamma_n(C)]$ called Berry's phase, where C is the closed contour of an adiabatic excursion in R. Actually this new phase has been observed in some laboratories.[3] A mathematical interpretation has been given by Simon,[4] who has shown that it is the holonomy in a Hermitian line bundle.

An interesting point is that Berry's phase has its classical counterpart. Hannay[5] has found that, in certain integrable systems, the usual angle variable $\theta=\int_0^T dt\, \partial H/\partial I$ in the classical adiabatic theorem[6] must be shifted by an extra angle $\Delta\theta(C)$ called Hannay's angle after an adiabatic excursion along C. Gozzi and Thacker[7] have shown that this angle can be understood as the holonomy in a phase-space bundle.

This indicates that the adiabatic holonomy has a certain classical-quantum correspondence. Actually this has been established based on the semiclassical approximation.[8]

In this paper, we study Pauli spin in a slowly varying external magnetic field playing a role of the adiabatic parameter. As stressed in Ref. 9, this system is a typical and important example that develops the nontrivial adiabatic holonomy. It has been studied only in the context of Berry's phase in energy eigenstates[2] and Hannay's angle in the classical Grassmann spin model.[9] Here we calculate the geometrical phase factor relative to the fermionic coherent state of the system. We find that Berry's gauge field strength is that of the monopole of a Grassmann charge. Such a charge is quantized based on the recently proposed Bohr-Sommerfeld-type rule for Grassmannian systems.[10]

We expect that our discussion gives some new insight into the correspondence in the holonomy effect, since coherent states have properties nearest to those of classical ones.[11,12] (A similar motivation will be found in Refs. 13 and 14.) Although we present some new relations, Berry's phase and Hannay's angle of the Pauli spin model themselves have been previously obtained. There are, therefore, no predictions of essentially new effects from the practical viewpoint. However, from the physical viewpoint, there is significance in recognizing the classical-quantum correspondence in various theoretical aspects. To the best of our knowledge, this is the first consistent direct semiclassical approach to fermions. In addition, our calculations may also have the meaning of a fermionic counterpart of the traditional investigations[15] on the evolution of coherent states, because they seem to have been done mainly for bosonic systems.

This paper is organized as follows. In Sec. II we briefly survey the Grassmann spin model. Our main results are proposed in Sec. III, where the adiabatic evolution of the fermionic coherent state is calculated and the geometrical phase and Hannay's angle are obtained. In Sec. IV the Bohr-Sommerfeld-type rule is examined in order to quantize the Grassmann charge appearing in Sec. III. Berry's phase is found to be reproduced there. Section V contains concluding remarks.

II. GRASSMANN SPIN IN A SLOWLY VARYING MAGNETIC FIELD

We start our discussion with the standard classical model of a Grassmann spin that becomes the Pauli spin after quantization. The Lagrangian of such a spin in a slowly varying uniform external magnetic field reads

$$L=\frac{i}{2}\xi\cdot\dot{\xi}+\frac{i}{2}\mathbf{B}(t)\cdot(\xi\times\xi)\ , \tag{1}$$

where the overdot denotes the time derivative. The components ξ_i $(i=1,2,3)$ of the vector ξ are Grassmannian: $\xi_i\xi_j+\xi_j\xi_i=0$. The definition of canonical momenta Π_i conjugate to ξ_i leads to the constraints $\chi_i\equiv\Pi_i+(i/2)\xi_i\approx 0$, which are known to be second class in Dirac's terminology. They are conveniently treated by Dirac's generalized canonical theory extended to dynamical systems including the Grassmann variables.

The Hamiltonian is

$$H(t)=\dot{\xi}\cdot\Pi-L=-\frac{i}{2}\mathbf{B}(t)\cdot(\xi\times\xi)\ , \tag{2}$$

which has the explicit time dependence due to **B**. After canonical quantization with the Dirac-brackets formalism, the operators $\hat{\xi}_i$ are shown to generate the Clifford algebra

$$\{\hat{\xi}_i,\hat{\xi}_j\}=\delta_{ij}\ , \tag{3}$$

and therefore they can be represented irreducibly by the Pauli matrices as $\hat{\xi}=\sigma/\sqrt{2}$. Since the intrinsic angular momentum, the generator of rotation, is given by $\hat{\mathbf{S}}=\hat{\xi}\times\hat{\Pi}=-(i/2)\hat{\xi}\times\hat{\xi}$, the quantum Hamiltonian is

$$\hat{H}(t)=\mathbf{B}(t)\cdot\hat{\mathbf{S}}=\tfrac{1}{2}\mathbf{B}(t)\cdot\sigma\ , \tag{4}$$

which describes the Pauli spin in the magnetic field.

In their study of Hannay's angle of the classical system mentioned above, Gozzi and Thacker[9] decomposed the variables in Eq. (1) into the normal modes in such a way that the tensor dual to B_k, i.e., $\tilde{B}_{ij}=\epsilon_{ijk}B_k$ becomes diagonal: $B_d=S^\dagger\tilde{B}S=\mathrm{diag}(-iB,iB,0)$, where $B=|\mathbf{B}|$ and S is the complex unitary matrix explicitly given by

$$S=\begin{bmatrix}\dfrac{B_1B_3+iB_2B}{B\sqrt{2(B_1^2+B_2^2)}} & \dfrac{B_1B_3-iB_2B}{B\sqrt{2(B_1^2+B_2^2)}} & \dfrac{B_1}{B}\\ \dfrac{B_2B_3-iB_1B}{B\sqrt{2(B_1^2+B_2^2)}} & \dfrac{B_2B_3+iB_1B}{B\sqrt{2(B_1^2+B_2^2)}} & \dfrac{B_2}{B}\\ -\dfrac{\sqrt{B_1^2+B_2^2}}{\sqrt{2}B} & -\dfrac{\sqrt{B_1^2+B_2^2}}{\sqrt{2}B} & \dfrac{B_3}{B}\end{bmatrix}. \tag{5}$$

Then in normal coordinates

$$\psi_i=(S^\dagger)_{ij}\xi_j\ , \tag{6}$$

Eq. (1) takes the form

$$L=\frac{i}{2}\psi_i^*\dot{\psi}_i+\frac{i}{2}\psi_i^*(S^\dagger\dot{S})_{i,j}\psi_j-\tfrac{1}{2}B(t)(\psi_2^*\psi_2-\psi_1^*\psi_1)\ . \tag{7}$$

As seen from Eq. (5), $\psi_2=\psi_1^*$ and $\psi_3=\psi_3^*$. The second term is negligible relative to the third one in the adiabatic approximation. So the Lagrangian can be classically written as

$$L=\psi^*\left[i\frac{d}{dt}-B(t)\right]\psi+\frac{i}{2}\psi_3\dot{\psi}_3 \tag{8}$$

with $\psi\equiv\psi_2$. It is interesting to see that the first term is just the $(0+1)$-dimensional Dirac theory (the fermionic oscillator) with a variable mass.

III. ADIABATIC EVOLUTION OF FERMIONIC COHERENT STATE

We proceed to constructing the fermionic coherent state in the quantized system. For this purpose, we introduce the operators b, $b^\dagger$, and c by

$$\hat{\xi}_1=\frac{1}{\sqrt{2}}(b+b^\dagger)\ ,\ \hat{\xi}_2=\frac{i}{\sqrt{2}}(b-b^\dagger)\ ,\ \hat{\xi}_3=c=c^\dagger\ . \tag{9}$$

They satisfy the algebra

$$\{b,b^\dagger\}=\{c,c\}=1\ ,\ \ \{b,b\}=\{b,c\}=0\ . \tag{10}$$

$b^\dagger$ and b play roles of the creation and annihilation operators of a fermionic oscillator, respectively. [In the matrix notation, $b^\dagger=\sigma_+$ and $b=\sigma_-$ with $\sigma_\pm=\frac{1}{2}(\sigma_1\pm i\sigma_2)$, and the Clifford number c is σ_3.]

Since we shall work in the Schrödinger picture, the operators in Eq. (9) do not evolve in time. On the other hand, the operators transformed by Eq. (6) have the time dependence through B. We write them simply as $\hat{\psi}_1=\hat{\psi}_2^\dagger=b^\dagger(t)$, $\hat{\psi}_2=b(t)$, and $\hat{\psi}_3=c(t)=c^\dagger(t)$, respectively. For later convenience, we give the explicit relation

$$\begin{bmatrix}b\\ b^\dagger\\ c\end{bmatrix}=\Sigma\begin{bmatrix}b(t)\\ b^\dagger(t)\\ c(t)\end{bmatrix}\ , \tag{11}$$

where the unitary matrix Σ similar to S in Eq. (5) is

$$\Sigma=\begin{bmatrix}\dfrac{(B_1-iB_2)(B+B_3)}{2B\sqrt{B_1^2+B_2^2}} & -\dfrac{(B_1-iB_2)(B-B_3)}{2B\sqrt{B_1^2+B_2^2}} & \dfrac{B_1-iB_2}{\sqrt{2}B}\\ -\dfrac{(B_1+iB_2)(B-B_3)}{2B\sqrt{B_1^2+B_2^2}} & \dfrac{(B_1+iB_2)(B+B_3)}{2B\sqrt{B_1^2+B_2^2}} & \dfrac{B_1+iB_2}{\sqrt{2}B}\\ -\dfrac{\sqrt{B_1^2+B_2^2}}{\sqrt{2}B} & -\dfrac{\sqrt{B_1^2+B_2^2}}{\sqrt{2}B} & \dfrac{B_3}{B}\end{bmatrix}. \tag{12}$$

Therefore the transformed operators obviously satisfy the algebra isomorphic to Eq. (10):

$$\{b(t),b^\dagger(t)\}=\{c(t),c(t)\}=1\ ,$$
$$\{b(t),b(t)\}=\{b(t),c(t)\}=0\ . \tag{13}$$

This holds only for the equal-time anticommutation relations. In terms of these operators, the quantum Hamiltonian associated with Eq. (7) is expressed as

$$\hat{H}(t)=B(t)[b^\dagger(t)b(t)-\tfrac{1}{2}]\ . \tag{14}$$

The second term in the brackets is nothing but the zero-point oscillation energy.

Now we define the time-dependent coherent state by

$$b(t)|\zeta(t)\rangle=\zeta(t)|\zeta(t)\rangle\ . \tag{15}$$

The eigenvalue $\zeta(t)$ is necessarily Grassmannian due to the nilpotency of $b(t)$. This state is constructed from the instantaneous vacuum $|0)_t$ satisfying $b(t)|0)_t=0$ by the unitary displacement operator in the following way:

$$|\zeta(t))=D(\zeta(t))|0)_t\ , \tag{16}$$

$$D(\zeta(t))=\exp[b^\dagger(t)\zeta(t)+b(t)\zeta^*(t)]\ . \tag{17}$$

The unitarity of D is a simple result of the convention: $(b^\dagger\zeta)^\dagger=\zeta^*b=-b\zeta^*$. [Note that our definition (15)–(17) is slightly different from that in Ref. 17.]

We must state here the notion of the adiabatic evolution of the system. Let us first define the stable adiabatic vacuum as

$$|0)_t=|0)_{t=0}\equiv|0)\ . \tag{18}$$

This is equivalent to the requirement that the creation and annihilation operators are not mixed with each other through their time evolution. This is indeed the case, since, from Eq. (12), we can find the following:

$$\frac{\partial}{\partial t}\begin{bmatrix} b(t)\\ b^\dagger(t)\\ c(t)\end{bmatrix}=\Theta\begin{bmatrix} b(t)\\ b^\dagger(t)\\ c(t)\end{bmatrix}=\dot{\mathbf{B}}(t)\cdot\nabla_B\begin{bmatrix} b(t)\\ b^\dagger(t)\\ c(t)\end{bmatrix}\ , \tag{19}$$

$$\Theta=\dot{\Sigma}^\dagger\Sigma=[\dot{\mathbf{B}}(t)\cdot\nabla_B\Sigma^\dagger]\Sigma=\begin{bmatrix} X & 0 & -Y^*\\ 0 & X^* & -Y\\ Y & Y^* & 0\end{bmatrix}\ , \tag{20a}$$

$$X=\frac{i(B_1\dot{B}_2-B_2\dot{B}_1)B_3}{B(B_1^2+B_2^2)}\ , \tag{20b}$$

$$Y=\frac{(B_1B_3-iB_2B)\dot{B}_1+(B_2B_3+iB_1B)\dot{B}_2-(B_1^2+B_2^2)\dot{B}_3}{B^2\sqrt{2(B_1^2+B_2^2)}}\ . \tag{20c}$$

Next let us consider the unitary time-evolution operator formally given by

$$U(t,0)=T\exp\left[-i\int_0^t ds\,\hat{H}(s)\right]\ . \tag{21}$$

The chronological symbol is required, since $[\hat{H}(t),\hat{H}(s)]\neq 0$ in general. However it can be discarded in the adiabatic approximation. To see this simply, let us employ the Hamiltonian (4), which gives rise to $[\hat{H}(t),\hat{H}(s)]=(i/2)[\mathbf{B}(t)\times\mathbf{B}(s)]\cdot\sigma$. For $s=t+\Delta t$, $\mathbf{B}(s)\simeq\mathbf{B}(t)+\Delta t\dot{\mathbf{B}}(t)$, and therefore the right-hand side of the commutation relation is second-order infinitesimal.

Therefore, using the evolution operator associated to Eq. (14), we obtain the adiabatic evolution

$$\begin{aligned}(\zeta(t)|b(0)|\zeta(t))&=(\zeta(0)|U^\dagger(t,0)b(0)U(t,0)|\zeta(0))\\ &=\zeta(0)\exp\left[-i\int_0^t ds\,B(s)\right] &(22a)\\ &=\zeta(0)\exp(-iBt)\ , &(22b)\end{aligned}$$

where B in the last line is understood as the time-averaged value of $B(t)$. This expectation value behaves just like the classical motion described by the Euler-Lagrange equation for ψ in Eq. (8). This is the reason why the coherent state is said to have properties nearest to the classical ones. Thus, for example, we have

$$\zeta(t)=\zeta(0)\exp(-iBt)\ , \tag{23a}$$

$$\zeta^*(t)\zeta(t)=\zeta^*(0)\zeta(0)\ . \tag{23b}$$

Equation (23b) shows that the quadratic form $\zeta^*\zeta$ is a constant of motion.

The state $|\zeta(t))$ itself cannot be a solution of the Schrödinger equation

$$i\frac{\partial}{\partial t}|\Psi(t)\rangle=\hat{H}(t)|\Psi(t)\rangle \tag{24}$$

with the Hamiltonian (14). However we can easily construct such a state $|\zeta(t)\rangle$ by the U(1) *gauge transformation*

$$|\zeta(t)\rangle=\exp[i\phi(t)]|\zeta(t))\ , \tag{25}$$

provided that the phase ϕ obeys the auxiliary equation

$$d\phi(t)=(\zeta(t)|i\partial/\partial t|\zeta(t))dt-(\zeta(t)|\hat{H}(t)|\zeta(t))dt\ . \tag{26}$$

The first term in the right-hand side is written in terms of the gauge potential A as

$$A=\mathbf{A}\cdot d\mathbf{B}=(\zeta(t)|i\nabla_{\mathbf{B}}|\zeta(t))\cdot d\mathbf{B}\ . \tag{27}$$

A is called Berry's connection one-form, which defines the parallel transport of the coherent state along the contour in R, the parameter **B**-space. Imposing the initial condition $\phi(0)=0$ [i.e., $|\zeta(0))=|\zeta(0)\rangle$], we integrate Eq. (26) as follows:

$$\phi(t)=\int_0^t ds[\dot{\mathbf{B}}(s)\cdot(\zeta(s)|i\nabla_B|\zeta(s))-H_{\rm cl}(s)+\tfrac{1}{2}B(s)]\ , \tag{28}$$

$$H_{\rm cl}(s)=B(s)\zeta^*(s)\zeta(s)\ . \tag{29}$$

Here $H_{\rm cl}(s)$ is the classical Hamiltonian associated with Eq. (8) with $\psi=\zeta$.

We are now interested in the physical situation originally stated by Berry: the parameter B travels along a closed contour C in R with the period T. Then the geometrical part of the phase for such a process is

$$\begin{aligned}\gamma(C)&=\int_0^T dt\,\dot{\mathbf{B}}(t)\cdot(\zeta(t)|i\nabla_B|\zeta(t))\\ &=\oint_C d\mathbf{B}\cdot\mathbf{A}\\ &=\int\int_S d\mathbf{S}\cdot\nabla_B\times\mathbf{A}\ , &(30)\end{aligned}$$

where the surface S has C as its boundary $\partial S=C$. Using Eqs. (16), (17), and (23), and the formula

$$\nabla_B e^G=e^G\left[\nabla_B G+\frac{1}{2!}[\nabla_B G,G]+\frac{1}{3!}[[\nabla_B G,G]G]+\cdots\right],$$

with $G=b^\dagger\zeta+b\zeta^*$, the gauge field strength is found to be

$$\nabla_B \times \mathbf{A} = i(\zeta(t)|\overleftarrow{\nabla}_B \times \overrightarrow{\nabla}_B|\zeta(t))$$
$$= i\zeta(0)\zeta^*(0)(0|[\nabla_B b^\dagger(t)]\times[\nabla_B b(t)] - [\nabla_B b(t)]\times[\nabla_B b^\dagger(t)]|0) . \quad (31)$$

It is important to note that the first term among the bra and ket vacua never vanishes. After some calculations with Eqs. (13), (19), and (20), this can be further reduced to

$$\nabla_B \times \mathbf{A} = \zeta(0)\zeta^*(0)\frac{\mathbf{B}}{B^3} . \quad (32)$$

This is just the *field strength of a monopole of a Grassmann charge*

$$Q = \zeta(0)\zeta^*(0) \quad (33)$$

located at $B=0$.

Consequently, we obtain

$$|\zeta(T)\rangle = \exp[i\phi(T)]|\zeta(0)\rangle , \quad (34)$$

$$\phi(T) = \gamma(C) - \int_0^T dt[H_{\text{cl}}(t) - \tfrac{1}{2}B(t)] , \quad (35)$$

$$\gamma(C) = Q\Omega(C) , \quad (36)$$

where the periodicity and initial conditions $|\zeta(T)) = |\zeta(0)) = |\zeta(0)\rangle$ has been imposed. $\Omega(C)$ stands for the solid angle spanned by C seen from the origin in R. These equations completely determine the adiabatic evolution of the fermionic coherent state along the circuit C.

It is straightforward to get Hannay's angle. Let us define the action variable (Ehrenfest's adiabatic invariant) for this purpose. From Eq. (8) with $\psi=\zeta$, the canonical momentum Π conjugate to ζ is $\Pi = -i\zeta^*$, and therefore the action variable is

$$I = \frac{1}{2\pi}\int_0^\tau dt\, \dot{\zeta}(t)\Pi(t) \quad (37a)$$

$$= \zeta^*(0)\zeta(0) , \quad (37b)$$

where $\tau = 2\pi/B$ is the period of motion. (So the charge Q is an adiabatic invariant.) Now we find

$$\theta(T) + \Delta\theta(C) = -\frac{\partial\phi(T)}{\partial I} = \int_0^T dt\, B(t) + \Omega(C) . \quad (38)$$

The first term in the second line is the ordinary angle variable that is the integral of the instantaneous frequency $\partial H_{\text{cl}}/\partial I$. While the second term is just equal to Hannay's angle of the present model.[9]

IV. BOHR-SOMMERFELD QUANTIZATION OF A GRASSMANN CHARGE

In this section, we wish to examine the Bohr-Sommerfeld-type quantization of the geometrical phase (i.e., the Grassmann charge) obtained in the previous section. To perform the semiclassical analysis of the adiabatic holonomy effect, such an old-fashioned quantum-mechanical treatment may be still interesting, although the exact quantization procedure and the WKB approximation have been completely established.

The usual Bohr-Sommerfeld rule consists of equating the adiabatic invariant such as Eq. (37a) with $n+\sigma$, where n runs over integers and σ is a constant including the Maslov-Keller index. It is however clear that there exist no such higher excitations for intrinsic dynamical degrees of freedom like a spin. In this point, let us focus our attention to the integrand in Eq. (37a). It satisfies an *identity*

$$dt\,\dot{\zeta}\Pi = d\zeta\,\Pi . \quad (39)$$

The integral of the left-hand side gives Ehrenfest's adiabatic invariant. On the other hand, the integral of the right-hand side is just the Berezin integration[18] over the Grassmann number. Obviously the integration suppresses higher excitations, since it gives simply a definite constant. We only know that this quantity is real. Therefore it is quite reasonable to set it as $\pm a$, where a is a real constant associated to the normalization of the Berezin integration. a is chosen as π in the present case. Consequently, we define

$$-i\int d\zeta\,\zeta^* = \pm\pi . \quad (40)$$

Then we have the *exact* result

$$\zeta^*(0)\zeta(0) = -Q = \pm\tfrac{1}{2} . \quad (41)$$

Accordingly the Hamiltonian (29) and the geometrical phase (36) become

$$E_\pm(t) = \pm\tfrac{1}{2}B(t) , \quad (42)$$

$$\gamma_\pm(C) = \mp\tfrac{1}{2}\Omega(C) , \quad (43)$$

respectively. $\gamma_\pm(C)$ is just equal to Berry's phase in the energy eigenstates.[2]

Closing this section, we make a comment on the rule mentioned above. Equating the integrals of both sides in Eq. (39) is somewhat unclear mathematically, since the integral of the left-hand side is an even Grassmann number, while that of the right-hand side is an ordinary number. The present requirement of equating those two integrals with each other might be regarded as a substitution rule, which is analogous to that in the ordinary Bohr-Sommerfeld rule: $I=n+\sigma$. In this respect, we wish to stress again that Eq. (39) itself is entirely an identity. We also feel that the choice (40) is model independent, although we have not known the general proof. The consistency of the normalization (40) is ascertained in the Pauli spin and supersymmetric Dirac particle models in Ref. 10.

V. CONCLUDING REMARKS

We have studied the model of Pauli spin in a slowly varying external magnetic field and its fermionic coherent state in order to elucidate a certain classical-quantum correspondence in the adiabatic holonomy effect. The adiabatic evolution of the coherent state along the closed contour in the adiabatic parameter space was shown to be specified not only by the dynamical phase but also by the geometrical phase. Hannay's angle was explicitly related to the latter one. Berry's gauge field strength was shown to be that of a monopole of a Grassmann charge. Based on the Bohr-Sommerfeld-type rule for Grassmannian systems, we proved that quantization of the charge reproduces Berry's phase in the energy eigenstates. Consequently the semiclassical implication of the effect was manifested without resort to the direct WKB expansion.

It has been pointed out that Berry's phase is closely related to anomalies in some kinds of gauge theories with fermions.[19] In this context, we expect that field-theoretical extensions of the present approach may cast light on the semiclassical or rather classical meaning of anomalies in a peculiar manner.

Note added. Since submittal of this work, two papers[20] have been published. In these papers, the nonadiabatic geometrical phases are discussed based on the bosonic coherent states.

ACKNOWLEDGMENTS

The author would like to thank Dr. S. Iida for illuminating discussions and Professor H. Kuratsuji for his useful comments and for sending me his paper (Ref. 20).

[1]A. Messiah, *Quantum Mechanics* (North-Holland, Amsterdam, 1962), Vol. 2.

[2]M. V. Berry, Proc. R. Soc. London **A392**, 45 (1984).

[3]A. Tomita and R. Y. Chiao, Phys. Rev. Lett. **57**, 937 (1986); R. Tycko, *ibid.* **58**, 2281 (1987); R. Y. Chiao, A. Antaramian, K. M. Ganga, H. Jiao, S. R. Wilkinson, and H. Nathel, *ibid.* **60**, 1214 (1988).

[4]B. Simon, Phys. Rev. Lett. **51**, 2167 (1983). See also E. Kiritsis, Commun. Math. Phys. **111**, 417 (1987); J. Anandan and L. Stodolsky, Phys. Rev. D **35**, 2597 (1987); Y. Aharonov and J. Anandan, Phys. Rev. Lett. **58**, 1593 (1987).

[5]J. H. Hannay, J. Phys. A **18**, 221 (1985).

[6]L. D. Landau and E. M. Lifshitz, *Mechanics,* 3rd ed. (Pergamon, Oxford, 1976).

[7]E. Gozzi and W. D. Thacker, Phys. Rev. D **35**, 2398 (1987).

[8]M. V. Berry, J. Phys. A **18**, 15 (1985).

[9]E. Gozzi and W. D. Thacker, Phys. Rev. D **35**, 2388 (1987).

[10]S. Abe and S. Naka, Prog. Theor. Phys. **72**, 881 (1984).

[11]R. J. Glauber, Phys. Rev. Lett. **10**, 84 (1963); Phys. Rev. **130**, 2529 (1963); **131**, 2766 (1963).

[12]J. R. Klauder and B. S. Skagerstam, *Coherent States* (World Scientific, Singapore, 1985).

[13]G. Ghosh and B. Dutta-Roy, Phys. Rev. D **37**, 1709 (1988).

[14]H. Kuratsuji and S. Iida, Phys. Rev. Lett. **56**, 1003 (1986).

[15]J. G. Hartley and J. R. Ray, Phys. Rev. D **25**, 382 (1982). For a recent review, see S. K. Bose and U. B. Dubey, Fortschr. Phys. **35**, 675 (1987).

[16]R. Casalbuoni, Nuovo Cimento **33A**, 389 (1976); F. A. Berezin and M. S. Marinov, Ann. Phys. (N.Y.) **104**, 336 (1977).

[17]Y. Ohnuki and T. Kashiwa, Prog. Theor. Phys. **60**, 548 (1978).

[18]F. A. Berezin, *The Method of Second Quantization* (Academic, New York, 1966).

[19]P. Nelson and L. Alvarez-Gaumé, Commun. Math. Phys. **99**, 103 (1985); A. J. Niemi and G. W. Semenoff, Phys. Rev. Lett. **55**, 927 (1985); H. Sonoda, Nucl. Phys. **B266**, 410 (1986). See also H. Kuratsuji and S. Iida, Phys. Rev. D **37**, 441 (1988).

[20]H. Kuratsuji, Phys. Rev. Lett. **61**, 1687 (1988); R. G. Littlejohn, *ibid.* **61**, 2159 (1988).