Dirac constraint quantization of a parametrized field theory by anomaly-free operator representations of spacetime diffeomorphisms

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We construct a consistent Dirac constraint quantization of a parametrized massless scalar field propagating on a two-dimensional cylindrical Minkowskian background. The constraints are taken in the form of "diffeomorphism Hamiltonians" whose Poisson-brackets algebra is homomorphic to the Lie algebra of spacetime diffeomorphisms. The fundamental canonical variables are represented by operators acting on an embedding-dependent Fock space H which is based on the Heisenberg modes that are geometrically specified with respect to the Killing vector structure of the background. In the Heisenberg picture, the constraints become the Heisenberg embedding momenta and their Abelian Poisson algebra is homomorphically mapped into the operator commutator algebra without any anomaly. The algebra of normal-ordered Heisenberg evolution generators (which propagate the field operators) develops a covariantly defined anomaly. This anomaly is an exact two-form on the space of embeddings $\mathrm{Emb}(\Sigma,M)$ and can thus be written as a functional curl of an anomaly potential on $\mathrm{Emb}(\Sigma,M)$. By subtracting this potential from the normal-ordered Heisenberg generators (which amounts to their embedding-dependent reordering) we arrive at a commuting set of operators which we identify with the Schrödinger embedding momenta. By smearing the Heisenberg and the Schrödinger embedding momenta by spacetime vector fields we obtain a pair of anomaly-free operator representations of L Diff$M$. The diffeomorphism Hamiltonians annihilate the physical states and the smeared reordered Heisenberg evolution generators propagate the fields. We present the operator transformation from the Schrödinger to the Heisenberg picture. The two operator representations of L Diff$M$, by diffeomorphism Hamiltonians and by smeared Heisenberg evolution generators, guarantee that the Dirac constraint quantization is consistent, covariant, and leads to foliation-independent dynamics both in the Heisenberg and in the Schrödinger pictures. The appropriate factor ordering of the Hamiltonian flux operator and of the constraints is rewritten in terms of the fundamental Schrödinger variables with help of a normal-ordering kernel which is reconstructed from the intrinsic metric and the extrinsic curvature on a given embedding. All operators are defined and dynamics takes place on a single function space which is then restricted by the constraints to the space of physical states with a Hilbert-space structure.

I. INTRODUCTION

Our aim is to show that there exists a consistent Dirac constraint quantization of parametrized field systems. By this we mean that classical constraints can be turned into operators and imposed as restrictions on physical states in such a way that (1) they do not beget other constraints by commutators, (2) the quantum evolution remains foliation independent, and (3) the quantum theory respects the spacetime covariance of parametrized field theories.

In a classical theory, the constraints can be written in two equivalent forms: in the Dirac form,[1] in which they are projected perpendicular and parallel to the embeddings and in which their Poisson brackets "close" according to the Dirac "algebra," or in the form of "diffeomorphism Hamiltonians,"[2] whose Poisson algebra is homomorphic to the Lie algebra L Diff$M$ of spacetime diffeomorphisms. The closure of either one of these Poisson algebras ensures the foliation independence of the classical dynamical evolution, but the spacetime covariance of the canonical theory is best expressed when using the second form of the constraints.

Our three objectives formally reduce to the task of finding a factor ordering of the diffeomorphism Hamiltonians which makes their operator commutator algebra homomorphic to L Diff$M$ without any anomaly. We shall illustrate the procedure on a parametrized system whose classical features we have studied in a previous paper:[3] a massless scalar field propagating on a two-dimensional cylindrical Minkowskian background. The generalization to more complicated parametrized systems will be given elsewhere.

Our starting point is the Heisenberg picture of the parametrized classical dynamics introduced in I. We represent the fundamental Heisenberg canonical variables by operators acting on an embedding-dependent Fock space H. The construction of the Fock space is based on the geometrically privileged modes related to the Killing vectors of the Minkowskian background. In the Heisenberg picture the constraints are identical with the fundamental embedding momenta and their quantum algebra is therefore automatically homomorphic to the classical Poisson-brackets algebra. Our task of representing L Diff$M$ by the algebra of diffeomorphism Hamiltonian operators is thus built into the quantum theory from the very beginning (Sec. II A).

In the Heisenberg picture, the physical states are embedding independent while the field operators $\phi(x),\pi_1(x)$ are developed by the Heisenberg evolution generators. These generators are differences between the Heisenberg embedding momenta and the Hamiltonian flux operators for the field. In the Heisenberg picture, the flux operators can be ordered once and for all by adopting the normal ordering of the geometrically privileged modes. However, when this is done, the algebra of the Heisenberg evolution generators acquires an anomaly.

This anomaly is a reflection of the standard Schwinger terms in the commutators of the components of the spacetime energy-momentum tensor[4] and, for our two-dimensional system, it is related to the Virasoro algebra.[5] It has two parts: one (called ${}_{\rm I}F$) depending on the first derivatives of the δ function and another (called ${}_{\rm III}F$) depending on the third derivatives of δ. However, because our modes and the restriction process to curved embeddings are specified by geometric structures which do not depend on the use of coordinates, our anomaly on curved embeddings is a geometric object whose form significantly differs (Sec. II D) from the anomaly obtained by a coordinate-dependent Schrödinger picture ordering adopted from string theory.[6,7] We exhibit the covariant form of the anomaly on curved embeddings as it appears in the algebra of the Hamiltonian flux operators (Sec. II B), in the algebra of the Heisenberg evolution generators (Sec. II C), and in the related Dirac algebras (Sec. II D).

The important features of the covariant anomaly are the following (Sec. II B): the anomaly depends only on the geometric data of an embedding (the intrinsic metric and the extrinsic curvature). To reconstruct ${}_{\rm I}F$ from the geometric data, one needs to know them on the whole embedding. On the other hand, ${}_{\rm III}F$ is a local functional of the intrinsic metric and the extrinsic curvature.

Because the Heisenberg evolution generators are not constrained to vanish on the physical space, such an anomaly in their operator algebra does not lead to any inconsistency in the Heisenberg picture Dirac constraint quantization (Sec. II E). However, the generators have the wrong commutation relations to become the Schrödinger embedding momenta. We must modify their factor ordering to remove the anomaly.

The required modification relies on an important observation: *the anomaly is an exact two-form on the space* $\mathrm{Emb}(\Sigma,M)$ *of embeddings* (Sec. III A). One can thus find a one-form A on $\mathrm{Emb}(\Sigma,M)$ (the anomaly potential) whose functional curl generates F. By subtracting the anomaly potential from the Heisenberg evolution generator (which amounts to reordering the Hamiltonian flux operator) we arrive at a commuting set of operators which we identify with the Schrödinger embedding momenta. We are then able to perform an operator transformation from the Heisenberg to the Schrödinger picture (Sec. III C). The fact that the Schrödinger embedding momenta commute amounts to the cancellation of the anomaly in the Dirac algebra of their projections (Sec. III D).

The anomaly potential can be gauged so that it depends on the same geometric data (the metric g_{11} and the extrinsic curvature K_{11} of an embedding) as the anomaly. Again, to reconstruct ${}_{\rm I}A$ one needs to know the geometric data on the whole embedding, while ${}_{\rm III}A$ is a local functional of g_{11} and K_{11} (Sec. III B). The anomaly potential can be interpreted as the Heisenberg vacuum expectation value of the reordered Hamiltonian flux ("the total Hamiltonian flux"). The Heisenberg normal-ordered Hamiltonian flux is a perpendicular projection of a conserved symmetric trace-free tensor, namely, of the Heisenberg normal-ordered energy-momentum tensor. This is no longer true of the total Hamiltonian flux: while ${}_{\rm I}A$ can be generated in this way from a geometric tensor constructed solely from the Killing vectors of the background, ${}_{\rm III}A$ eludes a similar interpretation (Sec. IV). The new factor ordering is thus intrinsically dependent on the choice of the hypersurface.

There is a neat physical description of the influence which the anomaly potentials ${}_{\rm I}A$ and ${}_{\rm III}A$ have on the distribution of the energy and momentum on a given hypersurface (Sec. IV). The potential ${}_{\rm I}A$ lowers the inertial energy density by a constant amount $-(12\pi)^{-1}$ corresponding to the Casimir effect introduced by the closing of S^1. It also lowers the energy density measured by the hypersurface observer by an amount which depends on the local speed of this observer relative to the inertial observer; as that speed approaches the speed of light, the negative contribution diverges. On the other hand, ${}_{\rm III}A$ leads to a redistribution of the inertial energy (and momentum) on the hypersurface while keeping their total values fixed. The inertial energy (and momentum) densities are thus indefinite; they are spatial gradients of terms containing the mean extrinsic curvature of the hypersurface. One can also express these densities in terms of the arc length derivatives of the Lorentz contraction factor corresponding to the speed v along the embedding.

The normal factor ordering of the Hamiltonian flux operator is initially given in terms of the fundamental Heisenberg operators. However, one can rewrite it entirely in terms of the fundamental Schrödinger variables $\phi(x)$ and $\pi_1(x)$ (Sec. VI). The normal-ordering kernel which achieves this task is a distribution which can be reconstructed from the geometric data (the intrinsic metric and the extrinsic curvature) on the embedding which carries the operator fields $\phi(x),\pi_1(x)$. From here, one can also express the factor ordering of the total Hamiltonian flux and of the constraint operators directly in terms of the fundamental Schrödinger operators. This puts the final touch on our description of the passage from the Heisenberg to the Schrödinger picture.

Our procedure gave us at the very beginning a commuting set of Heisenberg embedding momentum operators $\mathbf{P}_{1\alpha}(x)$. Furthermore, the elimination of the anomaly by reordering the Heisenberg evolution generators yielded a commuting set of the Schrödinger embedding momentum operators $P_{1\alpha}(x)$. These two sets of commuting operators enable us to reach our main goal of representing L Diff$M$ by a commutator algebra of operators on H. We obtain two such representations, $\mathbf{U}\rightarrow\mathbf{P}(\mathbf{U})$ and $\mathbf{U}\rightarrow P(\mathbf{U})$, by smearing either the Heisenberg or the Schrödinger momenta by all complete spacetime vector fields $\mathbf{U}$ restricted to the embeddings. The diffeomorphism Hamiltonians $\mathbf{P}(\mathbf{U})$ annihilate the physical states, and the diffeomorphism Heisenberg evolution generators $P(\mathbf{U})$ evolve the field operators $\phi(x),\pi_1(x)$. The first statement yields a many-fingered time Schrödinger equation in the Schrödinger picture and the embedding independence of the states in the Heisenberg picture. The second statement amounts to the embedding-independence of the Schrödinger field operators $\phi(x),\pi_1(x)$ in the Schrödinger picture, and to the Heisenberg equations of motion for $\phi(x)$ and $\pi_1(x)$ in the Heisenberg picture. The representation of L Diff$M$ realized by the mapping $\mathbf{U}\rightarrow P(\mathbf{U})$ ensures that the states are evolved from an initial embedding to a final embedding by the many-fingered time Schrödinger equation in a way which does not depend on the connecting foliation. Similarly, the representation of L Diff$M$ realized by the mapping $\mathbf{U}\rightarrow P(\mathbf{U})$ ensures that the evolution of the field according to the Heisenberg equations of motion is foliation independent. Each of these representations reflects the covariance of the quantum canonical formalism under spacetime diffeomorphisms. Taken together they ensure that the Dirac constraint quantization can be consistently carried out both in the Heisenberg and in the Schrödinger pictures. The diffeomorphism Hamiltonians $\mathbf{P}(\mathbf{U})$ and the diffeomorphism Heisenberg evolution generators $P(\mathbf{U})$ are well-defined operators on a *single* function space H. The diffeomorphism Hamiltonians select the physical states from H (which they annihilate) and describe how such states evolve along an arbitrary foliation. The space of physical states H_0 is then endowed with a Hilbert-space structure. Similarly, the diffeomorphism Heisenberg evolution generators are capable of evolving the field operators $\phi(x)$ and $\pi_1(x)$ along an arbitrary foliation.

One may ask how the two operator representations of L Diff$M$ are connected with those of the group of conformal isometries C. In the classical theory, C plays the role of a symmetry group of the diffeomorphism Hamiltonians $\mathbf{P}(\mathbf{U})$. The generators $\mathbf{u}\in LC$ are represented by the smeared Schrödinger momenta $P(\mathbf{u})$ which have vanishing Poisson brackets with $\mathbf{P}(\mathbf{U})$ and are thus constants of motion. Does our procedure eliminate the anomaly from the *quantum* algebra of such constants of motion? The answer is no. The operators $P(\mathbf{u})$ and $\mathbf{P}(\mathbf{U})$ no longer commute as the classical variables did; the $P(\mathbf{u})$'s still represent LC, but they do not keep the diffeomorphism Hamiltonians invariant and therefore they are not quantum constants of motion. On the other hand, the Heisenberg *normal*-ordered fluxes and the corresponding Heisenberg evolution generators yield nontrivial quantum constants of motion $-h(\mathbf{u})\approx\Pi(\mathbf{u})$. However, the operator algebra of these constants of motion is not homomorphic to LC, but to the central extension of LC. The symmetry group of the quantum system is thus different from the symmetry group of the classical system. This fact, however, has no bearing on the Dirac constraint quantization itself.

II. CANONICAL QUANTIZATION AND THE ANOMALY

A. What space are the constraints acting on?

In the Dirac constraint quantization the fundamental canonical variables are turned into operators whose commutator algebra is isomorphic to the classical Poisson-brackets algebra. They are supposed to act on a suitably defined function space H which does not need to have a Hilbert-space structure.[8] The classical constraints must also be turned into operators and factor ordered so that their commutators do not beget other constraints. The physical space H_0 is then defined as the set of states in H which are annihilated by the constraints. It is only this space (or its completion) which is endowed with a Hilbert-space inner product.

Let us specify what our function space H is and what our fundamental operators $\mathbf{X}^\alpha(x)$, $\mathbf{P}_\alpha(x)$, $\mathbf{q}$, $\mathbf{p}$, $\mathbf{a}_k$, $\mathbf{a}^*{}_k$ are in the Heisenberg picture (we now use the same symbols for the operators as we previously did for the classical dynamical variables). Start with the Fock space $\mathcal{F}$ on which $\mathbf{a}_k$ and $\mathbf{a}^*{}_k$ act as the standard annihilation and creation operators. Use the occupation number representation $\Psi(n_k)=\Psi(n_1,n_2,\ldots)$ in $\mathcal{F}$ (the sequence n_k gives the occupation numbers of the harmonic modes $k=\pm1,\pm2,\ldots$). Make Ψ also dependent on a real parameter $\mathbf{p}$ and functionally on the embedding variables $\mathbf{X}^\alpha(x)\in\mathrm{Emb}(\Sigma,M)$:

$$\Psi=\Psi(n_k,\mathbf{p};\mathbf{X}] \ . \tag{2.1}$$

The function(al) Ψ is an element of H. The action of the operators $\mathbf{q}$, $\mathbf{p}$, $\mathbf{a}_k$, $\mathbf{a}^*{}_k$, $\mathbf{X}^\alpha(x)$, and $\mathbf{P}_{1\alpha}(x)$ on Ψ is prescribed by the rules

$$\begin{aligned}
&\mathbf{q}\Psi(n_k,\mathbf{p};\mathbf{X}]=i\frac{\partial}{\partial\mathbf{p}}\Psi(n_k,\mathbf{p};\mathbf{X}] \ ,\\
&\mathbf{p}\Psi(n_k,\mathbf{p};\mathbf{X}]=\mathbf{p}\times\Psi(n_k,\mathbf{p};\mathbf{X}] \ ,\\
&\mathbf{a}_l\Psi(n_k,\mathbf{p};\mathbf{X}]=\sqrt{n_l}\,\Psi(n_k-\delta_{kl},\mathbf{p};\mathbf{X}] \ ,\\
&\mathbf{a}^*{}_l\Psi(n_k,\mathbf{p};\mathbf{X}]=\sqrt{n_l+1}\,\Psi(n_k+\delta_{kl},\mathbf{p};\mathbf{X}] \ ;
\end{aligned} \tag{2.2}$$

$$\begin{aligned}
&\mathbf{X}^\alpha(x)\Psi(n_k,\mathbf{p};\mathbf{X}]=\mathbf{X}^\alpha(x)\times\Psi(n_k,\mathbf{p};\mathbf{X}] \ ,\\
&\mathbf{P}_{1\alpha}(x)\Psi(n_k,\mathbf{p};\mathbf{X}]=-i\frac{\delta}{\delta\mathbf{X}^\alpha(x)}\Psi(n_k,\mathbf{p};\mathbf{X}] \ .
\end{aligned} \tag{2.3}$$

As a consequence of Eqs. (2.2) and (2.3), the operators $\mathbf{X}^\alpha(x)$, $\mathbf{P}_{1\alpha}(x)$, $\mathbf{q}$, $\mathbf{p}$, $\mathbf{a}_k$, and $\mathbf{a}^*{}_k$ satisfy on H commutation relations which are isomorphic to the Poisson algebra of these variables under the mapping $\{\ ,\ \}\leftrightarrow(1/i)[\ ,\ ]$.

One advantage of the Heisenberg picture is that the constraint functions are identical to the fundamental embedding variables and hence their quantum algebra is automatically isomorphic to the classical algebra (I.3.37) or (I.4.19):

$$\frac{1}{i}[\mathbf{P}_{1\alpha}(x),\mathbf{P}_{1'\beta}(x')]=0 \ ,$$

or (2.4)

$$\frac{1}{i}[\mathbf{P}(\mathbf{U}),\mathbf{P}(\mathbf{V})]=\mathbf{P}(-[\mathbf{U},\mathbf{V}]) \qquad \forall\mathbf{U},\mathbf{V}\in L\,\mathrm{Diff}M \ .$$

When the constraints are imposed on the states of the system,

$$\mathbf{P}_{1\alpha}(x)\Psi(n_k,\mathbf{p};\mathbf{X})=0 , \tag{2.5}$$

they imply that the Heisenberg states do not depend on the embedding:

$$\Psi=\Psi(n_k,\mathbf{p}) . \tag{2.6}$$

Such states are the elements of the physical space H_0. An inner product $\langle\Psi_1|\Psi_2\rangle$ is defined only between two states Ψ_1 and Ψ_2 in H_0, not between the states (2.1) in the big space H. We put

$$\langle\Psi_1|\Psi_2\rangle:=\sum_{n=0}^{\infty}\sum_{\{n_k\}}\left[\prod_{k=-\infty}^{\infty}{}' (k)^{n_k}\right] \times\int_{-\infty}^{\infty}d\mathbf{p}\,\Psi^*(n_k,\mathbf{p})\Psi(n_k,\mathbf{p}) , \tag{2.7}$$

where the infinite product $\prod'^{\infty}_{k=-\infty}$ excludes $k=0$, the sum $\sum_{\{n_k\}}$ is taken over all occupation number sequences with the same total number $\sum'^{\infty}_{k=-\infty}n_k=n$ of excited modes $k\neq0$, and the final sum $\sum_{n=0}^{\infty}$ takes care of different total numbers n. The operators $\mathbf{q},\mathbf{p}$, and $\mathbf{q}_k,\mathbf{p}_k$ given by Eqs. (I.3.74) are self-adjoint under this inner product.

Instead of the occupation number representation $\Psi(n_k,\mathbf{p};\mathbf{X}]$, the states in the big space H can equally well be given in the $\boldsymbol{\phi}_k^{(c)},\boldsymbol{\phi}_k^{(s)}$ representation $\Psi(\boldsymbol{\phi}_k^{(c)},\boldsymbol{\phi}_k^{(s)};\mathbf{X}]$ or, what amounts to the same thing, in the Schrödinger functional representation $\Psi[\boldsymbol{\phi}(x),\mathbf{X}(x)]$. Note that this Schrödinger functional *representation* is taken in the Heisenberg *picture*, i.e., that $\boldsymbol{\phi}(x)$ is the distribution of the scalar field on the initial embedding (I.3.40).

The rest of this paper is essentially devoted to two problems: (I) how to construct the Schrödinger operators $\phi(x)$, $\pi_1(x)$, $X^\alpha(x)$, and $P_{1\alpha}(x)$ from the Heisenberg operators $\mathbf{q}$, $\mathbf{p}$, $\mathbf{a}_k$, $\mathbf{a}^*{}_k$, $\mathbf{X}^\alpha(x)$, and $\mathbf{P}_{1\alpha}(x)$ so that they are well defined on the space H described above, and satisfy on H the correct fundamental commutator algebra and (II) how to express the constraints $\mathbf{P}_{1\alpha}(x)$ as properly ordered combinations of such Schrödinger picture operators. This is the crux of the problem as to whether the Dirac constraint quantization can be consistently carried out in the Schrödinger picture by imposing the constraints in the Schrödinger form,

$$\mathbf{P}_{1\alpha}(x;\phi,\pi,X,P]\Psi[\phi(x),X(x)]=0 , \tag{2.8}$$

on the Schrödinger states. We shall *not* explicitly solve the problem of how to transform a given Heisenberg state $\Psi[\boldsymbol{\phi}(x)]\in H_0$ into the corresponding Schrödinger state $\Psi[\phi(x),X(x)]$ which satisfies the Schrödinger equation (2.8).

We start the calculations in the Heisenberg picture and make our way step by step to the Schrödinger picture.

B. Anomaly in the algebra of the Hamiltonian flux operators

As in the corresponding classical calculation, we begin with the energy-momentum tensor in the null coordinates (I.3.8) and express it through Eqs. (I.3.5)–(I.3.7) in terms of the Heisenberg data $\mathbf{q},\mathbf{p},\mathbf{a}_k,\mathbf{a}^*{}_k$. We *normal order* the energy-momentum-tensor operator in the creation and annihilation operators $\mathbf{a}^*{}_k,\mathbf{a}_k$. Here, as for all other quadratic operators, we keep the original symbol $T_{\pm\pm}(T^{\pm};\mathbf{q},\mathbf{p},\mathbf{a}_k,\mathbf{a}^*{}_\mathbf{k})$ for the normal-ordered operator.

We smear the Hamiltonian flux by the basis vectors (I.4.10) of the Lie algebras $L_{(\pm)}C$ and arrive at its (normal-ordered) Virasoro components (I.4.29):

$$\begin{aligned} h^{(\pm)}{}_0&=\tfrac{1}{2}\mathbf{p}^2+\sum_{k=1}^{\infty}\mathbf{a}^*{}_{\mp k}\mathbf{a}_{\mp k} ,\\ h^{(\pm)}{}_n&=i\frac{1}{\sqrt{2}}\mathbf{p}\,\mathbf{a}^*{}_{\mp n}-\tfrac{1}{2}\sum_{k=1}^{n-1}\mathbf{a}^*{}_{\mp k}\mathbf{a}^*{}_{\mp(n-k)}\\ &\quad+\sum_{k=1}^{\infty}\mathbf{a}^*{}_{\mp(k+n)}\mathbf{a}_{\mp k},\quad n>0 ,\\ h^{(\pm)}{}_{-n}&=(h^{(\pm)}{}_n)^* . \end{aligned} \tag{2.9}$$

There is only a finite number of the double-creator terms in $h^{(\pm)}{}_n$ and of the double-annihilator terms in $h^{(\pm)}{}_{-n}$; there are no such terms in $h^{(\pm)}{}_0$. The expressions (2.9) are thus well defined as operators when they act on the states (2.1).

The standard calculation[5] yields the anomaly in the Virasoro algebra:

$$\begin{aligned} \frac{1}{i}[h^{(\pm)}{}_m,h^{(\pm)}{}_n]&=i(m-n)h^{(\pm)}{}_{m+n}\\ &\quad+i\tfrac{1}{12}m(m^2-1)\delta_{m+n,0} ,\\ \frac{1}{i}[h^{(+)}{}_m,h^{(-)}{}_n]&=0 . \end{aligned} \tag{2.10}$$

Because the only term sensitive to the factor ordering is $h^{(\pm)}{}_0$, the anomaly is proportional to $\delta_{m+n,0}$. When we reconstruct $T_{\pm\pm}(T^{\pm})$ from its Fourier components (I.4.28),

$$T_{\pm\pm}(T^{\pm})=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}e^{inT^{\pm}}h^{(\pm)}{}_n , \tag{2.11}$$

we learn that

$$\begin{aligned} \frac{1}{i}[T_{\pm\pm}(T^{\pm}),T_{\pm\pm}(T^{\pm\prime})] &=T_{\pm\pm}(T^{\pm})\delta_{,\pm}(T^{\pm}-T^{\pm\prime})-(T^{\pm}\leftrightarrow T^{\pm\prime})\\ &\quad+\frac{1}{12}\frac{1}{2\pi}(\delta_{,\pm\pm\pm}(T^{\pm}-T^{\pm\prime})-\delta_{,\pm}(T^{\pm}-T^{\pm\prime})) \end{aligned} \tag{2.12}$$

and

$$\frac{1}{i}[T_{++}(T^{+}),T_{--}(T^{-\prime})]=0 . \tag{2.13}$$

The term $\delta_{,\pm\pm\pm}$ is the standard Schwinger term, the $\delta_{,\pm}$ term is brought in by the finite size of the spatial sections of $M=\mathbb{R}\times S^1$.

By projections and restrictions of Eqs. (2.12) and (2.13) we get the algebra of the null components of the Hamiltonian flux operator:

$$h_{1\pm}(x):=T_{\pm\pm}(\mathbf{T}^{\pm}(x))n^{\pm}_{1}(x;\mathbf{T}^{\pm}]$$
$$=\pm T_{\pm\pm}(\mathbf{T}^{\pm}(x))\mathbf{T}^{\pm}{}_{,1}(x)\ . \qquad (2.14)$$

This algebra

$$\frac{1}{i}[h_{1\pm}(x),h_{1'\pm}(x')]$$
$$=(\mathbf{T}^{\pm}{}_{,1}(x))^{-1}h_{1\pm}(x)\delta_{1',1}(x,x')-(x\leftrightarrow x')$$
$$-F_{1\pm1'\pm}(x,x';T^{\pm}]\ , \qquad (2.15)$$
$$\frac{1}{i}[h_{1+}(x),h_{1'-}(x')]=0\ ,$$

differs from the corresponding classical algebra (I.3.83) by the anomaly

$$F_{1\pm1'\pm}(x,x';\mathbf{T}^{\pm}]={}_{\mathrm{I}}F_{1\pm1'\pm}(x,x')+{}_{\mathrm{III}}F_{1\pm1'\pm}(x,x';\mathbf{T}^{\pm}]\ , \qquad (2.16)$$

$${}_{\mathrm{I}}F_{1\pm1'\pm}(x,x')=\pm\frac{1}{12}\frac{1}{2\pi}\delta_{1',1}(x,x')\ , \qquad (2.17)$$

$${}_{\mathrm{III}}F_{1\pm1'\pm}(x,x';\mathbf{T}^{\pm}]$$
$$=\mp\frac{1}{12}\frac{1}{2\pi}(\mathbf{T}^{\pm}{}_{,1}(x))^{-2}(D^{(\pm)}{}_{1})^{2}\delta_{1',1}(x,x')$$
$$=\mp\frac{1}{12}\frac{1}{2\pi}\partial_1(\mathbf{T}^{\pm}{}_{,1}(x))^{-1}\partial_1(\mathbf{T}^{\pm}{}_{,1}(x))^{-1}\delta_{1',1}(x,x')\ . \qquad (2.18)$$

We used the identities (I.B2) and (I.B4) for the differentiated δ functions when projecting the anomaly. We split the anomaly into two pieces. The first piece (2.17) is brought in by the finite profile of the cylinder $M=\mathbb{R}\times S^1$; it contains only the first derivative of the spatial δ function and it does not depend on the embedding $\mathbf{T}^{\pm}(x)$. The second piece (2.18) is independent of the size of the cylinder $M=\mathbb{R}\times S^1$; it contains the third derivatives of the spatial δ function and it depends on the embedding $\mathbf{T}^{\pm}(x)$ through the covariant derivatives $D^{(\pm)}{}_1$. It is evident that the anomaly proportional to the third derivative of the δ function must depend on the embedding to preserve the spatial covariance of Eqs. (2.15)–(2.18). The concrete form of the expressions (2.17) and (2.18) was obtained by using the projection formulas (I.B2) and (I.B4) for the derivatives of the spacetime δ functions.

Equations (2.15)–(2.18) can be written in a form which is valid in an arbitrary spacetime coordinate system X^{α}. The regular part of the algebra is identical to the classical algebra (I.3.83) and it undergoes the same rearrangement (I.3.86):

$$\frac{1}{i}[h_{1\alpha}(x),h_{1'\beta}(x')]$$
$$=C^{1\gamma}{}_{\alpha\beta'}(x,x';\mathbf{X}]h_{1\gamma}(x)\delta_{1',1}(x,x')-(\alpha x\leftrightarrow\beta x')$$
$$-F_{1\alpha1'\beta}(x,x';\mathbf{X}]\ . \qquad (2.19)$$

The coefficient $C^{1\gamma}{}_{\alpha\beta'}$ is given by the classical expression (I.3.87). Our only task is to find the covariant transcription

$$F_{1\alpha1'\beta}(x,x';\mathbf{X}]={}_{\mathrm{I}}F_{1\alpha1'\beta}(x,x';\mathbf{X}]$$
$$+{}_{\mathrm{III}}F_{1\alpha1'\beta}(x,x';\mathbf{X}] \qquad (2.20)$$

of the anomaly.

For ${}_{\mathrm{I}}F_{1\alpha1'\beta}$ this is easily achieved by using the properties (I.2.8) of the privileged null basis $\mathbf{e}_{(\pm)}$:

$${}_{\mathrm{I}}F_{1\alpha1'\beta}(x,x';\mathbf{X}]$$
$$=-\frac{1}{12}\frac{1}{2\pi}(e_{(+)\alpha}(\mathbf{X}(x))e_{(+)\beta}(\mathbf{X}(x'))$$
$$-e_{(-)\alpha}(\mathbf{X}(x))e_{(-)\beta}(\mathbf{X}(x')))\delta_{1',1}(x,x')\ . \qquad (2.21)$$

Alternatively,

$${}_{\mathrm{I}}F_{1\alpha1'\beta}(x,x';\mathbf{X}]$$
$$=-\frac{1}{6}\frac{1}{2\pi}(t_{\alpha}(\mathbf{X}(x))s_{\beta}(\mathbf{X}(x'))$$
$$+s_{\alpha}(\mathbf{X}(x))t_{\beta}(\mathbf{X}(x')))\delta_{1',1}(x,x') \qquad (2.22)$$

in terms of the privileged spacetime basis (I.2.1).

In hypersurface dynamics, however, all quantities should be expressed directly in terms of the hypersurface data. To do that, we must use the nonlocal reconstruction (I.2.28) of the spacetime basis $\mathbf{e}_{(\pm)}$ from the hypersurface basis $\mathbf{n}_{(\pm)}$. The anomaly (2.22) then assumes the form

$$ {}_{\rm I}F_{1\alpha 1'\beta}(x,x';\mathbf{X}) = -\frac{1}{12}\frac{1}{2\pi}(\Lambda^{-1}(x)n_{(+)\alpha}(x)\Lambda^{-1}(x')n_{(+)\beta}(x') - \Lambda(x)n_{(-)\alpha}(x)\Lambda(x')n_{(-)\beta}(x'))\delta_{1',1}(x,x') \tag{2.23} $$

with the slope factor $\Lambda(x;\mathbf{X}]$ given by the integrals (I.2.31) and (I.2.35).

A manifestly covariant form of the second part of the anomaly is

$$ {}_{\rm III}F_{1\alpha 1'\beta}(x,x';\mathbf{X}] = -\frac{1}{12}\frac{1}{2\pi}(n_{(+)\alpha}^{\ \ 1}(x)n_{(+)\beta}^{\ \ 1}(x')D^{(-)}{}_1 D^{(-)}{}_{1'} - n_{(-)\alpha}^{\ \ 1}(x)n_{(-)\beta}^{\ \ 1}(x')D^{(+)}{}_1 D^{(+)}{}_{1'})\delta_{1',1}(x,x') \ . \tag{2.24} $$

To arrive at Eq. (2.24) from Eq. (2.18) we have used the properties (I.2.27) of the null hypersurface basis and the fact that

$$ D^{(+)}{}_1 n_{(-)\alpha}^{\ \ 1} = 0 = D^{(-)}{}_1 n_{(+)\alpha}^{\ \ 1} \ . \tag{2.25} $$

Note that ${}_{\rm III}F_{1\alpha 1'\beta}$ is expressed directly in terms of the hypersurface data and does not involve the slope factor Λ.

The algebra (2.19) of the Hamiltonian flux operators may be smeared by conformal Killing vector fields $\mathbf{u},\mathbf{v}\in LC$, with the result

$$ \frac{1}{i}[h(\mathbf{u}),h(\mathbf{v})] = h([\mathbf{u},\mathbf{v}]) - F(\mathbf{u},\mathbf{v}) \ . \tag{2.26} $$

We see that $h(\mathbf{u})$, $\mathbf{u}\in LC$, no longer represent the algebra LC of conformal isometries because of the smeared anomaly

$$ F(\mathbf{u},\mathbf{v}) := \int_\Sigma dx^1 \int_\Sigma dx^{1'} \times u^\alpha(\mathbf{X}(x))F_{1\alpha 1'\beta}(x,x';\mathbf{X}]u^\beta(\mathbf{X}(x')) \ . \tag{2.27} $$

However, $h(\mathbf{u})$ are still constants of motion, as in the classical case (I.4.25):

$$ \frac{1}{i}[h(\mathbf{u}),\mathbf{P}(\mathbf{V})] = 0, \quad \forall \mathbf{u}\in LC \text{ and } \forall \mathbf{V}\in L\ \mathrm{Diff}M \ . \tag{2.28} $$

The proof is exactly the same as in Eq. (I.4.26): $h(\mathbf{u})$ depends only on the Heisenberg data $\mathbf{q},\mathbf{p},\mathbf{a}_k,\mathbf{a}^*{}_k$, but not on the Heisenberg embedding. This is made entirely explicit in the case of the Virasoro generators (2.9). We shall discuss the significance of these two results, Eqs. (2.26) and (2.28), after we pass to the Schrödinger picture.

C. Anomaly in the algebra of the Heisenberg evolution generators

The quantum algebra (2.4) of the Heisenberg embedding momenta has no anomaly. On the other hand, the algebra of the Hamiltonian flux operators has an anomaly, Eqs. (2.19) or (2.26). Therefore it is not surprising that the anomaly finds its way into the algebra of the natural candidates

$$ \Pi_{1\alpha}(x) := \mathbf{P}_{1\alpha}(x) - h_{1\alpha}(x) \tag{2.29} $$

for the Schrödinger embedding momentum operators. Because of this anomaly, the expressions (2.29) do not commute among themselves and hence cannot be interpreted as momentum operators [this is why we denote them by a symbol $\Pi_{1\alpha}$ which differs from that used for the classical quantities (I.3.47)]. The operators $\Pi_{1\alpha}$ still retain the classical function of evolving the Schrödinger field variables in the Heisenberg picture, Eq. (I.3.53). For that reason we shall call them the Heisenberg evolution generators.

To evaluate the anomaly we smear Eq. (2.29) by a conformal Killing vector field $\mathbf{u}\in LC$ and use Eqs. (2.4), (2.26), and (2.28):

$$ \frac{1}{i}[\Pi(\mathbf{u}),\Pi(\mathbf{v})] = \Pi(-[\mathbf{u},\mathbf{v}]) - F(\mathbf{u},\mathbf{v}) \ . \tag{2.30} $$

The unsmeared form of Eq. (2.30) is

$$ \frac{1}{i}[\Pi_{1\alpha}(x),\Pi_{1'\beta}(x')] = -F_{1\alpha 1'\beta}(x,x';\mathbf{X}] \ ; \tag{2.31} $$

this can then be smeared by arbitrary vector fields $\mathbf{U},\mathbf{V}\in L\ \mathrm{Diff}M$ to yield

$$ \frac{1}{i}[\Pi(\mathbf{U}),\Pi(\mathbf{V})] = \Pi(-[\mathbf{U},\mathbf{V}]) - F(\mathbf{U},\mathbf{V}) \ . \tag{2.32} $$

D. Anomaly in the Dirac algebra

In parametrized field theories it is best to handle the constraints $\mathbf{P}_{1\alpha}(x)$, the Hamiltonian flux $h_{1\alpha}(x)$, and the Heisenberg evolution generators $\Pi_{1\alpha}(x) = \mathbf{P}_{1\alpha}(x) - h_{1\alpha}(x)$ in the unprojected form. Their commutators then naturally lead to the commutator algebras of the smeared quantities $\mathbf{P}(\mathbf{U})$, $\Pi(\mathbf{U})$, $\mathbf{U}\in L\ \mathrm{Diff}M$, and $h(\mathbf{u})$, $\mathbf{u}\in LC$. In general relativity and string theory, however, it is customary to work in terms of the projected quantities $\mathbf{P}_{11\perp},\mathbf{P}_{11}$; $h_{11\perp},h_{11}$; and $\Pi_{11\perp},\Pi_{11}$. In classical theory, the Poisson brackets of either set of these projected quantities follow the same pattern, namely, Eq. (I.3.34), which defines what is meant by the Dirac algebra. In quantum theory, the commutators $(1/i)[\ ,\ ]$ of the projected constraint operators $\mathbf{P}_{11\perp},\mathbf{P}_{11}$ follow exactly the Dirac algebra. However, the commutators $(1/i)[\ ,\ ]$ of the Hamiltonian flux operators $h_{11\perp},h_{11}$ or of the Heisenberg evolution generators $\Pi_{11\perp},\Pi_{11}$ develop an anomaly. To compare this anomaly with the expressions given in the literature, we must work out the commutator algebra of these projections. By projecting Eq. (2.19) we get the algebra

$$\frac{1}{i}[h_{11}(x),h_{1'1'}(x')]$$
$$=h_{11}(x)\delta_{1',1}(x,x')-(x\leftrightarrow x')-F_{11\,1'1'}(x,x')\ ,$$
$$\frac{1}{i}[h_{11\perp}(x),h_{1'1'}(x')]$$
$$=h_{11\perp}(x)\delta_{1',1}(x,x')-(x\leftrightarrow x')-F_{11\perp 1'1'}(x,x')\ ,$$
$$(2.33)$$
$$\frac{1}{i}[h_{11\perp}(x),h_{1'1'\perp}(x')]$$
$$=h_{11}(x)\delta_{1',1}(x,x')-(x\leftrightarrow x')-F_{11\perp 1'1'\perp}(x,x')\ ,$$

which differs from the classical Dirac algebra only by the projected anomaly

$$F_{11\,1'1'}(x,x'):=F_{1\alpha 1'\beta}(x,x')X^{\alpha}_{1}(x)X^{\beta}_{1'}(x')\ ,$$
$$F_{11\perp 1'1'}(x,x'):=F_{1\alpha 1'\beta}(x,x')n^{\alpha}_{1}(x)X^{\beta}_{1'}(x')\ , \qquad (2.34)$$
$$F_{11\perp 1'1'\perp}(x,x'):=F_{1\alpha 1'\beta}(x,x')n^{\alpha}_{1}(x)X^{\beta}_{1'}(x')\ .$$

By projecting Eq. (2.31) we get exactly the same anomalous algebra for the projections $\Pi_{11\perp}$ and Π_{11}.

We must evaluate expressions (2.34). From Eqs. (2.23) and (2.24) (using $n_{(\pm)}{}^{1}_{\alpha}X^{\alpha}_{1}=\pm 1$ and $n_{(\pm)}{}^{1}_{\alpha}n^{\alpha}_{1}=-1$) we obtain

$${}_{\mathrm{I}}F_{11\,1'1'}(x,x')={}_{\mathrm{I}}F_{11\perp 1'1'\perp}(x,x')$$
$$=\frac{1}{12}\frac{1}{2\pi}(\Lambda^{2}(x)-\Lambda^{-2}(x))$$
$$\times g_{1}(x)\delta_{1',1}(x,x')g_{1'}(x')\ ,$$
$$(2.35)$$
$${}_{\mathrm{I}}F_{11\perp 1'1'}(x,x')=\frac{1}{12}\frac{1}{2\pi}(\Lambda^{2}(x)+\Lambda^{-2}(x))$$
$$\times g_{1}(x)\delta_{1',1}(x,x')g_{1'}(x')\ ,$$

and

$${}_{\mathrm{III}}F_{11\,1'1'}(x,x')$$
$$={}_{\mathrm{III}}F_{11\perp 1'1'\perp}(x,x')$$
$$=\frac{1}{12}\frac{1}{2\pi}(D^{(+)}{}_{1}D^{(+)}{}_{1'}-D^{(-)}{}_{1}D^{(-)}{}_{1'})\delta_{1',1}(x,x')\ , \qquad (2.36)$$

$${}_{\mathrm{III}}F_{11\perp 1'1'}(x,x')$$
$$=-\frac{1}{12}\frac{1}{2\pi}(D^{(+)}{}_{1}D^{(+)}{}_{1'}+D^{(-)}{}_{1}D^{(-)}{}_{1'})\delta_{1',1}(x,x')\ .$$

The combinations of the slope factor which occur in Eq. (2.35) can be expressed in terms of the velocity $v(x)$ of the hypersurface observer **n** relative to the privileged inertial observer **t**:

$$\Lambda^{2}-\Lambda^{-2}=4\frac{v}{1-v^{2}}\quad\text{and}\quad\Lambda^{2}-\Lambda^{-2}=2\frac{1+v^{2}}{1-v^{2}}\ . \qquad (2.37)$$

Furthermore, the $D^{(\pm)}{}_{1}$ derivatives in Eqs. (2.36) can be replaced by the metric covariant derivative D_{1} and the extrinsic curvature. Each of the $D^{(\pm)}{}_{1}$ and $D^{(\pm)}{}_{1'}$ derivatives in Eq. (2.36) acts on a covector. Under such an action,

$$D^{(\pm)}{}_{1}=D_{1}\pm g_{1}K \qquad (2.38)$$

in accordance with Eq. (I.2.24). Hence,

$$(D^{(+)}{}_{1}D^{(+)}{}_{1'}-D^{(-)}{}_{1}D^{(-)}{}_{1'})\delta_{1',1}(x,x')$$
$$=2K_{,1}(x)\delta_{1',1}(x,x')g_{1'}(x')-(x\leftrightarrow x') \qquad (2.39)$$

and

$$(D^{(+)}{}_{1}D^{(+)}{}_{1'}+D^{(-)}{}_{1}D^{(-)}{}_{1'})\delta_{1',1}(x,x')$$
$$=2(D_{1}D_{1'}+K(x)g_{1}(x)K(x')g_{1}(x'))\delta_{1',1}(x,x')$$
$$=2g_{11}(x)\Delta D_{1'}\delta_{1',1}(x,x')$$
$$+(K^{2}(x)g_{1}(x)\delta_{1',1}(x,x')g_{1'}(x')-(x\leftrightarrow x'))\ .$$
$$(2.40)$$

The equivalence of the last two forms of Eq. (2.40) can best be proved by smearing this equation by two test vector fields $M^{1}(x)$ and $N^{1'}(x')$ on Σ.

The rearrangement (2.39) and (2.40) leads us to the final form for the projected anomaly (2.36): namely,

$${}_{\mathrm{III}}F_{11\,1'1'}(x,x')={}_{\mathrm{III}}F_{11\perp 1'1'\perp}(x,x')$$
$$=-\frac{1}{6}\frac{1}{2\pi}(K_{,1}(x)\delta_{1',1}(x,x')g_{1'}(x')$$
$$-(x\leftrightarrow x'))\ , \qquad (2.41)$$

$${}_{\mathrm{III}}F_{11\perp 1'1'}(x,x')$$
$$=-\frac{1}{6}\frac{1}{2\pi}(D_{1}D_{1'}$$
$$+g_{1}(x)K(x)g_{1'}(x')K(x'))\delta_{1',1}(x,x')\ .$$
$$(2.42)$$

Note that the third derivative of the δ function disappears from the projections (2.41) and persists only in the mixed projection (2.42).

Let us compare our covariant factor ordering and the resulting anomaly (2.35), (2.41), and (2.42) with the prescription discussed by Teitelboim.[7] By adapting a procedure common in string theory,[6] Teitelboim expands the fields $\phi(x)$ and $\pi_1(x)$ on a given embedding $X:\Sigma\to M$ in Fourier components with respect to a given coordinate system x^1 on Σ, and normal orders the operators $h_{11\perp}(x)$ and $h_{11}(x)$ in the corresponding creation and annihilation operators. Unlike our normal ordering based on the Heisenberg modes, Teitelboim's prescription is not covariant. It depends (among other things) on the choice of the spatial coordinate x^1. This is reflected in the structure of the anomaly. Teitelboim's anomaly resides entirely in the $F_{11\perp 1'1'}(x,x')$ component, while the other two components $F_{11\,1'1'}(x,x')$ and $F_{11\perp 1'1'\perp}(x,x')$ vanish. The form of $F_{11\perp 1'1'}(x,x')$ itself is also different from our expressions. Teitelboim's ${}_{\rm I}F_{11\perp 1'1'}(x,x')$ is proportional to $\delta_{1',1}(x,x')$, without the geometric correcting factor $(\Lambda^2(x)+\Lambda^{-2}(x))g_1(x)g_{1'}(x')$, but with a coefficient which depends on the inverse square of the *coordinate* length. Similarly, Teitelboim's ${}_{\rm III}F_{11\perp 1'1'}(x,x')$ is equal to $-(12\pi)^{-1}$ times the third *partial* derivative $\delta_{1',111}(x,x')$ of the δ function. Our expression (2.24) uses the covariant derivatives $D_1D_{1'}$ which ensure covariance under DiffΣ, and in addition contains the terms which depend on the extrinsic curvature (bending) of the embedding. It is obvious that the two prescriptions do not coincide on a generic embedding even if one uses the geometrically privileged arc length parameter σ in the role of x^1. The only case in which the two expressions for the anomaly equal each other is on maximal circles parametrized by the arc length where everything reduces to the result originally obtained by Boulware and Deser.[4]

E. Does the anomaly matter in the Dirac constraint quantization?

In the Heisenberg picture, the states Ψ are embedding independent,

$$\mathbf{P}(\mathbf{U})\Psi[\mathbf{X},\boldsymbol{\phi}]=0 \quad \forall \mathbf{U}\in L\ \mathrm{Diff}M\ , \tag{2.43}$$

and the field $\phi(x;\mathbf{X},\boldsymbol{\phi},\boldsymbol{\pi}],\pi_1(x;\mathbf{X},\boldsymbol{\phi},\boldsymbol{\pi}]$ is propagated by $\Pi(\mathbf{U})[\mathbf{X},\mathbf{P},\boldsymbol{\phi},\boldsymbol{\pi}]$ by the Heisenberg equations of motion

$$\frac{1}{i}[\phi(x),\Pi(\mathbf{U})]=0=\frac{1}{i}[\pi_1(x),\Pi(\mathbf{U})]\ . \tag{2.44}$$

Because $\mathbf{P}(\mathbf{U})$ represent L Diff$M$ without any anomaly [Eq. (2.4)] the constraints (2.43) on the physical states $\Psi\in H_0$ of the system are consistent. Were there an anomaly in the commutator (2.4) of the smeared Heisenberg momenta, $(1/i)[\mathbf{P}(\mathbf{U}),\mathbf{P}(\mathbf{V})]=\mathbf{P}(-[\mathbf{U},\mathbf{V}])-\bar{F}(\mathbf{U},\mathbf{V})$, the constraints (2.43) would force the physical states Ψ to vanish. For the consistency of the Dirac constraint quantization it is vital that the quantum commutator algebra of the constraint operators not have any anomaly. The same kind of argument can of course be carried out with the Dirac algebra of the projected constraints; the consistency of the constraint equation again depends on the fact that the quantum Dirac algebra of the projected constraint operators does not have any anomaly.

We must now decide whether the consistency of the field evolution (2.44) is in jeopardy because the quantum algebra (2.32) of the Heisenberg evolution generators $\Pi(\mathbf{U})$ has an anomaly. Let us evolve the field first by deforming the embedding in the steps $\mathbf{U}(\mathbf{X}(x)),\mathbf{V}(\mathbf{X}(x))$, then, equivalently, in the steps $\mathbf{V}(\mathbf{X}(x))$, $\mathbf{U}(\mathbf{V}(x))$, and $[\mathbf{U},\mathbf{V}](\mathbf{X}(x))$, and see whether the final result is the same. In other words, we must check the validity of the equation

$$\frac{1}{i}[(1/i)[\phi(x),\Pi(\mathbf{U})],\Pi(\mathbf{V})]-(\mathbf{U}\leftrightarrow\mathbf{V}) = \frac{1}{i}[\phi(x),\Pi(-[\mathbf{U},\mathbf{V}])]\ . \tag{2.45}$$

Now, by the Jacobi identity and the anomalous representation equation (2.32) the left-hand side of Eq. (2.45) reduces to

$$\frac{1}{i}[\phi(x),(1/i)[\Pi(\mathbf{U}),\Pi(\mathbf{V})]] = \frac{1}{i}[\phi(x),\Pi(-[\mathbf{U},\mathbf{V}])]+\frac{1}{i}[F(\mathbf{U},\mathbf{V}),\phi(x)]\ ; \tag{2.46}$$

the last commutator is indeed equal to zero because the anomaly depends only on the embeddings and the field $\phi(x)$ does not depend on the conjugate embedding momentum. The same reasoning, of course, applies to the evolution of $\pi_1(x)$.

The evolution of the field variables $\phi(x),\pi_1(x)$ by $\Pi(\mathbf{U})$ is thus consistent in spite of the anomaly in the algebra of the $\Pi(\mathbf{U})$'s. A fundamental difference between the Heisenberg evolution generators $\Pi(\mathbf{U})$ and the constraints $\mathbf{P}(\mathbf{U})$ is that the former are not supposed to vanish. Therefore, there is no equation for $\Pi(\mathbf{U})$ analogous to Eq. (2.43) in which the anomaly would matter.

In the Heisenberg picture we can live with the generators $\Pi(\mathbf{U})$ whose algebra has an anomaly. However, if we acquiesce in this, we must give up any hope that we can ever pass to the Schrödinger picture: the generators $\Pi_{1\alpha}(x)$ have wrong commutation relations to become the Schrödinger embedding momenta. The question remains whether we can replace the Heisenberg evolution generators $\Pi(\mathbf{U})$ by some other operators $P(\mathbf{U})$ which evolve the fields equally well, but which have the commutators appropriate for the smeared momenta. We shall show that this is indeed possible and thereby open the way from the Heisenberg to the Schrödinger picture.

III. REMOVAL OF THE ANOMALY

A. Integrability of the anomaly

Our trouble is that the Heisenberg evolution generators (2.29) do not commute among themselves [Eq. (2.31)] and thus cannot serve as the Schrödinger embedding momenta. The commutator in Eq. (2.31), however, must satisfy the Jacobi identity. Because the anomaly depends only on the embedding $\mathbf{X}(x)$ but not on the dynamical variables $\mathbf{q},\mathbf{p},\mathbf{a}_k,\mathbf{a}^*{}_k$, we get

$$\frac{1}{i}[(1/i)[\Pi_{1\alpha}(x),\Pi_{1'\beta}(x')],\Pi_{1''\gamma}(x'')]+\text{cyclic permutations }(\alpha x,\beta x',\gamma x'')$$
$$=-\frac{1}{i}[F_{1\alpha 1'\beta}(x,x';\mathbf{X}],\mathbf{P}_{1''\gamma}(x'')]+\text{cyclic permutations }(\alpha x,\beta x',\gamma x'')$$
$$=\{F_{1\alpha 1'\beta}(x,x';\mathbf{X}],\mathbf{P}_{1''\gamma}(x'')\}+\text{cyclic permutations }(\alpha x,\beta x',\gamma x'')$$
$$=\delta F_{1\alpha 1'\beta}(x,x';\mathbf{X}]/\delta \mathbf{X}^{\gamma}(x'')+\text{cyclic permutations }(\alpha x,\beta x',\gamma x'')=0\ . \quad (3.1)$$

This is the statement that the functional exterior derivative of $F_{1\alpha 1'\beta}(x,x';\mathbf{X}]$ with respect to the embedding variables $\mathbf{X}^{\gamma}(x)$ vanishes. [Note that in Eq. (3.1) it is permissible to pass from the operators $\mathbf{X}^{\alpha}(x)$ and $\mathbf{P}_{1''\gamma}(x'')$ to the classical variables while replacing the commutator by the corresponding Poisson bracket.]

The final situation resembles that of a charged particle $e=1$ moving in a configuration space Q with coordinates q^a in an external magnetic field $F_{ab}(q)$ with a vector potential $A_a(q)$. The generalized momentum P_a of the particle has the standard Poisson brackets,

$$\{q^a,P_b\}=\delta^a_b,\quad \{P_a,P_b\}=0\ , \quad (3.2)$$

but the mechanical momentum

$$\Pi_a=P_a+A_a \quad (3.3)$$

of the particle does not. Indeed,

$$\{\Pi_a,\Pi_b\}=-F_{ab} \quad (3.4)$$

yields the field strength

$$F_{ab}=-\{A_a,P_b\}+\{A_b,P_a\}=-A_{a,b}+A_{b,a}\ . \quad (3.5)$$

By its construction (3.5), F_{ab} is a closed two-form:

$$\{F_{ab},P_c\}+\text{cyclic permuations }(abc)$$
$$=F_{ab,c}+F_{bc,a}+F_{ca,b}=0\ . \quad (3.6)$$

The argument can be inverted. Let us have the mechanical momentum Π_a which satisfies Eq. (3.4) in which the field strength is closed, Eq. (3.6). On a simply connected configuration space Q there exists a vector potential A_a whose curl generates F_{ab}, Eq. (3.5). Of course, the potential A_a is determined by Eq. (3.5) only up to a gauge transformation

$$A_a\to\overline{A}_a=A_a+\Omega_{,a}\ . \quad (3.7)$$

When we complement the mechanical momentum Π_a by the vector potential $-A_a$ we get the generalized momentum

$$P_a:=\Pi_a-A_a \quad (3.8)$$

with correct Poisson brackets (3.2).

In our original problem, $\text{Emb}(\Sigma,M)$ plays the role of Q, the embedding variables $\mathbf{X}^{\alpha}(x)$ the role of the coordinates q^a, and the anomaly $F_{1\alpha 1'\beta}(x,x';\mathbf{X}]$ the role of the field strength $F_{ab}(q)$. The anomaly two-form is closed: Eq. (3.1). Our space of orientation-preserving spacelike embeddings $\text{Emb}(\Sigma,M)$ is simply connected because each continuous closed curve $\mathbf{X}^{\alpha}(x,t)$ in $\text{Emb}(\Sigma,M)$ can be continuously deformed into a point $\mathbf{X}^{\alpha}(x)$. Therefore, we expect that there exists a potential $A_{1\alpha}(x;\mathbf{X}]$ in $\text{Emb}(\Sigma,M)$ whose functional curl generates the anomaly

$$F_{1\alpha 1'\beta}(x,x';\mathbf{X}]$$
$$=-\{A_{1\alpha}(x),\mathbf{P}_{1'\beta}(x')\}+\{A_{1'\beta}(x'),\mathbf{P}_{1\alpha}(x)\}$$
$$=-\delta A_{1\alpha}(x;\mathbf{X}]/\delta\mathbf{X}^{\beta}(x')+\delta A_{1'\beta}(x';\mathbf{X}]/\delta\mathbf{X}^{\alpha}(x)\ . \quad (3.9)$$

We shall call such a potential $A_{1\alpha}(x;\mathbf{X}]$ the anomaly potential. Of course, $A_{1\alpha}(x;\mathbf{X}]$ is determined by Eq. (3.9) only up to a functional gradient,

$$A_{1\alpha}(x;\mathbf{X}]\to\overline{A}_{1\alpha}(x;\mathbf{X}]=A_{1\alpha}(x;\mathbf{X}]+\delta\Omega[\mathbf{X}]/\delta\mathbf{X}^{\alpha}(x)\ , \quad (3.10)$$

of an arbitrary gauge potential $\Omega[\mathbf{X}]$ which depends solely on embeddings. Once we find an anomaly potential which generates the anomaly by Eq. (3.9) we can subtract it from the Heisenberg evolution generator $\Pi_{1\alpha}(x)$ and thereby construct the operators

$$P_{1\alpha}(x;\mathbf{X},\mathbf{P},\mathbf{q},\mathbf{p},\mathbf{a}_k,\mathbf{a}^*{}_k]:=\Pi_{1\alpha}(x)-A_{1\alpha}(x;\mathbf{X}] \quad (3.11)$$

which commute among themselves:

$$\frac{1}{i}[P_{1\alpha}(x),P_{1'\beta}(x')]=0 \ . \tag{3.12}$$

To prevent any misinterpretation of our analogy, let us stress the physical difference between the two situations. In the case of a charged particle the coordinates q^a are true dynamical variables; in our problem the embedding variables $\mathbf{X}^\alpha(x)$ play the role of time, which is a canonical coordinate in the extended phase space, but is not one of the physical degrees of freedom. Second, the field strength F_{ab} already exists at the classical level, prior to quantization, while the anomaly $F_{1\alpha 1'\beta}(x,x';\mathbf{X}]$ arises only in the quantized theory as a consequence of the normal ordering and vanishes in the classical limit $\hbar\to 0$. It is not our intention to stretch the analogy beyond its formal limits.

B. The anomaly potential

We now know that an anomaly potential exists; our next task is to find it. As so often in the past we find it convenient to carry out our calculations in null coordinates and to cast the result into a covariant form only in the end.

In the null coordinates Eq. (3.9) reduces to

$$\delta A_{1\pm}(x)/\delta\mathbf{T}^{\pm}(x')-(x\leftrightarrow x)=-F_{1\pm 1'\pm}(x,x';\mathbf{T}^{\pm}] \tag{3.13}$$

and

$$\delta A_{1\pm}(x)/\delta\mathbf{T}^{-}(x')-\delta A_{1'-}(x')/\delta\mathbf{T}^{+}(x)=0 \ . \tag{3.14}$$

The anomaly in Eq. (3.13) is given by the expressions (2.16)–(2.18). Equation (3.14) can easily be satisfied by putting

$$A_{1\pm}(x;\mathbf{X}]=A_{1\pm}(x;\mathbf{T}^{\pm}] \ . \tag{3.15}$$

This is consistent with Eq. (3.13) because $F_{1+1'+}$ depends only on $\mathbf{T}^+(x)$ and $F_{1-1'-}$ only on $\mathbf{T}^-(x)$. A particular solution of these equations can be found by inspection:

$$A_{1\pm}(x;\mathbf{T}^{\pm}]={}_{\mathrm{I}}A_{1\pm}(x;\mathbf{T}^{\pm}]+{}_{\mathrm{III}}A_{1\pm}(x;\mathbf{T}^{\pm}] \ , \tag{3.16}$$

with

$${}_{\mathrm{I}}A_{1\pm}(x;\mathbf{T}^{\pm}]=\mp\frac{1}{24}\frac{1}{2\pi}\mathbf{T}^{\pm}{}_{,1}(x) \ , \tag{3.17}$$

$${}_{\mathrm{III}}A_{1\pm}(x;\mathbf{T}^{\pm}]=\mp\frac{1}{24}\frac{1}{2\pi}(T^{\pm}{}_{,1}(x))^{-1}{}_{,11} \ . \tag{3.18}$$

While ${}_{\mathrm{I}}A_{1\pm}(x)$ is a covector under spatial transformations, ${}_{\mathrm{III}}A_{1\pm}(x)$ is not. One cannot therefore meaningfully subtract ${}_{\mathrm{III}}A_{1\pm}(x)$ from the covector $\Pi_{1\pm}(x)$ to obtain $P_{1\pm}(x)$. To improve this, write

$$\begin{aligned}{}_{\mathrm{III}}A_{1\pm}(x;\mathbf{T}^{\pm}]&=-\frac{1}{24}\frac{1}{2\pi}\left[(\mathbf{T}^{\pm}{}_{,1})^{-1}\left[\frac{\mathbf{T}^{-}{}_{,11}}{\mathbf{T}^{-}{}_{,1}}-\frac{\mathbf{T}^{+}{}_{,11}}{\mathbf{T}^{+}{}_{,1}}\right]\right]_{,1}\pm\frac{1}{24}\frac{1}{2\pi}\left[\frac{\mathbf{T}^{\mp}{}_{,11}}{\mathbf{T}^{-}{}_{,1}\mathbf{T}^{+}{}_{,1}}\right]_{,1}\\&=-\frac{1}{12}\frac{1}{2\pi}((\mathbf{T}^{\pm}{}_{,1})^{-1}g_1K)_{,1}+\delta\Omega[\mathbf{T}^+,\mathbf{T}^-]/\delta\mathbf{T}^{\pm}(x) \ ,\end{aligned} \tag{3.19}$$

where

$$\Omega[\mathbf{T}^+,\mathbf{T}^-]:=-\frac{1}{24}\frac{1}{2\pi}\int_\Sigma dx^1\ln(\mathbf{T}^+{}_{,1}(x))\partial_1\ln(-\mathbf{T}^-{}_{,1}(x)) \ , \tag{3.20}$$

and the extrinsic curvature was introduced by Eq. (I.2.23). We see that we can gauge the old potential (3.18) into a new potential

$${}_{\mathrm{III}}A_{1\pm}(x;\mathbf{X}]=-\frac{1}{12}\frac{1}{2\pi}((\mathbf{T}^{\pm}{}_{,1})^{-1}g_1K)_{,1} \ , \tag{3.21}$$

which is a spatial covector and still generates the same anomaly. Note that while the + component of the old potential depended only on $\mathbf{T}^+(x)$ and the − component only on $\mathbf{T}^-(x)$, each component of the new potential (3.21) depends on both embedding variables $\mathbf{T}^{\pm}(x)$. It is impossible to construct an anomaly potential which would be a spatial covector and whose ± components would depend only on the corresponding $\mathbf{T}^{\pm}(x)$ variables.

The potentials (3.17) and (3.21) can be written in a form which holds in an arbitrary system of spacetime coordinates. For the potential (3.17), this form relies on the privileged $\mathbf{e}_{(\pm)}$ basis,

$$\begin{aligned}{}_{\mathrm{I}}A_{1\alpha}(x;\mathbf{X}]&\\&=-\frac{1}{24}\frac{1}{2\pi}(e_{(+)\alpha}e_{(+)\beta}+e_{(-)\alpha}e_{(-)\beta})n_1^\beta\\&=-\frac{1}{24}\frac{1}{2\pi}(\Lambda^{-2}n_{(+)\alpha}n_{(+)\beta}+\Lambda^2n_{(-)\alpha}n_{(-)\beta})n_1^\beta \ .\end{aligned} \tag{3.22}$$

Like the anomaly ${}_{\mathrm{I}}F_{1\alpha 1'\beta}(x,x';\mathbf{X}]$ which it generates, the potential (3.22) can be obtained from the hypersurface data only with help of a nonlocally constructed slope factor Λ. On the other hand, the covariant form of the potential ${}_{\mathrm{III}}A_{1\pm}$ is a local function of g_{11} and K_{11}:

$$\begin{aligned}{}_{\mathrm{III}}A_{1\alpha}(x;\mathbf{X}]&=-\frac{1}{6}\frac{1}{2\pi}\mathbf{X}_1^\beta\nabla_\beta(K\,g_1\mathbf{X}_\alpha^1)\\&=-\frac{1}{6}\frac{1}{2\pi}g_1(K_{,1}\mathbf{X}_\alpha^1-K^2n_\alpha) \ .\end{aligned} \tag{3.23}$$

In the last rearrangement of Eq. (3.23) we have used the Gauss-Weingaarten equation

$$\mathbf{X}^{\alpha}_{1}\nabla_{\alpha}\mathbf{X}^{\beta}_{1}=-g_{11}K\,n^{\beta}+\Gamma^{1}{}_{11}\mathbf{X}^{\beta}_{1}\ . \qquad (3.24)$$

C. Quantum canonical transformation to the Schrödinger picture

The Heisenberg evolution generators $\Pi_{1\alpha}(x)$ correctly propagate the quantum field $\phi(x),\pi_1(x)$ by the Heisenberg equations of motion (2.44), but their commutator has an anomaly, Eq. (2.31). When we subtract from them the anomaly potential, we construct a set of commuting operators (3.11). These operators still correctly propagate the field operators by the Heisenberg equations of motion,

$$\frac{1}{i}[\phi(x),P_{1'\alpha}(x')]=0=\frac{1}{i}[\pi_1(x),P_{1'\alpha}(x')]\ , \qquad (3.25)$$

because $\phi(x)$ and $\pi_1(x)$ commute with the anomaly potential $A_{1\alpha}(x;\mathbf{X}]$. Moreover, they retain the correct commutators

$$\frac{1}{i}[X^{\alpha}(x),P_{1'\beta}(x')]=0 \qquad (3.26)$$

with the embedding variables

$$X^{\alpha}(x)=\mathbf{X}^{\alpha}(x)\ . \qquad (3.27)$$

These properties enable us to accomplish a "quantum canonical transformation"[9]

$$\mathbf{X}^{\alpha}(x),\mathbf{P}_{1\alpha}(x),\mathbf{q},\mathbf{p},\mathbf{a}_k,\mathbf{a}^*{}_k\rightarrow X^{\alpha}(x),P_{1\alpha}(x),\phi(x),\pi_1(x) \qquad (3.28)$$

from the Heisenberg picture to the Schrödinger picture. First, the Schrödinger field operators $\phi(x),\pi_1(x)$ are defined in terms of the Heisenberg mode operators $\mathbf{q},\mathbf{p},\mathbf{a}_k,\mathbf{a}^*{}_k$ and the Heisenberg embedding $\mathbf{X}^{\alpha}(x)$ as in the classical theory, Eqs. (I.3.5), (I.3.6), (I.3.68), and (I.3.70). Because they are linear in the Heisenberg mode operators and no factor ordering is thus involved, the classical derivation of their fundamental Poisson brackets (I.3.69), (I.3.72), and (I.3.73) still holds for the commutators:

$$\frac{1}{i}[\phi(x),\phi(x')]=0=\frac{1}{i}[\pi_1(x),\pi_{1'}(x')]\ ,$$
$$\frac{1}{i}[\phi(x),\pi_{1'}(x')]=\delta_{1'}(x,x')\ . \qquad (3.29)$$

Second, the Heisenberg evolution generators are introduced by Eq. (2.29). Their definition involves normal ordering of the Hamiltonian flux operator in the Heisenberg mode operators. Their commutators (2.44) with the field operators nevertheless preserve the classical algebra (I.4.21) because the field operators are linear in the Heisenberg mode operators. On the other hand, the commutator algebra of two Heisenberg evolution generators contains an anomaly because each of them is quadratic in the Heisenberg mode operators and involves a normal ordering. We learned, however, that this anomaly can be canceled by subtracting from the Heisenberg evolution generators an anomaly potential and thereby passing to the Schrödinger embedding momentum operators (3.11). This is done at no expense to their commutators (3.25) with the Schrödinger field operators and their commutators (3.26) with the embedding. We are thus led to the conclusion that the transformation (3.28) from the Heisenberg picture to the Schrödinger picture accomplished by Eqs. (I.3.5), (I.3.6), (I.3.68), and (I.3.70) for the field variables and by Eqs. (3.27), (2.29), (3.11), (3.22), and (3.23) for the embedding variables leads to the correct commutators (3.29), (3.25), (3.26), and (3.12) between the fundamental Schrödinger operators. This is what makes the transition to the Schrödinger picture and the Dirac constraint quantization in that picture possible.

D. Cancellation of anomaly in the Dirac algebra

When the operators $P_{1\alpha}:=\mathbf{P}_{1\alpha}-h_{1\alpha}-A_{1\alpha}$ commute, their projections $P_{11\perp}:=P_{1\alpha}n^{\alpha}_{1}$ and $P_{11}:=P_{1\alpha}X^{\alpha}_{1}$ must necessarily obey the Dirac algebra [cf. (I.3.34)]

$$\frac{1}{i}[P_{11}(x),P_{1'1'}(x')]=P_{11}(x)\delta_{1',1}(x,x')-(x\leftrightarrow x')\ ,$$
$$\frac{1}{i}[P_{11\perp}(x),P_{1'1'}(x')]=P_{11\perp}(x)\delta_{1',1}(x,x')-(x\leftrightarrow x')\ , \qquad (3.30)$$
$$\frac{1}{i}[P_{11\perp}(x),P_{1'1'\perp}(x')]=P_{11}(x)\delta_{1',1}(x,x')-(x\leftrightarrow x')\ .$$

This implies that the projected anomaly (2.35), (2.41), and (2.42) must be related to the projected anomaly potential

$${}_{\mathrm{I}}A_{11}=-\frac{1}{24}\frac{1}{2\pi}g_{11}(\Lambda^2-\Lambda^{-2})\ ,$$
$${}_{\mathrm{I}}A_{11}=-\frac{1}{24}\frac{1}{2\pi}g_{11}(\Lambda^2+\Lambda^{-2})\ , \qquad (3.31)$$
$${}_{\mathrm{III}}A_{11}=-\frac{1}{6}\frac{1}{2\pi}g_1K_{,1}\ ,$$
$${}_{\mathrm{III}}A_{11\perp}=-\frac{1}{6}\frac{1}{2\pi}g_{11}K^2 \qquad (3.32)$$

by a set of three identities:

$$\{A_{11}(x),\mathbf{P}_{1'1'}(x')\}-(x\leftrightarrow x')=2F_{11\,1'1'}(x,x')\ , \qquad (3.33)$$
$$\{A_{11\perp}(x),\mathbf{P}_{1'1'\perp}(x')\}=F_{11\,1'1'\perp}(x,x')\ , \qquad (3.34)$$

and

$$\{A_{11\perp}(x),\mathbf{P}_{1'1'\perp}(x')\}-(x\leftrightarrow x')=2F_{11\perp1'1'\perp}(x,x')\ . \qquad (3.35)$$

The argument that these identities must be satisfied is always the same and we shall present it for only one of them, say, Eq. (3.35).

We use the definition (3.11) of the Schrödinger embedding momentum and write Eq. (3.30) in the form

$$\frac{1}{i}[\Pi_{11\perp}(x),\Pi_{1'1'\perp}(x')]-\left[\frac{1}{i}[A_{11\perp}(x),\mathbf{P}_{1'1'\perp}(x')]-(x\leftrightarrow x')\right]=\Pi_{11}(x)\delta_{1',1}(x,x')-(x\leftrightarrow x')$$
$$-(A_{11}(x)\delta_{1',1}(x,x')-(x\leftrightarrow x')) \ . \tag{3.36}$$

In Sec. II D we have noticed that the Π's satisfy the same anomalous Dirac algebra (2.33) as the h's. The first terms on each side of Eq. (3.36) thus bring in the anomaly $F_{11\perp 1'1'\perp}(x,x')$. The last term in Eq. (3.36), when we compare it with expressions (2.35) and (2.41), again yields the same anomaly. This leads us to the identity (3.35). Of course, one can also prove the identities (3.33)–(3.35) directly. For example, the identity (3.35) for the ${}_{\text{III}}A_{11\perp}$ part of the potential is a consequence of the Mainardi equation (I.A6).

The argument, of course, can also be run backward. By virtue of the identities (3.33)–(3.35), when we subtract the projected anomaly potential $A_{11}, A_{11\perp}$ from the projected Heisenberg evolution generator $\Pi_{11},\Pi_{11\perp}$, we cancel the anomaly in the Dirac algebra of $\Pi_{11},\Pi_{11\perp}$ and arrive at the Dirac algebra (3.30) for the projections $P_{11},P_{11\perp}$ of the Schrödinger embedding momentum.

IV. THE TOTAL HAMILTONIAN FLUX

By Eqs. (3.11) and (2.29) the Schrödinger embedding momentum operator is related to the Heisenberg embedding momentum operator through the relation

$$P_{1\alpha}(x)=\mathbf{P}_{1\alpha}(x)-H_{1\alpha}(x) \ , \tag{4.1}$$

in which

$$H_{1\alpha}(x):=h_{1\alpha}(x)+A_{1\alpha}(x) \ . \tag{4.2}$$

In either picture the expression (4.2) depends only on the field operators and the embedding. We can interpret Eqs. (4.1) and (4.2) by saying that our old interpretation of the Hamiltonian flux operator as the projected normal-ordered energy-momentum tensor $h_{1\alpha}=T_{\alpha\beta}n_1^\beta$ was inappropriate and that the true Hamiltonian flux operator is to be identified with the expression $H_{1\alpha}(x)$. The two flux operators $H_{1\alpha}(x)$ and $h_{1\alpha}(x)$ can be considered as two different factor orderings of the same classical flux. The anomaly potential $A_{1\alpha}(x)$ is a correction which must be added to the normal-ordered flux $h_{1\alpha}(x)$ to yield the new factor ordering. At a given point X, this correction depends on the embedding $\mathbf{X}(x)$ passing through that point. To distinguish it from $h_{1\alpha}(x)$ we shall call $H_{1\alpha}(x)$ the total Hamiltonian flux operator.

Because the expectation value of $h_{1\alpha}(x)$ vanishes in the Heisenberg vacuum state $\Psi_0=\Psi(n_k=0,\mathbf{p}=0)$,

$$\langle\Psi_0|h_{1\alpha}(x)|\Psi_0\rangle=0 \ , \tag{4.3}$$

we can also identify $A_{1\alpha}(x;\mathbf{X}]$ with the vacuum expectation value of the total Hamiltonian flux operator $H_{1\alpha}(x)$:

$$A_{1\alpha}(x;\mathbf{X}]=\langle\Psi_0|H_{1\alpha}(x)|\Psi_0\rangle \ . \tag{4.4}$$

We want to understand how the potential $A_{1\alpha}(x;\mathbf{X}]$ contributes to the energy and momentum distributions.

The field Hamiltonian flux is a projection of a conserved symmetric trace-free tensor, namely, the energy-momentum tensor $T_{\alpha\beta}$:

$$h_{1\alpha}(x)=T_{\alpha\beta}(\mathbf{X}(x))n_1^\beta(x;\mathbf{X}] \ . \tag{4.5}$$

Does there exist a purely geometric conserved symmetric and trace-free tensor $\Theta_{\alpha\beta}(X)$ such that

$$A_{1\alpha}(x)=\Theta_{\alpha\beta}(\mathbf{X}(x))n_1^\beta(x;\mathbf{X}]? \tag{4.6}$$

We shall show that the answer is "yes" for ${}_{\text{I}}A_{1\alpha}(x)$ and "no" for ${}_{\text{III}}A_{1\alpha}(x)$.

From the covariant form (3.22) of ${}_{\text{I}}A_{1\alpha}(x)$ we see that ${}_{\text{I}}\Theta_{\alpha\beta}(X)$ must be given by

$${}_{\text{I}}\Theta_{\alpha\beta}(X)=-\frac{1}{24}\frac{1}{2\pi}(e_{(+)\alpha}e_{(+)\beta}+e_{(-)\alpha}e_{(-)\beta}) \ . \tag{4.7}$$

This tensor is obviously symmetric and trace free. It is also conserved, because the vectors $\mathbf{e}_{(\pm)}$ are teleparallel and hence $\nabla_\gamma\,{}_{\text{I}}\Theta_{\alpha\beta}=0$. More generally, if $\mathbf{k}$ is a Killing vector field, the tensor field

$$\Theta^{\beta}_{\alpha}:=k_\alpha k^\beta+\tfrac{1}{2}(k_\gamma k^\gamma)\delta^\beta_\alpha \tag{4.8}$$

is conserved by virtue of the Killing equation. So is the sum of two such tensors. The tensor (4.7) is the sum of two tensors (4.8) generated by the null Killing vector fields $\mathbf{e}_{(\pm)}$ and it is therefore conserved. The tensor ${}_{\text{I}}\Theta_{\alpha\beta}$ can also be expressed in terms of the Killing vectors $\mathbf{t}$ and $\mathbf{s}$:

$${}_{\text{I}}\Theta_{\alpha\beta}=-\frac{1}{6}\frac{1}{2\pi}(t_\alpha t_\beta+s_\alpha s_\beta) \ . \tag{4.9}$$

Let us calculate the energy densities corresponding to ${}_{\text{I}}\Theta_{\alpha\beta}$ measured by different observers. The energy "density" measured by the privileged inertial observer $\mathbf{t}$ on any maximal hypersurface $T=$const is

$${}_{\text{I}}A(\mathbf{t},\mathbf{t}):={}_{\text{I}}\Theta_{\alpha\beta}t^\alpha t^\beta=-\frac{1}{6}\frac{1}{2\pi}<0 \ . \tag{4.10}$$

This constant negative energy density corresponds to the Casimir effect introduced by the closing of S^1. The energy density measured by the hypersurface observer $\mathbf{n}$ is

$$\begin{aligned}{}_{\text{I}}A_1(\mathbf{n},\mathbf{n}):=h_{1\alpha}n^\alpha={}_{\text{I}}\Theta_{\alpha\beta}n^\alpha n_1^\beta\\ =-\frac{1}{12}\frac{1}{2\pi}g_1(\Lambda^2+\Lambda^{-2})\\ =-\frac{1}{6}\frac{1}{2\pi}\frac{1+v^2}{1-v^2}g_1 \ ,\end{aligned} \tag{4.11}$$

where $v(x)$ is the velocity of the hypersurface observer $\mathbf{n}$ with respect to the privileged inertial observer $\mathbf{t}$. The density (4.11) is always negative; it has its maximal value for $v=0$, i.e., for $\mathbf{n}=\mathbf{t}$, and it goes to $-\infty$ as $v\to\pm 1$.

Unless $\mathbf{n}$ is a Killing vector field $\mathbf{n}=\mathbf{t}$, the total energy

$\int_\Sigma dx^1{}_I A_1(\mathbf{n},\mathbf{n})$ is not conserved. However, from Eq. (I.4.23) we know that

$$_{\rm I}A(\mathbf{u}):=\int_\Sigma dx^1 u^\alpha{}_{\rm I}\Theta_{\alpha\beta}n_1^\beta \tag{4.12}$$

is a constant of motion for any conformal Killing vector field $\mathbf{u}$. In particular, for $\mathbf{u}=\mathbf{t}$, $_{\rm I}A(\mathbf{t})=-\frac{1}{6}$ (the value which the total energy has on a maximal hypersurface). The energy density

$$_{\rm I}A_1(\mathbf{t},\mathbf{n}):={}_{\rm I}\Theta_{\alpha\beta}t^\alpha n_1^\beta=-\frac{1}{6}\frac{1}{2\pi}\frac{1}{1-v^2}g_1 \tag{4.13}$$

which the observer $\mathbf{t}$ measures on the hypersurface with the normal $\mathbf{n}$ differs from the privileged density (4.10) by the square of the Lorentz contraction factor $\gamma:=(1-v^2)^{-1/2}$.

The Virasoro components of $_{\rm I}A_{1\alpha}(x)$ are

$$_{\rm I}A^{(\pm)}{}_n:=\int_{-\pi}^{\pi}dT^\pm e^{-inT^\pm}{}_{\rm I}\Theta_{\pm\pm}=-\tfrac{1}{24}\delta_{n0} \ . \tag{4.14}$$

This term modifies the anomaly of the Virasoro algebra of the quantities

$$_{\rm I}H^{(\pm)}{}_n:=h^{(\pm)}{}_n+{}_{\rm I}A^{(\pm)}{}_n \ . \tag{4.15}$$

Indeed,

$$\frac{1}{i}[{}_{\rm I}H^{(\pm)}{}_m,{}_{\rm I}H^{(\pm)}{}_n]=i(m-n){}_{\rm I}H^{(\pm)}{}_{m+n}+i\tfrac{1}{12}m^3\delta_{m+n,0} \ ; \tag{4.16}$$

the extra term $_{\rm I}A^{(\pm)}{}_m$ cancels that part of the anomaly in Eq. (2.10) which is proportional to m.

The situation with $_{\rm III}A_{1\alpha}(x)$ is different. From Eq. (3.23) we can identify the symmetric trace-free tensor

$$\begin{aligned}{}_{\rm III}\Theta_{\alpha\beta}(x;\mathbf{X}]&=\frac{1}{12}\frac{1}{2\pi}((g^1K_{,1}-K^2)n_{(+)\alpha}n_{(+)\beta}\\&\quad-(g^1K_{,1}+K^2)n_{(-)\alpha}n_{(-)\beta})\\&=\frac{1}{12}\frac{1}{2\pi}((g^1K_{,1}-K^2)\Lambda^2e_{(+)\alpha}e_{(-)\alpha}\\&\quad-(g^1K_{,1}+K^2)\Lambda^{-2}e_{(-)\alpha}e_{(-)\beta})\end{aligned} \tag{4.17}$$

whose projection yields $_{\rm III}A_{1\alpha}(x;\mathbf{X}]$:

$$_{\rm III}A_{1\alpha}(x;\mathbf{X}]={}_{\rm III}\Theta_{\alpha\beta}(x;\mathbf{X}]n_1^\beta(x;\mathbf{X}] \ . \tag{4.18}$$

However, there is no spacetime tensor field $_{\rm III}\Theta_{\alpha\beta}(X)$ such that

$$_{\rm III}\Theta_{\alpha\beta}(x;\mathbf{X}]={}_{\rm III}\Theta_{\alpha\beta}(\mathbf{X}(x)) \ . \tag{4.19}$$

To see this, note first that expression (4.17) is a local functional of the embedding: if we keep a small stretch of $\mathbf{X}^\alpha(x)$ in the vicinity of a point $_0x$ fixed, but modify the embedding outside that stretch, $_{\rm III}\Theta_{\alpha\beta}(x;\mathbf{X}]$ at $_0x$ remains unchanged. For simplicity, choose $_0x$ such that $\mathbf{T}^\pm(_0x)=0$ (the embedding passes through the origin of the null coordinate system). Boost a piece of the embedding in the vicinity $(_0x-\epsilon,\,_0x+\epsilon)$ of the origin by a constant λ,

$$\overline{\mathbf{T}}^\pm(x)=\lambda^{\pm1}\mathbf{T}^\pm(x) \ , \tag{4.20}$$

and complete the boosted piece $\overline{\mathbf{T}}^\pm(x)$, $(_0x-\epsilon,\,_0x+\epsilon)$, arbitrarily into a new complete embedding $\overline{\mathbf{T}}^\pm(x)$. If $_{\rm III}\Theta_{\alpha\beta}(x;\mathbf{X}]$ were a restriction of a spacetime tensor field, Eq. (4.19), $_{\rm III}\Theta_{\alpha\beta}(_0x;\mathbf{T}^+,\mathbf{T}^-]$ should equal $_{\rm III}\Theta_{\alpha\beta}(_0x;\overline{\mathbf{T}}^+,\overline{\mathbf{T}}^-]$. The boost changes neither the spatial metric nor the extrinsic curvature as a function of x about $_0x$, but it changes the slope factor (I.2.29):

$$\Lambda(x)\to\overline{\Lambda}(x)=\lambda\Lambda(x) \ . \tag{4.21}$$

The null basis $e_{(\pm)}{}^\alpha(\mathbf{X}(_0x))$ at $_0x$ stays fixed. From the second form of Eq. (4.17) it follows that $_{\rm III}\Theta_{\alpha\beta}(_0x;\mathbf{T}^+,\mathbf{T}^-]\neq{}_{\rm III}\Theta_{\alpha\beta}(_0x;\overline{\mathbf{T}}^+,\overline{\mathbf{T}}^-]$. There is no spacetime tensor field $_{\rm III}\Theta_{\alpha\beta}(X)$ which would yield $_{\rm III}\Theta_{\alpha\beta}(x;\mathbf{X}]$ by restriction, Eq. (4.19).

We could come to the same conclusion by bending the embedding at $_0X^\alpha=\mathbf{X}^\alpha(_0x)$ without boosting it, i.e., by changing K and/or $K_{,1}$ at $_0x$ while keeping $\mathbf{X}_1^\alpha(_0x)$, $n_1^\alpha(_0x)$, and $\Lambda(_0x)$ fixed. Again, this changes $_{\rm III}\Theta_{\alpha\beta}(_0x;\mathbf{X}]$ as given by Eq. (4.17), and hence Eq. (4.19) must fail. Finally, the argument would still hold if we admitted $_{\rm III}\Theta_{\alpha\beta}(x;\mathbf{X}]$ with a trace.

For completeness let us note that, by virtue of the Mainardi equation (I.A6),

$$\begin{aligned}n^\alpha(x)\partial_\alpha K(x)&:=\int dx^{1\prime}n^\alpha(x')\delta K(x)/\delta X^\alpha(x')\\&=K^2(x) \ ,\end{aligned} \tag{4.22}$$

and hence

$$g^1K_{,1}\pm K^2=n_{(\pm)}{}^\alpha\partial_\alpha K \ . \tag{4.23}$$

This enables us to write $_{\rm III}\Theta_{\alpha\beta}(x;\mathbf{X}]$ in the form

$$\begin{aligned}{}_{\rm III}\Theta_{\alpha\beta}=\frac{1}{12}\frac{1}{2\pi}(&n_{(+)\alpha}n_{(+)\beta}n_{(-)}{}^\gamma\\&-n_{(-)\alpha}n_{(-)\beta}n_{(+)}{}^\gamma)\partial_\gamma K\end{aligned} \tag{4.24}$$

The energy density

$$_{\rm III}A_{1\alpha}(x;\mathbf{X}]n^\alpha(x,\mathbf{X}]=-\frac{1}{6}\frac{1}{2\pi}g_1K^2\le0 \tag{4.25}$$

measured by the hypersurface observer is always negative unless a piece of the embedding is extrinsically flat (unbent). On the other hand, the energy density measured on the hypersurface by the privileged inertial observer $\mathbf{t}$ is indefinite:

$$_{\rm III}A_{1\alpha}(x;\mathbf{X}]t^{\alpha}(\mathbf{X}(x)) = -\frac{1}{6}\frac{1}{2\pi}(Kg_1 t^1)_{,1} \overset{>}{\underset{<}{=}} 0 \ ; \tag{4.26}$$

indeed, the total anomaly energy vanishes,

$$_{\rm III}A(\mathbf{t}) := \int_{\Sigma} dx^1 \, _{\rm III}A_{1\alpha}(x) t^{\alpha}(\mathbf{X}(x)) = 0 \ , \tag{4.27}$$

and hence it is trivially conserved. Similarly, the inertial momentum density

$$_{\rm III}A_{1\alpha}(x;\mathbf{X}]s^{\alpha}(\mathbf{X}(x)) = -\frac{1}{6}\frac{1}{2\pi}(Kg_1 s^1)_{,1} \overset{>}{\underset{<}{=}} 0 \tag{4.28}$$

yields the vanishing total anomaly momentum

$$_{\rm III}A(\mathbf{s}) := \int_{\Sigma} dx^1 \, _{\rm III}A_{1\alpha}(x;\mathbf{X}]s^{\alpha}(\mathbf{X}(x)) = 0 \ . \tag{4.29}$$

Adding $_{\rm III}A_{1\alpha}$ to $h_{1\alpha}$ thus amounts to redistributing the inertial energy and momentum on a hypersurface while keeping their total values fixed. On the other hand, adding $_{\rm I}A_{1\alpha}$ to $h_{1\alpha}$ not only changes the energy distribution (4.13), but also lowers its total value by the Casimir energy $-\frac{1}{6}$.

The energy density (4.26) can be given a nice intuitive interpretation. Parametrize the embedding by the arc length σ. Then

$$t^1 = -dT/d\sigma = -v/\sqrt{1-v^2} \tag{4.30}$$

and

$$Kg_1 = d\ln\Lambda/d\sigma = \frac{dv(\sigma)}{d\sigma}\Big/(1-v^2) \ , \tag{4.31}$$

where $v(\sigma)$ is the velocity of the hypersurface observer with respect to the inertial observer. From here we see that the energy density

$$_{\rm III}A_{1\alpha}(\sigma;\mathbf{X}]t^{\alpha}(\mathbf{X}(\sigma)) = \frac{1}{6}\frac{1}{2\pi}\frac{d^2}{d\sigma^2}\gamma(\sigma) \tag{4.32}$$

is proportional to the second arc length derivative of the Lorentz contraction factor $\gamma(\sigma) := (1-v^2(\sigma))^{-1/2}$. A similar result holds for the momentum density

$$_{\rm III}A_{1\alpha}(\sigma;\mathbf{X}]s^{\alpha}(\mathbf{X}(\sigma)) = \frac{1}{6}\frac{1}{2\pi}\frac{d}{d\sigma}\frac{1}{v(\sigma)}\frac{d}{d\sigma}\frac{1}{\gamma(\sigma)} \ . \tag{4.33}$$

While $_{\rm III}A_{1\alpha}(x)$ smeared by the privileged Killing vector fields $\mathbf{t}$ and $\mathbf{s}$ [and thus also by an arbitrary Killing vector field (I.2.9)] is zero and hence conserved, $_{\rm III}A_{1\alpha}(x)$ smeared by a conformal Killing vector field $\mathbf{u}\in LC$ is not a constant of motion. This distinguishes $_{\rm III}A_{1\alpha}(x)$ from $_{\rm I}A_{1\alpha}(x)$; we have seen that $_{\rm I}A(\mathbf{u})$ *is* a constant of motion. To study what happens, multiply Eq. (3.23) by $u^{\alpha}(\mathbf{X}(x))$ and use the conformal Killing equation (I.4.2):

$$_{\rm III}A_{1\alpha}(x;\mathbf{X}]u^{\alpha}(\mathbf{X}(x)) = -\frac{1}{6}\frac{1}{2\pi}(Kg_1 u^1)_{,1} + \frac{1}{12}\frac{1}{2\pi}g_1 K\, W[\mathbf{u}] \ . \tag{4.34}$$

Integrating,

$$_{\rm III}A(\mathbf{u}) = \frac{1}{12}\frac{1}{2\pi}\int dx^1 g_1 K\, W[\mathbf{u}] \ . \tag{4.35}$$

This is a spatial invariant and hence its Poisson bracket with $\mathbf{P}_{11}(x)$ vanishes. However, the Poisson bracket of $_{\rm III}A(\mathbf{u})$ with $\mathbf{P}_{1\perp}(x)$ does not vanish; by the Mainardi equation (I.A6),

$$\begin{aligned}\{_{\rm III}A(\mathbf{u}), \mathbf{P}_{1\perp}(x)\} &= \frac{1}{12}\frac{1}{2\pi}(-g_1\Delta W[\mathbf{u}] + g_1 K\, W_{,\perp}[\mathbf{u}]) \\ &= -\frac{1}{12}\frac{1}{2\pi}g^1(D^{(+)}{}_1 D^{(+)}{}_1 u^{+}{}_{,+}(\mathbf{T}^{+}(x)) \\ &\quad + D^{(-)}{}_1 D^{(-)}{}_1 u^{-}{}_{,-}(\mathbf{T}^{-}(x))) \ . \end{aligned} \tag{4.36}$$

This means that while $h(\mathbf{u})$, $\mathbf{u}\in LC$ is a quantum constant of motion, $H(\mathbf{u})$ is not.

V. SPACETIME DIFFEOMORPHISMS AND CONFORMAL ISOMETRIES IN THE DIRAC CONSTRAINT QUANTIZATION

A. Operator representations of *L* Diff*M* and of the Dirac algebra

We have seen that spacetime diffeomorphisms play the role of a dynamical group of the classical theory. The generators $\mathbf{U}$ of $L\,\mathrm{Diff}M$ can be homomorphically represented either by the smeared Schrödinger momenta $P(\mathbf{U})$ or the smeared Heisenberg momenta $\mathbf{P}(\mathbf{U})$, Eqs. (I.4.18) and (I.4.19). The dynamical variables $P(\mathbf{U})$ generate the evolution of classical states under infinitesimal diffeomorphisms $\mathbf{U}$, Eq. (I.4.20). The representation equation (I.4.19) ensures that this evolution is foliation independent. Similarly, the dynamical variables $P(\mathbf{U})$ generate, by the Heisenberg equations of motion (I.4.21), the evolution of the field variables $\phi(x)$ and $\pi_1(x)$. Again, the representation equation (I.4.18) ensures the foliation independence of this evolution.

Our definition (2.3) of the commuting Heisenberg operators $\mathbf{P}_{1\alpha}(x)$ and the subsequent construction (2.29) and (3.11) of the commuting Schrödinger operators $P_{1\alpha}(x)$ enable us to take these classical relations over into the quantum theory. By smearing the momentum operators $\mathbf{P}_{1\alpha}(x)$ and $P_{1\alpha}(x)$ by the vector fields $\mathbf{U}(X)$ restricted to the embedding, we map each generator $\mathbf{U}\in L\,\mathrm{Diff}M$ into an operator, $\mathbf{P}(\mathbf{U})$ or $P(\mathbf{U})$, acting on the function space described in Sec. II A. From the fundamental commutation relations (2.4) and (3.12) it follows that the smeared operators homomorphically represent the Lie algebra L Diff$M$ by the commutator algebra on H:

$$\frac{1}{i}[\mathbf{P}(\mathbf{U}),\mathbf{P}(\mathbf{V})]=\mathbf{P}(-[\mathbf{U},\mathbf{V}]) , \tag{5.1}$$

$$\frac{1}{i}[P(\mathbf{U}),P(\mathbf{V})]=P(-[\mathbf{U},\mathbf{V}]) . \tag{5.2}$$

The smeared form of Eq. (2.5) imposes the constraints as the restrictions

$$\mathbf{P}(\mathbf{U})\Psi=0 \quad \forall \mathbf{U}\in L \text{ Diff}M \tag{5.3}$$

on the physical states $\Psi\in H_0$ of the system. Similarly, the smeared form of Eq. (3.25) tells us how the field operators $\phi(x)$ and $\pi_1(x)$ are evolved by the generators $P(\mathbf{U})$:

$$\frac{1}{i}[\phi(x),P(\mathbf{U})]=0=\frac{1}{i}[\pi_1(x),P(\mathbf{U})] \quad \forall \mathbf{U}\in L \text{ Diff}M . \tag{5.4}$$

These equations are quantum counterparts of the classical equations (I.4.20) and (I.4.21).

In the Schrödinger picture, Eq. (5.4) is an expression of the familiar fact that the Schrödinger field operators do not explicitly depend on the Schrödinger time (the Schrödinger embedding). On the other hand, Eq. (5.3) acquires the dynamical meaning of the Schrödinger equation

$$i\int_\Sigma dx^1 U^\alpha(X(x))\frac{\delta\Psi[X,\phi]}{\delta X^\alpha(x)}=H(\mathbf{U})\Psi[X,\phi] . \tag{5.5}$$

Equation (5.5) specifies how the Schrödinger state $\Psi[X,\phi]$ evolves from an embedding $X(x)$ to a nearby embedding $X(x)+\mathbf{U}(X(x))$ which is a result of displacing $X(x)$ by an infinitesimal diffeomorphism $\mathbf{U}(X)$. The representation equation (5.1) ensures that when we deform an initial embedding into a final embedding either by a two-step process $\mathbf{U}(X(x)),\mathbf{V}(X(x))$, or by an equivalent three-step process $\mathbf{V}(X(x))$, $\mathbf{U}(X(x))$, and $[\mathbf{U},\mathbf{V}](X(x))$, the initial state always evolves into the same final state up to terms of second order in the displacements $\mathbf{U}$ and $\mathbf{V}$. From here we can conclude by a familiar argument that the evolution of the state does not depend on the foliation connecting a given initial embedding with a given final embedding.

Because the deformation vector $\mathbf{U}(X)$ is arbitrary, the Schrödinger equation (5.5) can also be written as a variational differential equation (2.8):

$$i\frac{\delta\Psi[X,\phi]}{\delta X^\alpha(x)}=H_{1\alpha}(x)\Psi[X,\phi] . \tag{5.6}$$

This form of the Schrödinger equation underscores the role of the embedding as a many-fingered time variable.

Another possibility is to look at Eqs. (5.3) and (5.4) from the point of view of the Heisenberg picture. Equation (5.3) then reduces to the statement that the Heisenberg states do not depend on the embedding, Eq. (2.6). It is now Eq. (5.4) which has a dynamical meaning: it tells us how the field operators $\phi(x)$ and $\pi_1(x)$ evolve under an infinitesimal diffeomorphism $\mathbf{U}(X)$ which displaces the embedding $\mathbf{X}(x)$ into $\mathbf{X}(x)+\mathbf{U}(\mathbf{X}(x))$. In the same way in which the representation equation (5.1) ensures that the evolution of the Schrödinger states did not depend on the choice of foliation, the representation equation (5.2) now ensures that the evolution of the field operators $\phi(x)$ and $\pi_1(x)$ does not depend on such a choice.

The representation equations (5.1) and (5.2) crown our effort to find a covariant constraint quantization of a parametrized field theory. They ensure that the Dirac constraint quantization can be consistently carried out both in the Heisenberg and in the Schrödinger pictures. The key to our solution of the problem was the elimination of the anomaly from the operator representation equations. The generators of spacetime diffeomorphisms are then represented by the operators $\mathbf{P}(\mathbf{U})$ and $P(\mathbf{U})$ which generate evolutions along all possible foliations. These operators are well defined on a *single* function space H which, after the constraints are imposed, reduces to the physical space H_0 with a Hilbert structure.

In our discussion we took advantage of the formal and conceptual simplifications brought in by working with the unprojected constraints and the associated spacetime diffeomorphism algebra, instead of with the more usual projected constraints (the super-Hamiltonian and supermomentum constraints) and the associated Dirac algebra. All arguments, however, can easily be repeated in this slightly more cumbersome language, smearing the quantum Dirac algebra of the projected operators $P_{11\perp}(x),P_{11}(x)$ [or $\mathbf{P}_{11\perp}(x),\mathbf{P}_{11}(x)$] by externally prescribed lapse and shift functions $N^{1\perp}(x)$ and $N^1(x)$. Again, the key element in performing a consistent Dirac constraint quantization of our parametrized system is the ability to construct the Dirac operator algebra (3.30) without any anomaly.

B. Conformal isometries and quantum constants of motion

In classical theory, the group C of conformal isometries can be considered either as a subgroup of the dynamical group Diff$M$ or as a symmetry group of the diffeomorphism Hamiltonians. In the first way of looking at C, the generators $\mathbf{u}$ of LC are represented by the smeared Heisenberg momenta $\mathbf{P}(\mathbf{u})$ and they satisfy the representation equation (I.4.19). In the second way of looking at C, the generators of LC are represented by the smeared Schrödinger momenta $P(\mathbf{u})$ and they satisfy the representation equation (I.4.18). Moreover, the mixed Poisson brackets between $P(\mathbf{u}),\mathbf{u}\in LC$ and $\mathbf{P}(\mathbf{V})$, $\mathbf{V}\in L$ Diff$M$ (weakly) vanish: Eq. (I.4.33). The diffeomorphism Hamiltonians $\mathbf{P}(\mathbf{V})$ are thus left conditionally symmetric under a conformal motion generated by $P(\mathbf{u})$, and $P(\mathbf{u})$ is a constant of motion. The dynamical variable $P(\mathbf{u})$ is weakly equal to the smeared Hamiltonian flux $-h(\mathbf{u})$ which is also a constant of motion; indeed, because $h(\mathbf{u})$ does not depend on the embedding momenta, its Poisson bracket with the diffeomorphism Hamiltonian $\mathbf{P}(\mathbf{V})$ vanishes strongly rather than weakly, Eq. (I.4.25). The constants of motion $h(\mathbf{u})$ form an antihomomorphic representation of LC, Eq. (I.4.27).

Let us see how much of this classical structure survives the Dirac constraint quantization. Of course, to consider C as a subgroup of the dynamical group Diff$M$ is straightforward; all that is needed is to restrict the elements $\mathbf{U}\in L$ Diff$M$ in the operator representation equations (5.1) and (5.2) to the elements $\mathbf{u}\in LC$. Problems arise only when we try to consider C as a symmetry group of diffeomorphism Hamiltonian operators: the operators $P(\mathbf{u})$ which represent the elements $\mathbf{u}\in LC$, Eq. (5.2), no longer weakly commute with the $\mathbf{P}(\mathbf{V})$. Indeed by Eqs. (2.28), (4.1), (4.2), and (5.1),

$$\frac{1}{i}[P(\mathbf{u}),\mathbf{P}(\mathbf{V})]=\mathbf{P}(-[\mathbf{u},\mathbf{V}])-\{A(\mathbf{u}),\mathbf{P}(\mathbf{V})\}\ . \quad (5.7)$$

When we apply the operator (5.7) to a physical state $\Psi\in H_0$, the Poisson bracket term yields a nonvanishing contribution because the anomaly potential ${}_{\rm III}A(\mathbf{u})$ is not conserved under normal deformations, Eq. (4.36):

$$\frac{1}{i}[P(\mathbf{u}),\mathbf{P}(\mathbf{V})]\Psi\neq 0 \quad \text{for } \Psi\in H_0\ . \quad (5.8)$$

We must conclude that the diffeomorphism Hamiltonian operators are not left invariant by the generators $P(\mathbf{u})$ of conformal isometries, and that these generators are not quantum constants of motion on the physical space H_0.

On the other hand, when we smear the original Heisenberg evolution generator (2.29) based on the normal-ordered flux by a conformal Killing vector $\mathbf{u}\in LC$, we do get a quantum constant of motion. Indeed, from Eqs. (2.28), (2.29), and (5.1),

$$\frac{1}{i}[\Pi(\mathbf{u}),\mathbf{P}(\mathbf{V})]=\mathbf{P}(-[\mathbf{u},\mathbf{V}])$$
$$\Longrightarrow \frac{1}{i}[\Pi(\mathbf{u}),\mathbf{P}(\mathbf{V})]\Psi=0 \quad \forall\Psi\in H_0\ . \quad (5.9)$$

However, because of the anomaly, Eq. (2.32), the operators $\Pi(\mathbf{u})$ do not represent the Lie algebra LC. Note also that $\Pi(\mathbf{u})$ are equivalent to the Hamiltonian flux operators $-h(\mathbf{u})$. This brings us back to our old results of Sec. II B: because the Hamiltonian flux operators $h(\mathbf{u})$ do not depend on the embedding momenta, they are quantum constants of motion not only on the physical space H_0, but also on the big function space H, Eq. (2.28). However, as their weak equivalents $-\Pi(\mathbf{u})$, they do not represent the Lie algebra LC, due to the anomaly in Eq. (2.26).

To summarize this part of our discussion, the generators $P(\mathbf{u})$ of LC no longer leave the diffeomorphism Hamiltonian operators invariant on the physical Hilbert space H_0, and as a consequence they are no longer (quantum) constants of motion. One can find an infinite system of constants of motion, namely, $-h(\mathbf{u})\approx\Pi(\mathbf{u})$. These generate quantum canonical transformations which leave the diffeomorphism Hamiltonian operators invariant, but their operator algebra differs from LC by the anomaly. The symmetry group of the quantum system is thus different from the symmetry group of the classical system. Fortunately, this does not effect in any way the consistency of the Dirac constraint quantization.

VI. FACTOR ORDERING OF THE CONSTRAINTS IN THE SCHRÖDINGER PICTURE

In the Heisenberg picture the constraints require no factor ordering because they turn out to be the fundamental Heisenberg operators $\mathbf{P}_{1\alpha}(x)$. On the other hand, the operators

$$P_{1\alpha}(x;\mathbf{X},\mathbf{P},\mathbf{p},\mathbf{q},\mathbf{a}_k,\mathbf{a}^*{}_k]$$
$$:=\mathbf{P}_{1\alpha}(\mathbf{x})-h_{1\alpha}(x;\mathbf{X},\mathbf{p},\mathbf{q},\mathbf{a}_k,\mathbf{a}^*{}_k]-A_{1\alpha}(x;\mathbf{X}]\ , \quad (6.1)$$

which in the Heisenberg picture evolve the field variables $\phi(x),\pi_1(x)$, must have a definite ordering to be well-defined operators on the function space H and to commute with each other, Eq. (3.12). We have shown that this factor ordering amounts to taking $A_{1\alpha}(x;\mathbf{X}]$ as the multiplication operator, $\mathbf{P}_{1\alpha}(x)$ as the variational derivative operator, and performing the normal ordering of the Heisenberg mode operators $\mathbf{a}_k,\mathbf{a}^*{}_k$ in the field Hamiltonian flux $h_{1\alpha}(x)$.

In the Schrödinger picture the factor ordering of the constraints

$$\mathbf{P}_{1\alpha}(x):=P_{1\alpha}(x)+h_{1\alpha}(x;X,\phi,\pi]+A_{1\alpha}(x;X] \quad (6.2)$$

becomes nontrivial. The Schrödinger operators $A_{1\alpha}(x;X]$ and $P_{1\alpha}(x)$ are now the multiplication and the variational derivative operators; our task is to find the factor ordering of the field Hamiltonian flux $h_{1\alpha}(x;X,\phi,\pi]$ in the Schrödinger operators $X^\alpha(x)$, $\phi(x)$, and $\pi_1(x)$ which would ensure that the constraints $\mathbf{P}_{1\alpha}(x)$ commute: Eq. (3.12). The correct ordering of $h_{1\alpha}(x)$ of course amounts to a transcription of the Heisenberg normal ordering to the Schrödinger set of variables.

We start from Eq. (I.3.35),

$$h_{1\alpha}(x)=\tfrac{1}{2}n_{(-)}{}^{1}_{\alpha}(x;X]h_{11(+)}(x)$$
$$+\tfrac{1}{2}n_{(+)}{}^{1}_{\alpha}(x;X]h_{11(-)}(x)\ , \quad (6.3)$$

which reduces our task to finding the correct ordering of the hypersurface null components (I.3.30) of the flux,

$$h_{11(\pm)}(x)=\tfrac{1}{2}(\pi_{1(\pm)}(x))^2\ . \quad (6.4)$$

In the Schrödinger picture, the null momenta

$$\pi_{1(\pm)}(x):=\pi_1(x)\pm\phi_{,1}(x) \quad (6.5)$$

are simple combinations of the fundamental Schrödinger field variables. On the other hand, they are related by Eqs. (I.3.29), (I.2.27), and (I.2.37) to the spacetime fields $\phi_{,\pm}(T^{\pm};\mathbf{q},\mathbf{p},\mathbf{a}_k,\mathbf{a}^*{}_k)$ of the Heisenberg picture:

$$\pi_{1(\pm)}(x)=\pm 2T^{\pm}{}_{,1}(x)\phi_{,\pm}(T^{\pm}(x))$$
$$=2\Lambda^{\pm 1}(x)g_1(x)\phi_{,\pm}(T^{\pm}(x))\ . \quad (6.6)$$

The slope factor $\Lambda(x;X]$ is given by the integrals (I.2.31) and (I.2.35).

Equations (6.4) and (6.6) set the framework for what we want to do. In the first step we define kernels of integral operators which separate the spacetime fields $\phi_{,\pm}(T^{\pm})$ into the positive-frequency (the Heisenberg annihilator $\mathbf{a}_k$) and negative-frequency (the Heisenberg creator $\mathbf{a}^*_k$) parts:

$$\phi_{,\pm}(T^{\pm})={}_{(+)}\phi_{,\pm}(T^{\pm})+{}_{(-)}\phi_{,\pm}(T^{\pm}) . \tag{6.7}$$

In the second step we restrict these kernels to obtain the corresponding split of the null Schrödinger momenta (6.6),

$$\pi_{1(\pm)}(x)={}_{(+)}\pi_{1(\pm)}(x)+{}_{(-)}\pi_{1(\pm)}(x) . \tag{6.8}$$

In the third and the final step we place the positive-frequency parts before the negative-frequency parts in the flux components (6.4) and thereby obtain their normal-ordering kernels.

The separation (6.7) of the positive- and negative-frequency parts of $\phi_{,\pm}(T^{\pm})$ is achieved by the positive- and negative-frequency parts ${}_{(\pm)}\delta$ of the δ functions $\delta(T^{\pm})$:

$$\begin{aligned} {}_{(\pm)}\phi_{,+}(T^{+})&=\int_{-\pi}^{\pi} dT^{+'}\,{}_{(\pm)}\delta(T^{+}-T^{+'})\phi_{,+}(T^{+'}) ,\\ {}_{(\pm)}\phi_{,-}(T^{-})&=\int_{-\pi}^{\pi} dT^{-'}\,{}_{(\pm)}\delta(T^{-}-T^{-'})\phi_{,-}(T^{-'}) . \end{aligned} \tag{6.9}$$

These functions are defined by the formulas

$${}_{(\pm)}\delta(T^{+}):=\frac{1}{2\pi}\left[\tfrac{1}{2}+\sum_{k=1}^{\infty} e^{\mp ikT^{+}}\right] \tag{6.10}$$

and

$${}_{(\pm)}\delta(T^{-}):=\frac{1}{2\pi}\left[\tfrac{1}{2}+\sum_{k=1}^{\infty} e^{\mp ikT^{-}}\right] . \tag{6.11}$$

In each of these equations, the homogeneous mode is divided equally between the positive- and the negative-frequency parts. Let us note that

$$\delta={}_{(+)}\delta+{}_{(-)}\delta \quad\text{and}\quad {}_{(+)}\delta={}_{(-)}\delta^{*} \tag{6.12}$$

for both of the expressions (6.10) and (6.11).

We use Eq. (6.6) to induce the splitting (6.8) of the null Schrödinger momenta:

$$\begin{aligned} {}_{(\pm)}\pi_{1(+)}(x)&=2\int_{\Sigma} dx^{1'}\,{}_{(\pm)}\delta_1(x',x;T^{+}]\pi_{1'(+)}(x') ,\\ {}_{(\pm)}\pi_{1(-)}(x)&=2\int_{\Sigma} dx^{1'}\,{}_{(\pm)}\delta_1(x',x;T^{-}]\pi_{1'(-)}(x') . \end{aligned} \tag{6.13}$$

Here,

$${}_{(\pm)}\delta_1(x',x;T^{+}]:=\Lambda(x)g_1(x)\,{}_{(\pm)}\delta(T^{+}(x)-T^{+}(x')) \tag{6.14}$$

and

$${}_{(\pm)}\delta_1(x',x;T^{-}]:=-\Lambda^{-1}(x)g_1(x)\,{}_{(\pm)}\delta(T^{-}(x)-T^{-}(x')) . \tag{6.15}$$

Unlike the δ functions themselves, the positive- and negative-frequency parts (6.14) and (6.15) depend on the embedding. We can write these functions in a form which does not require knowledge of the privileged null coordinates $T^{\pm}$:

$${}_{(\pm)}\delta_1(x',x;T^{+}] =\Lambda(x)g_1(x)\,{}_{(\pm)}\delta\left[\int_{x'}^{x} dx^{1''}g_{1''}(x'')\Lambda(x'')\right] \tag{6.16}$$

and

$${}_{(\pm)}\delta_1(x',x;T^{-}] =-\Lambda^{-1}(x)g_1(x)\,{}_{(\pm)}\delta\left[-\int_{x'}^{x} dx^{1''}g_{1''}(x'')\Lambda^{-1}(x'')\right] . \tag{6.17}$$

The functional form of the spacetime functions ${}_{(\pm)}\delta(\ )$ in Eqs. (6.16) and (6.17) is the same, namely, (6.10) and (6.11). Their arguments are given by Eq. (I.2.38). Like the slope factor $\Lambda(X)$ itself, the functions (6.16) and (6.17) are reconstructed from the intrinsic geometry and the extrinsic curvature of the embedding.

The normal-ordered fluxes (6.4) can now be expressed in terms of the functions (6.16) and (6.17). We get

$$h_{11(\pm)}(x)=\int_{\Sigma} dx^{1'}\int_{\Sigma} dx^{1''}N_{11(\pm)}(x,x',x'';T^{\pm}] \times\pi_{(\pm)}(x')\pi_{(\pm)}(x'') . \tag{6.18}$$

The kernels $N_{11(\pm)}$ which enforce the normal ordering have the form

$$\begin{aligned} N_{11(\pm)}(x,x',x'';T^{\pm}]&=2\delta_1(x',x)\delta_1(x'',x)\\ &\quad+2i(\delta_1(x',x)\bar{\delta}_{(\pm)1}(x'',x;T^{\pm}]\\ &\qquad-\delta_1(x'',x)\bar{\delta}_{(\pm)1}(x',x;T^{\pm}]) \end{aligned} \tag{6.19}$$

in which the dependence on the embedding enters through a set of two real functions:

$$\bar{\delta}_{(\pm)1}(x',x;T^{\pm}]=\frac{1}{2i}\left({}_{(+)}\delta_1(x',x;T^{\pm}] -{}_{(+)}\delta^{*}{}_1(x',x;T^{\pm}]\right) . \tag{6.20}$$

We see that the normal-ordering kernels (6.19) are Hermitian in the arguments x',x'':

$$N^{*}{}_{11(\pm)}(x,x'',x')=N_{11(\pm)}(x,x',x'') ; \tag{6.21}$$

this ensures the self-adjointness of the Hamiltonian flux operator (6.18).

To summarize, Eqs. (6.3) and (6.18) express the Hamiltonian flux operator normal ordered in the Heisenberg modes in terms of the Schrödinger field variables (6.5). The kernels (6.19) which arrange the normal ordering on an embedding $X(x)$ are invariantly constructed from the intrinsic geometry and the extrinsic curvature of that embedding by Eqs. (6.20), (6.16), (6.17), (6.10), (6.11), (I.2.31), and (I.2.35). The factor ordering leading to the total Hamiltonian flux operator differs from the factor ordering (6.3) and (6.18) by the anomaly potential, Eq. (4.2). Either one of these two covariant factor orderings is considerably more involved than the straightforward noncovariant factor ordering suggested by the string theory.

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[1]P. A. M. Dirac, Can. J. Math. **2**, 129 (1950); *Lectures on Quantum Mechanics* (Academic, New York, 1965).

[2]C. J. Isham and K. V. Kuchař, Ann. Phys. (N.Y.) **164**, 288 (1985).

[3]K. Kuchař, Phys. Rev. D **39**, 1579 (1989). We call this paper I and refer to the equations from I by inserting I. before the equation number.

[4]D. G. Boulware and S. Deser, J. Math. Phys. **8**, 1468 (1967).

[5]See, e.g., M. B. Green *et al.*, *Superstring Theory* (Cambridge University Press, Cambridge, England, 1987).

[6]C. Rebbi, Phys. Rep. **12**, 1 (1974); or the discussion given in Ref. 5.

[7]C. Teitelboim, in *Quantum Theory of Gravity*, edited by S. Christensen (Hilger, Bristol, 1984).

[8]K. V. Kuchař, Phys. Rev. D **34**, 3044 (1986).

[9]We prefer this phrase to the term "unitary" because the space H on which the fundamental operators act is not a Hilbert space.