Probabilistic time in quantum gravity

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The concept of time and its relation with probability in quantum gravity is studied. It is argued that the notion of probability is better defined when the topology is fixed. A "probabilistic time" is then introduced. It coincides with the proper time, in the classical limit, and yields a parabolic field equation, i.e., the Schrödinger equation.

I. INTRODUCTION

As is well known there is no well-defined notion of time in quantum gravity, and it is also known that we do not have a well-based probabilistic interpretation. The aim of this paper is to show that both problems are intimately related, and both can be solved if we endow space-time with a fixed topology.

Usually, in quantum gravity we foliate space-time, but we cannot give a time label to each shell, with a precise and distinct physical meaning. Every component of h_{ij}, or a function of all of them, such as $h^{1/2}$ (h_{ij} is the metric of each space surface of the foliation and $h = \det h_{ij}$), can be used as a "time," to label the sheets. But none of these parameters are privileged. None of them could claim to be something like proper time. Furthermore, h_{ij} can take many arbitrary values, thus this parameter can also change almost arbitrarily. On the contrary, at the classical, and also at the semiclassical level, we can draw an orthogonal trajectory to the foliation and define its proper time parameter or use other methods of labeling with a physical basis.

The origin of this amgibuity is our incomplete understanding of quantum gravity. To quantize gravity is a venerable problem of theoretical physics. Two kinds of difficulties prevent the formulation of a complete satisfactory theory. The first and main trouble is that we do not have yet a well-behaved gravitational Lagrangian for short distances, and, therefore, we do not have a renormalizable theory. We shall not study this issue in this paper and we shall use the Hilbert-Einstein Lagrangian, a bad short-distance behaved Lagrangian, to develop a model theory only.

But, there is a second kind of problem, the well-known list of interpretation problems. (a) What is the Hilbert space of states of the theory? And intimately related with this question, (b) what is the probability interpretation of the states? Finally, the last, and perhaps more important question, (c) what variable plays the role of time? If we do not have a well-defined and unique notion of time, Hamiltonian quantization is not very satisfactory. We shall study these problems and try, at least, to sketch a new solution.

The paper is thus organized as follows.

In Sec. II, we shall see that, perhaps, there exists a solution to problems (a) and (b), and also problem (c) if we fix the topology of space-time, and that, in fact, there are not compelling reasons to vary the topology.

We study the well-known inner product derived from the Feynman path integral, in Sec. III and explain how this inner product promotes the states space to a Hilbert space, if the topology is fixed. Nevertheless, we shall see that this product does not have the traditional role of an ordinary inner product of quantum theory, because it is not related, in the usual way, to the field equation of quantum gravity, the Wheeler-DeWitt equation. Furthermore the corresponding normalization is, in general, divergent. We shall try to overcome these difficulties and to define a new convergent normalization, in Sec. IV, that will allow us to define probabilities and to introduce a "probabilistic time"; the notion that we believe plays the role of Hamiltonian time in quantum gravity.

In Sec. V we shall study the problem with the Born-Oppenheimer approximation and show that probabilistic time coincides with proper time up to leading order. For gravitation, coupled in a sufficiently general way with a scalar field, we shall demonstrate that de Sitter space-time is the model of the Universe (up to this order) as a rigorous consequence of the formalism.

In Sec. VI we find a Schrödinger equation (where probabilistic time plays the role of ordinary time) for the wave function. This fact shows that probabilistic time is a good candidate to become Hamiltonian time. In a natural way, we define a new inner product, related in the usual fashion with the Schrödinger equation, that completes the analogy with an ordinary quantum theory.

In Sec. VII we will see that our interpretation of the problem coincides with the one of Vilenkin,[1] based on the Klein-Gordon probability current.

In Sec. VIII we draw our main conclusions and try to see if the good features of probabilistic time are mainly due to the approximation we are using, as seems to be the case for some of them.

In addition, as probabilistic time is defined exactly, and not through an approximate method, at least provisionally, all the problems stated at the beginning are solved. Further research will show if the introduced concepts are valid.

II. THE FIXED TOPOLOGY

As we have said in the Introduction, one of the most important problems in the formulation of quantum gravity theory is how to define an inner product in the state space, and how to promote this space to a Hilbert space.[2] To avoid this problem we can forget the whole Hilbert space idea and adopt the powerful formalism of the Feynman path integral.[3–5] Then we can use a similar interpretation to Everett's[6] that substitutes the probabilistic interpretation.

On the other hand, the Feynman integration is performed, at least in principle over the whole set of possible topologies. This method has been strongly criticized[7,8] because the change of toplogy might yield nonunitary problems. In this work we shall use the Feynman integral, but adopt a fixed toplogy (which we shall always choose to be a simple one). Thus, we shall see that we can find a Hilbert space and reestablish the probabilistic interpretation.

Einstein's elegant and primordial idea was that gravity is associated with the geometry of space-time. However, "geometry" is a word that summarizes several different mathematical structures: topological, differential, conformal, affine, projective, and metric structures.[9–12] Which one of these structures is the one really associated with the gravitational field? Certainly it is associated with the metric, and through the metric, with the conformal, affine, and projective structures, via the compatibility principle. But it is not, by all means, obvious that the toplogical or the differential structures should be associated with the gravitational field. We begin to believe that it is not at all related with these last structures. This fact is evident, at least locally and at the classical level, because gravitation is a local effect. In addition, when we study the Feynman integral of a scalar spin-zero field (or a vectorial spin-one field, etc.) we do not make the integral for different topologies. All of the theory turns out to be correct using one toplogy only. Why must we change the topology when we study the gravitational spin-two field? Perhaps the change of topology could bring a new and interesting insight to the theory.[13] However, it is evident that we can try to formulate a theory with a fixed topology, at least, as a first approach. This attitude allows us to define the wave function of the Universe, as in Ref. 4, and to obtain most interesting results.[14,15] Nevertheless, in these papers the fixed topology is only accepted as a working hypothesis. On the contrary, in this paper, we shall suppose that it is an essential property of space-time, i.e., the Universe has a certain differential manifold structure that we shall not change. We shall see that with this hypothesis we cannot only obtain the results of Refs. 4, 14, and 15, but also define a "usual" normalization.

Of course, we can take a less controversial view. As the probability of change of topology is normally considered very small (as in Ref. 4) or vanishing (because an infinite amount of energy is needed to change the topology, as in Refs. 7 and 8), we can disregard this phenomenon in a great number of problems. Thus, a more conservative version of our statement will be that a normalization can be introduced in the cases where the probability of a change of topology would be negligible. The reader can take the version of our idea that he (or she) likes better.

III. THE HILBERT SPACE

Let us suppose that space-time is a unique and well-defined connected differentiable manifold M and that there is also a neutral-scalar field $\phi(x)$ that symbolizes matter.

The set of all possible histories is the set of all metric $g_{\mu\nu}(x)$ that M can have, and all the fields $\phi(x)$ that can be defined on M.

Let us define two arbitrary regions in the manifold M [see Figs. 1(a) and 1(b)]: M_C, where we shall establish the conditions $\{C\}$ that the histories $(g_{\mu\nu}(x),\phi(x))$ must satisfy; M_O, where we shall perform our observation. $\{O\}$ symbolizes some particular results for these observations.

The probability amplitude $\psi(\{O\}/\{C\})=\langle O \mid C\rangle$ is

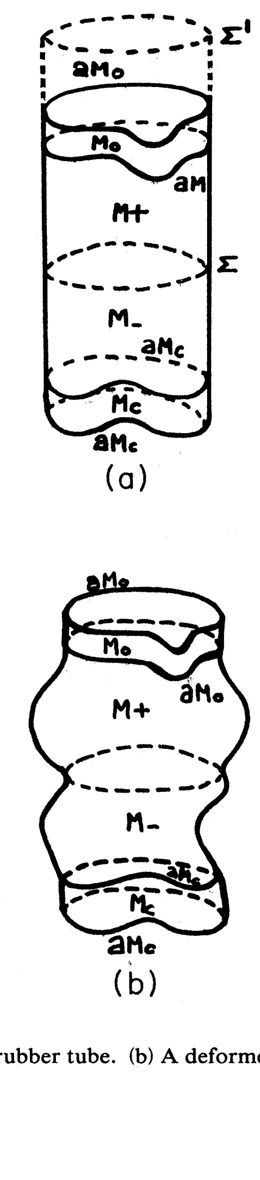

FIG. 1. (a) A rubber tube. (b) A deformed rubber tube.

$$\psi(\{O\}/\{C\})=\langle O \mid C\rangle = \int \mathcal{D}[g_{\mu\nu}]\mathcal{D}[\phi]\exp\{iS_O^C[g_{\mu\nu}(x),\phi(x)]\} , \quad (3.1)$$

where the integral is over the set of histories that satisfy $\{O\}$ and $\{C\}$, $S[g_{\mu\nu}(x),\phi(x)]$ is the action we adopt from fields $g_{\mu\nu}(x)$ and $\phi(x)$, integrated in the connected region that lies between ∂M_C and ∂M_O. The Feynman integration is performed on the set of all the histories that satisfy conditions $\{C\}$ and yield observations $\{O\}$, counting the histories only once each, because two different $g_{\mu\nu}$ could correspond to one and only one history if they are related by a coordinate transformation. Of course, there exists part of the histories outside M_O,M_C and the region between ∂M_O and ∂M_C. However, as in Ref. 5, if we considered this part of the histories, we should assign a constant value to their contribution.

Let us now consider a hypersurface Σ of M with no intersection with the regions M_C and M_O. Let us suppose that the topology is such that Σ is a connected three-surface [as in Fig. 1(a)]. On Σ each four-metric $g_{\mu\nu}$ defines a three-metric h_{ij}. Let us, in addition, suppose that the topology is such a one that Σ defines in M two connected regions M^-, with boundary Σ and ∂M_C, and M^+, with boundary Σ and ∂M_O.

Then we can define the wave functions

$$\psi_C[h_{ij},\phi^\Sigma]=\langle h_{ij},\phi^\Sigma \mid C\rangle = \int \mathcal{D}[g_{\mu\nu}]\mathcal{D}[\phi]\exp(iS_C^\Sigma[g_{\mu\nu},\phi]) , \quad (3.2)$$

where the integral is over the set of all histories that satisfy $\{C\}$ and yield h_{ij} and ϕ^Σ on Σ, and

$$\psi_O[h_{ij},\phi^\Sigma]=\langle h_{ij},\phi^\Sigma \mid O\rangle = \int \mathcal{D}[g_{\mu\nu}]\mathcal{D}[\phi]\exp(iS_O^\Sigma[g_{\mu\nu},\phi]) , \quad (3.3)$$

where the integral is over the set of all histories that satisfy $\{O\}$ and yield h_{ij} and ϕ^Σ on Σ, S_C^Σ (S_O^Σ) is the action integrated in M^- (M^+), and ϕ^Σ is the restriction of ϕ to Σ.

Now, let us observe that all the histories $\{g_{\mu\nu}(x);\phi(x)\}$ define a metric $h_{ij}(x)$ and a field $\phi^\Sigma(x)$ on Σ. If we count each metric structure of Σ just once (i.e., all the metric h_{ij} related by coordinates transformation are counted just once because they represent only one metric structure), the set of metric and fields, h_{ij},ϕ^Σ is an exclusive, complete set, because it contains all the possible metric structures and all possible fields counted just once. Then,

$$\psi(\{O\}/\{C\})=\langle O \mid C\rangle = \int \mathcal{D}[h_{ij}]\mathcal{D}[\phi^\Sigma]\int \mathcal{D}[g_{\mu\nu}]\mathcal{D}[\phi]\int \mathcal{D}[g_{\mu\nu}]\mathcal{D}[\phi]\exp\{i(S_C^\Sigma[g_{\mu\nu},\phi]-S_O^\Sigma[g_{\mu\nu},\phi])\}$$
$$= \int \mathcal{D}[h_{ij}]\mathcal{D}[\phi^\Sigma]\psi_O^*[h_{ij},\phi^\Sigma]\psi_C[h_{ij},\phi^\Sigma] , \quad (3.4)$$

where the first integral is over the set of all metrics and all fields in Σ, the second over the set of histories that satisfy $\{C\}$ and yield h_{ij} and ϕ^Σ on Σ, the third over the set of all histories that satisfy $\{O\}$ and yield h_{ij} and ϕ^Σ on Σ, and the final integral is over the set of all metrics and all fields in Σ.

It is needless to say that all these manipulations are only a symbolic algorithm, and that we cannot rigorously compute any one of these integrals until we know the definition of the corresponding measure. We have only presented a general formalism that can be used in the particular cases where the manifold M is defined and we know the measures (as in the case of minisuperspace). Only in these cases is the formalism rigorous.

But let us now remark, as in Ref. 4, that Eq. (3.4) allows us to define an inner product in the Hilbert space of states, at least at our symbolic level. In fact, let us define the product

$$\langle \psi_O,\psi_C\rangle=\langle O \mid C\rangle =\psi(\{O\}/\{C\}) = \int \mathcal{D}[h_{ij}]\mathcal{D}[\phi^\Sigma]\psi_O^*[h_{ij},\phi^\Sigma]\psi_C[h_{ij},\phi^\Sigma] , \quad (3.5)$$

where the integral is over the set of all metrics and all fields in Σ. Of course this inner product is independent of the surface (i.e., "time" independent) because it is directly defined by Eq. (3.1) where there is not a trace of Σ. In fact, the wave functions $\psi_C[h_{ij},\phi^\Sigma]$ and $\psi_O[h_{ij},\phi^\Sigma]$ of Eqs. (3.2) and (3.3) do not actually depend on the location of the surface Σ, provided Σ lies between ∂M_C and ∂M_O, but only the intrinsic metric h_{ij} and the scalar field ϕ^Σ. Hence, the inner product (3.5) does not depend on the precise location of Σ. In addition, it is Hermitian and positive definite.

Thus, this inner product promotes, at least symbolically, the space of states to a Hilbert space.

Of course, as stated in Ref. 4, the product (3.5) is not the one that would be required by a canonical theory to define a Hilbert space of physical states. It is an "unusual" inner product. We know that the wave field equation of the theory, the Wheeler-DeWitt equation, is hyperbolic and the product (3.5) is of a parabolic type. We would prefer to have a different product related, in the usual way, to the wave field equation. We shall find this modified product in Sec. VI.

For the moment, let us only say that the inner product (3.5) is just a mathematical construction that naturally stems from the Feynman integral formulation of quantum gravity. It can be used to normalize the wave function as in Ref. 4, asking that

$$\langle \psi_C,\psi_C\rangle=1 . \quad (3.6)$$

This kind of product can also be computed in principle because Σ can be "in the future" of M_O and M_C, as Σ' in Fig. 1(a). The problem is that the integral on the left-hand side (LHS) of Eq. (3.6) turns out to be divergent in general. Thus this normalization is not only "unusual" but also divergent. But we can overcome these problems, as we shall see. Then with this normalization $\psi_C[h_{ij},\phi^\Sigma]$ could be interpreted as a probability amplitude and $|\psi_C[h_{ij},\phi^\Sigma]|^2$ as the probability density to find the metric h_{ij} and the field configuration ϕ^Σ, if the quantum state satisfies conditions $\{C\}$. We shall use this interpretation in the next section.

Let us now study if the definition of the inner product (3.5) is possible when the topology is not fixed. In fact, in Refs. 6 and 16 it is shown that serious problems exist when we try to define this kind of inner product, based on the Feynman integral, if the histories can pass more than once through surface Σ, and this phenomenon can occur if we allow the topology to vary. Therefore, our worries are in order.

To fix the ideas let us consider a rubber tube, the manifold M [Fig. 1(a)], and let us draw a dotted line in the tube, the hypersurface Σ. Let us also draw M_C at the bottom of the tube and M_O on the upper part. M^+ and M^- are perfectly defined. We can vary the metric of the tube with no change in the topology, i.e., to deform the tube in all possible ways [Fig. 1(b)]. In this case all the definitions we have given work all right.

Of course we can also use other kinds of topologies, e.g., the Hartle-Hawking "no-boundary" where the topology must be compact in the past and all compact geometries bounded by Σ should be summed over (i.e., to use a closed tube in "the past").

However, troubles begin if we try to use *all* possible topologies. Because if all kinds of topoligies are allowed we can also add a torus to our cylinder, i.e., to break the tube and add a handle to it as in Fig. 2. Then it is impossible to define zones M^+ and M^- and to compute the wave functions of Eqs. (3.2) and (3.3). In fact, "the set of all histories that satisfy $\{C\}$ and yield h_{ij} and ϕ^Σ on Σ" are now not well defined because they can now pass by the handle and reach M_O and they are not bounded to satisfy any condition there. Therefore, Eq. (3.4) is not valid in this case.

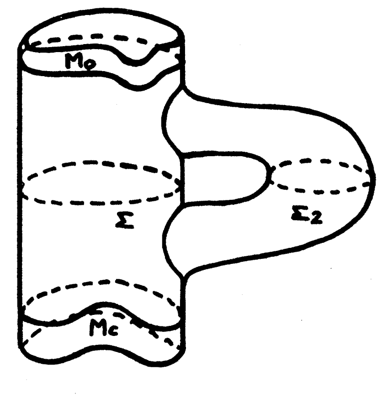

FIG. 2. The cut tube, to build a cylinder with a handle.

Of course, the solution to this problem seems very simple. We must allow the surface Σ to be nonconnected. Then surface Σ will be really $\Sigma=\Sigma_1\cup\Sigma_2$ where Σ_1 is the old Σ and Σ_2 a new surface that splits the handle (Fig. 2). Now our wave functions will really be

$$\psi_C[h_{ij}^{(1)},\phi_{(1)}^{\Sigma},h_{ij}^{(2)},\phi_{(2)}^{\Sigma}]\ ,$$

$$\psi_O[h_{ij}^{(1)},\phi_{(1)}^{\Sigma},h_{ij}^{(2)},\phi_{(2)}^{\Sigma}]\ ,$$

where $h_{ij}^{(1)},\phi_{(1)}^{\Sigma}$ $(h_{ij}^{(2)},\phi_{(2)}^{\Sigma})$ are the metric and the scalar field on Σ_1 (Σ_2). Doing so we can reestablish Eq. (3.4) and everything seems all right. But we can add a new handle to the torus and repeat the reasoning introducing a third component of Σ, Σ_3, and so on, up to Σ_n. Thus to use all topologies yields a surface Σ with n disconnected components $\Sigma_1,\Sigma_2,\ldots,\Sigma_n$. Then, our wave functions are really

$$\psi_C[h_{ij}^{(1)},\phi_{(1)}^{\Sigma},\ldots,h_{ij}^{(n)},\phi_{(n)}^{\Sigma},\Sigma]\ ,$$

$$\psi_O[h_{ij}^{(1)},\phi_{(1)}^{\Sigma},\ldots,h_{ij}^{(n)},\phi_{(n)}^{\Sigma},\Sigma]\ ,$$

and the inner product (3.4) now reads

$$\psi[\{O\}/\{C\}]=\langle O\mid C\rangle=\sum_n\int\mathcal{D}[h_{ij}^{(1)}]\mathcal{D}[\phi_{(1)}^{\Sigma}]\cdots\mathcal{D}[h_{ij}^{(n)}]\mathcal{D}[\phi_{(n)}^{\Sigma}]$$
$$\times\psi_O^*[h_{ij}^{(1)},\phi_{(1)}^{\Sigma},\ldots,h_{ij}^{(n)},\phi_{(n)}^{\Sigma},\Sigma]\psi_C[h_{ij}^{(1)},\phi_{(1)}^{\Sigma},\ldots,h_{ij}^{(n)},\phi_{(n)}^{\Sigma},\Sigma]\ , \qquad (3.4')$$

where n is the numer of components of Σ. This inner product is the analog, for quantum gravity, of the inner product of Eq. (2.27) of Ref. 6, i.e., the inner product of ordinary one-particle quantum mechanics for a surface S that can be crossed and recrossed many times. And this inner product cannot be defined. In fact, in Ref. 16 Hartle showed, using the theory of stochastic processes, that this inner product does not have a continuum limit.

The same pathology exists in our case because surface Σ can be crossed and recrossed many times, e.g., using the topology of Fig. 3 (that is the quantum gravity analog of Fig. 2 of Ref. 6).

Anyhow a closer study of the problem shows that this

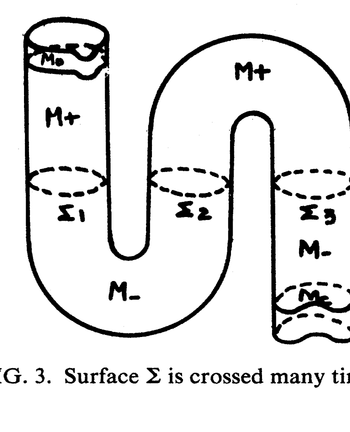

FIG. 3. Surface Σ is crossed many times.

disease can be cured at least in some particular cases,[17] but in order to show that the inner product (3.4′) can be used we must overcome at least three problems.

(1) To sum over all possible topologies we must know all of them. All possible topologies are well known if the manifold has dimension two, because we have the popular "genus" clasification. But we are far from having a similar knowledge when the dimension is greater than two (see, e.g., Ref. 18).

(2) If we use all possible topologies we will have the possibility of the creation of universes, i.e., "trousers topologies." In Refs. 7, 8, and 19 it is shown that an infinite amount of energy is needed to produce these changes of topology. Even if these papers are not completely conclusive we certainly have a new problem here.

(3) Finally, the normalization based on the inner product (3.4′) [and also the one of Eq. (3.5) when the topology is fixed] are not satisfactory because in general they are divergent[1] and therefore a probabilistic interpretation becomes impossible. In fact one of the variables of $\mathcal{D}(h_{ij})$ must be considered a time, then integral (3.4′) or (3.5) must be evaluated for $t=-\infty$ or $t=0$ (the "big bang") to $t=+\infty$ and therefore it diverges. Also as we said (3.4′) or (3.5) are unusual because the field equation of the theory is a hyperbolic one and these products are of parabolic type, and also because there is a "time" integration while in quantum mechanics the inner product is only obtained integrating over the space variables.

As we shall see we can solve all these problems if the topology is fixed because in this case we can define a foliation and a "probabilistic time." Of course we cannot define a foliation in the set of all topologies, therefore the problems of (3.5) can be solved but not those of (3.4′).

For all these reasons we shall work with a fixed topology from now on.

We shall conclude that the real difficulty in defining a satisfactory inner product, when the topology is fixed, only has a mathematical origin, our ignorance of how to define a rigorous measure for the Feynman path integral. On the contrary, if topology fluctuates, we cannot, in principle, define a normalization with the necessary properties to build a theory with the usual probabilistic interpretation. In the cases such as minisuperspace, where we do not have these problems, we can define the normalization with a full physical meaning.

IV. PROBABILISTICAL TIME AND THE "USUAL" NORMALIZATION

We have seen how a fixed topology allows us to define an inner product in states space and to introduce a probability interpretation. Let us now see how this interpretation allows us to define a time.

One of the typical problems of quantum gravity is the absence of a well-defined physical time. One of the coordinates of h_{ij}, or a function of all of them, such as the square root of the determinant of $h_{ij}=h^{1/2}$, plays the role of "time." However, there are infinitely many ways to choose this time, but none of them is privileged nor has a clear physical interpretation, such as proper time in general relativity, or classical time in mechanics, etc. The lack of a physical time makes it impossible to implement a parabolic Schrödinger equation, and makes the interpretation of the results obscure.

Let us see how we can fill the gap and define a "probabilistic time" with a clear physical motivation. We shall develop our ideas in minisuperspace to avoid further problems with the definition of the measure. The natural generalization to a more general situation will be given at the end of this section. We shall use the notation of Refs. 4 and 5 [with small modifications to make explicit the Planck length $l=(16\pi G)^{1/2}$ everywhere].

Let us study minisuperspace with metric

$$ds^2=-N^2(t)dt^2+a^2(t)d\Omega_3^2\ , \tag{4.1}$$

where $a(t)$ is the radius of the Universe, $N(t)$ is an arbitrary lapse function, and $d\Omega_3^2$ is the metric of the unitary three-sphere. The corresponding Euclidean metric is

$$ds^2=N^2(\tau)d\tau^2+a^2(\tau)d\Omega_3^2\ , \tag{4.2}$$

where τ is the Euclidean time such that $t=i\tau$.

The Hilbert-Einstein Lorentzian action, plus the usual action for the scalar field, in metric (4.1), reads

$$S=\sigma^{-2}\left[\tfrac{1}{2}\int dt(N/a)[(a\dot{a}/N)^2-a^2+H^2a^4]\right] +2\pi^2\left[\tfrac{1}{2}\int dt\{-(a^4/N^2)[\dot{\phi}+6\xi(\dot{a}/a)\phi]^2 +6\xi a^2\phi^2+[M^2\phi^2+V(\phi)]a^4\}\right]\ , \tag{4.3}$$

where $\sigma^2=l^2/24\pi^2$, $H^2=\Lambda/3$, i.e., one-third of the cosmological constant, M is the mass of field ϕ, ξ its coupling constant, and $V(\phi)$ an arbitrary potential (the Euclidean action I can be obtained making the change $t=i\tau$). The overdot symbolizes the derivative with respect to t (or τ) in the Lorentzian (Euclidean) case. Let us change the scalar field as

$$\phi\rightarrow\chi=(2\pi^2)^{1/2}a^{6\xi}\phi\ . \tag{4.4}$$

We obtain the action

$$S=(\sigma^{-2}/2)\int dt(N/a)[(a\dot{a}/N)^2-\sigma^2(a^{2-6\xi}\dot{\chi}/N)^2 +u(a)+\sigma^2v(a,\chi)]\ , \tag{4.5}$$

where

$$u(a)=-a^2+H^2a^4\ ,\qquad v(a,\chi)=2\pi^2\{6\xi a^2\phi^2+a^2[M^2\phi^2+V(\phi)]\}\ . \tag{4.6}$$

The metric of minisuperspace is

$$G_{ab}=\sigma^{-6}\begin{bmatrix}-a & 0\\ 0 & \sigma^2a^{3-12\xi}\end{bmatrix}\ , \tag{4.7}$$

where $a,\ b=1,2$ is an index for the components a and χ. The Wheeler-DeWitt equation for the Lorentzian case is

$$\tfrac{1}{2}\{-(\sigma^4/a^{2-6\xi})(\partial/\partial a)[a^{1-6\xi}(\partial/\partial a)]+[\sigma^2/a^{3-12\xi}(\partial^2/\partial\chi^2)]-u(a)/a-\sigma^2[v(a,\chi)/a]\}\psi(a,\chi)=0\ , \tag{4.8}$$

where $\psi(a,\chi)$ is the wave function $\psi(h_{ij},\phi^\Sigma)$ in the minisuperspace case and we have chosen the Laplace operator of metric (4.7) as a natural solution for the order problem. The signs of the potentials u and v change between the Lorentzian and Euclidean regions.

We shall now introduce the notion of "probabilistic time." Let us consider a volume element of minisuperspace $\sqrt{-G(a)}da\,d\chi$, where $G(a)=\det G_{ab}$. This volume element is, in fact, invariant under changes of coordinates in minisuperspace. From the discussion of the previous section $|\psi(a,\chi)|^2$ can be interpreted as the probability density to find the metric and the field in the volume element $\sqrt{-G(a)}da\,d\chi$ containing the "point" (a,χ), because the heuristic normalization (3.6) turns out to be the rigorous one:[17]

$$\int_0^{a_0}\int |\psi(a,\chi)|^2\sqrt{-G(a)}da\,d\chi=1\ , \tag{4.9}$$

in this case. [To avoid for the moment the problem of the divergency of Eq. (3.5) we integrate between $a=0$, the radius of the Universe at the "big bang" and a_0, a final radius of the Universe, i.e., a radius big enough to be completely outside the quantum-gravity domain. We will return to this problem below.] Thus, the probability of finding the metric and the field at (a,χ) is

$$d^2p=|\psi(a,\chi)|^2\sqrt{-G(a)}da\,d\chi\ . \tag{4.10}$$

Then the probability of finding the metric in the interval $a,a+da$, for all possible values of the field χ is

$$dp=da\int|\psi(a,\chi)|^2\sqrt{-G(a)}d\chi\ . \tag{4.11}$$

Therefore dp symbolizes, somehow, a quantity proportional to the number of possible metrics in the interval $a,a+da$. Now, as the metric is defined by the sole parameter a, we can intuitively think that the Universe must stay in metric a for a period of time proportional to dp. In other words, if we give to each metric a certain unitary time period (e.g., a second or, more likely, a Planck time), the metric of the Universe must lay between a and $a+da$, a period proportional to dp multiplied by this unitary time (e.g., dp sec or $dp\times$Planck time). Therefore, we can define the element of probabilistic time as

$$d\theta=c\,da\int|\psi(a,\chi)|^2\sqrt{-G(a)}d\chi\ , \tag{4.12}$$

or probabilistic time as

$$\theta=c\int_0^a da\int|\psi(a,\chi)|^2\sqrt{-G(a)}d\chi\ , \tag{4.13}$$

where $a=0$ corresponds to an eventual "big bang." We shall choose the proportionality constant c below.

Therefore, based on an intuitive idea, the Universe stops in each one of the possible metrics for the same time, we have defined a time parameter, the probabilistic time, and we claim that this definition has some physical sense. In fact, it seems a valid way to give a physical meaning to the probability density $|\psi(a,\chi)|^2$. The three-metrics are defined by a unique parameter a, but they are really part of a space-time four-metric, as they define the metric, not only on an hypersurface but also in a timelike neighborhood of this hypersurface through Eq. (4.1). How long does the Universe stay in the metric defined by a? What is the timelike thickness of the slice? It seems reasonable to give the same time thickness to every slice. In other words, if an evolution $a(\theta)$ is given and if someone wants to "count" the number of metrics in the interval $a,a+da$, it seems natural to divide the evolution into equal time steps, and then to count the number of the steps in the interval $a,a+da$. The inverted procedure yields precisely the definition of the probabilistic time. Of course, only a systematic study of this notion in a great number of examples will show if it is a useful idea or not. We shall see some of these examples below.

Of course, the definition of probabilistic time can be extended to the general case. Using the time parameters $h^{1/2}$ or K and the notation of Ref. 4, the general definition reads

$$\theta=c\int_0^{h^{1/2}}dh_0^{1/2}\int\mathcal{D}[h^{1/2}]\mathcal{D}[\tilde{h}_{ij}]\mathcal{D}[\phi^\Sigma]\times|\psi[h^{1/2},\tilde{h}_{ij},\phi^\Sigma]|^2\ , \tag{4.14}$$

$$\theta=c\int_{-\infty}^{K}dK_0\int\mathcal{D}[K]\mathcal{D}[\tilde{h}_{ij}]\mathcal{D}[\phi^\Sigma]|\psi(K,\tilde{h}_{ij},\phi^\Sigma)|^2\ , \tag{4.15}$$

where we have taken a particular trajectory that crosses all the hypersurfaces of the foliation. $h_0^{1/2}$ (or K_0) are the values of $h^{1/2}$ (or K) at the point of intersection of this trajectory and the hypersurfaces. It is used as a parameter to label the hypersurfaces. $\mathcal{D}[h^{1/2}]$ (or $\mathcal{D}[K]$) is the functional integration taking into account the values of $h^{1/2}$ (or K) in all the other points of the hypersurface except the intersection point.

A foliation of space-time is used to define h_{ij}. If no foliation is possible, as in the case of variable topology, this definition cannot be written.

As we really do not know how to perform these integrations we prefer to follow the study of probabilistic time in minisuperspace, using the Born-Oppenheimer approximation, in the next section.

We must remark that the definition (4.12) also means that we have chosen the "cosmological arrow of time."[20] Whether this arrow of time coincides or not with other arrows of time, in the quantum era, is a difficult problem that we do not discuss in this paper. (Perhaps an auxiliary scalar massless field can be introduced to fix a monotonically increasing time definition, as in Ref. 21.) But essentially, as DeWitt pointed out,[22] all clocks, being parts of the Universe, are in principle also described by the wave function of the Universe. To measure time one just chooses one of these clocks in a more or less arbitrary way. When we try to establish the law of motion of one "spatial" variable, what we are really doing is computing the correlation of the "spatial" variable and the "time" variable that we have chosen as a clock, i.e., another "spatial" variable: the clock hand point. The bigger the clock is, the smaller the quantum fluctuations; in this sense the best clock is the entire Universe, and the cosmological arrow of time seems the most reliable one.

Let us try to clarify the probabilistic time definition, studying how the same definition works in the simplest example: Let us consider the motion of a one-dimensional, nonrelativistic particle with a general Lagrangian $\mathcal{L}(x,dx/dt,t)$ and a wave function $\psi(x,t)$, moving in the domain

$$0<x<x_0, \quad 0<t<t_0 \; . \tag{4.16}$$

The wave function will be normalized as

$$\int_0^{t_0}\int_0^{x_0} |\psi(x,t)|^2 dx\, dt = 1 \; . \tag{4.17}$$

t_0 plays the role of a_0 now. We can see, in fact, that the integral diverges if $t_0 \to \infty$ and the normalization becomes impossible.

Let us remark that this normalization is "unusual" because we integrate over the time variable. Now, let us erase the notion of time. Let us change t for an arbitrary time $\tau=\tau(t)$.

Let us also change the time measure, introducing an arbitrary time measure $\mu(\tau)$. Thus, the action is

$$\begin{aligned} I &= \int \mathcal{L}(x,dx/dt,t)dt \\ &= \int [\mathcal{L}(x,(dx/d\tau)(d\tau/dt),t(\tau))\mu(\tau)^{-1}(dt/d\tau)] \\ &\quad \times \mu(\tau)d\tau \\ &= \int L(x,dx/d\tau,\tau)\mu(\tau)d\tau \; . \end{aligned} \tag{4.18}$$

L is the new Lagrangian and $\mu(\tau)$ is an arbitrary measure, but, of course, the physics is the same. Thus,

$$|\psi(x,t)|^2 dx\, dt = |\psi_\mu(x,\tau)|^2 \mu(\tau) dx\, d\tau \; , \tag{4.19}$$

where $\psi_\mu(x,\tau)$ is the new wave function (that depends on the measure μ). Let us now suppose that we only know L and we compute ψ_μ, and we want to reconstruct the notion of classical time t. In order to do that we can integrate the last equation in the variable x:

$$dt\int_0^{x_0} |\psi(x,t)|^2 dx = \mu(\tau)\, d\tau \int_0^{x_0} |\psi_\mu(x,\tau)|^2 dx \; . \tag{4.20}$$

Now we may introduce the "usual" normalization for the wave function: $|\psi(x,t)|^2$ must be the probability density to find the particle at x "normalized for every time." Thus,

$$\int_0^{x_0} |\psi(x,t)|^2 = \text{const} = t_0^{-1} \; . \tag{4.21}$$

The constant must be t_0^{-1} in order to satisfy normalization (4.17). Therefore, we have

$$dt = t_0 \mu(\tau) d\tau \int_0^{x_0} |\psi_\mu(x,\tau)|^2 dx \tag{4.22}$$

and

$$t = t_0 \int_0^t \mu(\tau)d\tau \int_0^{x_0} |\psi_\mu(x,\tau)|^2 dx \; , \tag{4.23}$$

i.e., equations similar to (4.12) and (4.13) [with $c=t_0$, as will be the case, cf. Eq. (5.23)].

Thus, we obtain classical time with the same definition as probabilistic time in our toy model. But now let us remark that from all possible time parameters τ, and for all possible measures $\mu(\tau)$, the classical time is singled out via the normalization

$$\mu(\tau)\int_0^{x_0} |\psi_\mu(x,\tau)|^2 dx = t_0^{-1} \; , \tag{4.24}$$

because from Eq. (4.20) we can see that in this case τ becomes t. Thus this will be the "usual" normalization, written for an arbitrary time parameter τ and an arbitrary measure $\mu(\tau)$, that substitutes the "unusual"one (4.17). It becomes the really usual one (4.21), if we use the privileged classical time t and the trivial measure $\mu(\tau)=1$.

As we can see the time definition and the way to obtain the usual normalization are intimately related and both problems can be solved at the same time.

Finally Eq. (4.24) can be written as

$$\mu(\tau)\int_0^{x_0} t_0 |\psi_\mu(x,\tau)|^2 dx = 1 \; . \tag{4.25}$$

This equation shows that what we are really normalizing is not $|\psi_\mu|^2$ but $t_0|\psi_\mu|^2$ and that the last integral is convergent even if $t_0 \to \infty$. This will be the recipe to overcome the divergence problem of Eq. (4.17) when $t_0 \to \infty$: (4.25) does not have this problem.

Let is also remark that (4.21) is a t constant, as is well known, as a consequence of the Schrödinger equation. On the contrary, in the general case, the second factor of (4.25) is not a τ constant, but it becomes a constant multplied by $\mu(\tau)$. The moral is the following: even if some factors of the normalization equation are not constant as a consequence of the field equation, they become a constant multiplied by the measure. This fact will be useful in the quantum gravity case.

Let us now use all the knowledge obtained in the toy model in our minisuperspace problem. We have the Wheeler-DeWitt equation (4.3), where the time variable is a, and the "unusual" (divergent) normalization (4.9). Let us obtain a privileged time variable and a "usual" (convergent) normalization as before. We must find a time variable

$$\theta = \theta(a) \tag{4.26}$$

that would play the role of classical time. As $\sqrt{-G(a)}$ is only a function of a the measure $\mu(\theta)$ of θ will be defined by

$$\sqrt{-G(a)}da = \mu(\theta)d\theta \; . \tag{4.27}$$

Thus we can compute $\mu(\theta)$ as soon as we know transformation (4.26).

As the physics must be the same using a or θ we can write [as in Eq. (4.19)]

$$|\psi(a,\chi)|^2\sqrt{-G(a)}da\, d\chi = |\psi(\theta,\chi)|^2 \mu(\theta) d\theta\, d\chi \; . \tag{4.28}$$

[Let us note that using Eq. (4.27) we do not actually change the measure, thus we do not need the subindex μ as in Eq. (4.19): there is no arbitrary measure.] Now we can integrate over the variable χ that plays the role of the spatial variable x:

$$\sqrt{-G(a)}da\int_{-\infty}^{+\infty}|\psi(a,\chi)|^2 d\chi = \mu(\theta)d\theta\int_{-\infty}^{+\infty}|\psi(\theta,\chi)|^2 d\chi \ . \quad (4.29)$$

The privileged time is singled out as in Eq. (4.24) by the condition

$$\mu(\theta)\int_{-\infty}^{+\infty}|\psi(\theta,\chi)|^2 d\chi=\theta_0^{-1} \ . \quad (4.30)$$

This is the "usual" normalization obtained integrating only over the spatial variable χ. The privileged time is defined as in Eqs. (4.22) and (4.23) by

$$d\theta=\theta_0\sqrt{-G(a)}\int_{-\infty}^{+\infty}|\psi(a,\chi)|^2 d\chi\, da \ , \quad (4.31)$$

$$\theta=\theta_0\int_0^{a_0}\sqrt{-G(a)}\int_{-\infty}^{+\infty}|\psi(a,\chi)|^2 d\chi\, da \ . \quad (4.32)$$

But these are the definitions (4.12) and (4.13) of the probabilistic time obtained in an independent way (if $c=\theta_0$, and this will be the case as we shall see in the next section [cf. Eqs. (5.23) and (5.24)]—precisely $c=\theta_0=t_0$ the classical time; this fact relates (4.22) and (4.23) to (4.31) and (4.32) even more strongly). Thus probabilistic time seems a logical interesting notion because it can be defined through, at least, two different considerations.

The "usual" normalization equation (4.31) can be written as

$$\mu(\theta)\int_{-\infty}^{+\infty}\theta_0|\psi(a,\chi)|^2 d\chi=1 \ . \quad (4.33)$$

This fact shows that what we really normalize is $\theta_0|\psi|^2$ and not $|\psi|^2$. If $\theta_0\to\infty$, Eq. (4.33) is still convergent. Moreover in the definition of probabilistic time $|\psi|^2$ appears as $\theta_0|\psi|^2$, thus $\theta_0|\psi|^2$ is the relevant physical quantity. When $\theta_0\to\infty$, $|\psi|^2\to 0$ but $\theta_0|\psi|^2$ remains finite, and the probabilistic time definition is thus independent of θ_0 if θ_0 is big enough, i.e., it belongs to the classical period of the Universe.

As in the previous case the second factor in Eq. (4.3) or (4.33) is not a constant as a consequence of the field equation of the theory, the Wheeler-DeWitt equation. In fact, this was one of the main criticisms of this kind of normalization. But the second factor multiplied by $\mu(\theta)$ yields a constant and therefore a reasonable normalization. And this feature of the method is not surprising since it also appears in Eqs. (4.21) and (4.25) where we are dealing with a perfectly well-understood toy model. We conclude that this particular feature is a consequence of the fact that the times τ or θ have nontrivial measures $\mu(\tau)$ or $\mu(\theta)$, that play a very important role in the normalization formulas.

All our considerations up to now are exact; no approximation was used. But the probabilistic time also has several interesting properties at the classical and semiclassical level that we shall study in the following sections.

V. PROBABILISTIC TIME IN THE SEMICLASSICAL APPROXIMATION

Let us compute the probability amplitude $\psi(a,x)$ and the probabilistic time θ using the Born-Oppenheimer approximation, as in Ref. 6. Thus, we introduce the decomposition

$$\psi(a,\chi)=e^{iK(a)}J(a,\chi) \ , \quad (5.1)$$

and we expand $K(a)$ and $J(a,\chi)$ in powers of σ^2:

$$K=\sigma^{-2}K_0+K_1+\sigma^2K_2+\cdots \ , \qquad J=J_0+\sigma^2J_1+\sigma^4J_2+\cdots \ . \quad (5.2)$$

We replace Eqs. (5.1) and (5.2) in the Wheeler-DeWitt equation (4.8) and we separate the terms with equal power in σ^2. Thus, we obtain an equation for each power or σ^2. For the leading order σ^0 the obtained equation reads

$$[dK_0(a)/da]^2-u(a)=0 \ . \quad (5.3)$$

This is, in fact, the Hamilton-Jacobi equation for the vacuum, i.e., with no matter field, because the potential $v(a,\chi)$ [Eq. (4.6)] corresponding to this field is missing in Eq. (5.3). From Eq. (5.3) we can obtain $K_0(a)$ as

$$K_0(a)=\pm\int[u(a)]^{1/2}da \ . \quad (5.4)$$

For the next order σ^2 we have

$$\tfrac{1}{2}[i(d^2K_0/da^2)-2(dK_0/da)(dK_1/da)+i(1-6\xi)a^{-1}(dK_0/da)]J_0 + i(dK_0/da)(\partial J_0/\partial a)-\tfrac{1}{2}a^{-2+12\xi}(\partial^2J_0/\partial\chi^2)+vJ_0/2=0 \ , \quad (5.5)$$

where we have two unknowns, K_1 and J_0. This ambiguity is not strange because decomposition (5.1) is, in fact, ambigous: we can take an arbitrary function of a, from the first factor to the second one, with no change in the form of the equation. Thus, we have an extra degree of freedom and we can adopt an extra equation, to choose the decomposition in a unique way, as we shall do below.

For the moment, let us return to classical time t of Eq. (4.1). If we make $N(t)=1$, t becomes proper time. From Ref. 6 we know that, if the metric has the form

$$ds^2=-dt^2+h_{ij}(t,x)dx^idx^j \ , \quad (5.6)$$

the gravitational field equation can be written as

$$dh_{ij}/dt=G_{ijkl}\delta K/\delta h_{kl} \ , \quad (5.7)$$

where $K[h_{ij}]$ is a functional solution of the corresponding Hamilton-Jacobi equation and G_{ijkl} the superspace metric

$$G_{ijkl}=\tfrac{1}{2}h^{-1/2}(h_{ik}h_{jl}+h_{il}h_{jl}-h_{ij}h_{kl}) \ . \tag{5.8}$$

For example, in the vacuum and for the metric (4.1) $K[h_{ij}]$ will be $K_0[h_{ij}]=K_0(a)$ given by Eq. (5.4). Equation (5.7) in our particular case is simply

$$da/dt=-(1/a)dK/da \ , \tag{5.9}$$

and in the particular case that K is only computed up to the leading order σ^0 using Eq. (5.4), we find

$$da/dt=\pm\mu^{1/2}/a \ . \tag{5.10}$$

In fact, this last equation, with the definition (4.6), yields the de Sitter space-time as a solution, i.e., the classical solution of the problem in the vacuum corresponding to the leading order σ^0 (as we shall show explicitly at the end of the section).

The inner product (3.5) now reads

$$\begin{aligned}\langle\psi,\psi'\rangle&=\int_0^{a_0}\int\psi^*\psi'\sqrt{-G}\,da\,d\chi\\&=\int_0^{a_0}dt\,(dK/da)(\sqrt{-G}/a)\int\psi^*\psi'd\chi\\&=t_0^{-1}\int_0^{t_0}dt\langle\psi,\psi'\rangle_{(\chi)} \ ,\end{aligned} \tag{5.11}$$

where t_0 is a "final time" corresponding to the final radius a_0 and where we have introduced the partial inner product

$$\langle\psi,\psi'\rangle_{(\chi)}=t_0(\sqrt{-G}/a)(dK/da)\int\psi^*\psi'd\chi \ . \tag{5.12}$$

In the particular case that we compute K only up to the leading order, this product reads

$$\langle\psi,\psi'\rangle_{(\chi)}=t_0u^{1/2}a^{1-6\xi}\int\psi^*\psi'd\chi \ . \tag{5.13}$$

The partial inner product (5.12) is naturally related to the computation of the probability of finding the metric in the interval $t,t+dt$ [Eq. (4.11)] that can now be written as

$$dp=t_0^{-1}\langle\psi,\psi\rangle_{(\chi)}dt \ , \tag{5.14}$$

where we now introduce proper time t, instead of "time" a, to compute the probability density.

We can now impose the missing condition. The product $\langle\psi,\psi\rangle_{(\chi)}$ is, in principle, a function of t (or a). We ask that the decomposition (5.1) must be such that the inner product

$$\langle J,J'\rangle_{(\chi)}=t_0(dK/da)(\sqrt{-G}/a)\int J^*J'd\chi \tag{5.15}$$

should be a constant. There is an important physical motivation for this condition. If $\langle J,J'\rangle_{(\chi)}=\text{const}$ and we normalize the J of Eq. (5.1) as [cf. Eq. (4.21) of the toy model]

$$\langle J,J\rangle_{(\chi)}=1 \ , \tag{5.16}$$

Eq. (5.14) reads

$$dp=t_0^{-1}dt\,e^{i[-K^*(a)+K(a)]} \ , \tag{5.17}$$

i.e., the probability of finding the metric in the interval $(a,a+da)$, only depends on the function $K(a)$; while $J(\chi,a)$ properly normalized by Eq. (5.16), is the probability amplitude of finding the field in the interval $(\chi,\chi+d\chi)$ if the metric is in the interval $(t,t+dt)$.

If we make the computation only up to the leading order and we only work in the case

$$u(a)=H^2a^4-a^2>0 \ , \tag{5.18}$$

i.e., when

$$a>H^{-1} \tag{5.19}$$

in the Lorentzian case and when

$$a<H^{-1} \tag{5.20}$$

in the Euclidean case, from Eq. (5.3), $K(a)$ turns out to be real and Eq. (5.17) is only

$$dp=t_0^{-1}dt \ . \tag{5.21}$$

Then probabilistic time [Eq. (4.12)] reads

$$d\theta=c\,dt/t_0 \ . \tag{5.22}$$

Thus, in the leading order of the semiclassical approximation, probabilistic time is proportional to proper time; i.e., when the Universe leaves the quantum era and goes over to the semiclassical regime, probabilistic time becomes proportional to classical time. It is logical to ask that probabilistic time be equal to proper time, in the classical limit. This prescription allows us to choose the constant c,

$$c=t_0 \tag{5.23}$$

and therefore, up to the leading order,

$$\theta=t \ . \tag{5.24}$$

Thus, the probabilistic time is proper time in the leading order of the semiclassical approximation.[23]

Now we can compare normalization (4.33) and (5.16), they turn out to be the same.

In fact, from Eq. (4.7) we can see that $a^{1/2}$ is the measure corresponding to variable a while $a^{(3-2\xi)/2}$ is the one of variable χ. Thus Eq. (4.27), for variable t, reads

$$\sqrt{a}\,da=\mu(t)dt \tag{5.25}$$

and from this equation and Eq. (5.9) we have

$$\mu(t)=a^{-1/2}(dK/da) \ , \tag{5.26}$$

where the minus sign disappears, because we must consider only the absolute value when variables are changed, i.e., the measure must be positive. The normalization (4.33) reads

$$t_0(\sqrt{-G}/a)(dK/da)\int|\psi(t,\chi)|^2d\chi=1 \tag{5.27}$$

and (4.33) coincides with (5.16), i.e., the "usual" normalization turns out to be the natural one at the classical level, as it was expected.

Let us finally review whether the evolution of the radius of the Universe is, up to leading order, as a function of proper or probabilistic time. We must integrate Eq. (5.10) (we choose the + sign):

$$da/d\theta=[\pm(H^2a^2-1)]^{1/2}\ , \qquad (5.28)$$

with the + (−) for the Lorentzian (Euclidean) case.

For $a<H^{-1}$ we are in the Euclidean case and the solution is

$$a=H^{-1}\cos(H\theta)\ , \qquad (5.29)$$

where we have placed the "big bang" at $\theta=-\pi H^{-1}/2$ for simplicity. This solution is correct up to time $\theta=0$ where $a=H^{-1}$. For $a>H^{-1}$ we are in the Lorentzian sector and the solution is

$$a=H^{-1}\cosh(H\theta)\ . \qquad (5.30)$$

This solution is correct from $\theta=0$ (where $a=H^{-1}$) to $\theta\rightarrow+\infty$. As θ is proper time in the leading order, these solutions correspond to a sphere of radius H^{-1} and to a de Sitter space of the same radius, respectively. Actually, only the second solution is physically important, because the leading order could be considered a good approximation near the classical limit only. In addition, we have shown that a solution of the problem exists where, after the quantum period, we have a classical solution that becomes a de Sitter expansion in the far future, for all M, ξ, and $v(a,\chi)$. (This is logical because we have taken $\Lambda\neq0$.)

VI. THE SCHRÖDINGER EQUATION

Let us continue the study to the next order σ^2, rewriting Eq. (5.5) in the form

$$(i/2)[d^2K_0/da^2+(1-6\xi)a^{-1}dK_0/da+2(dK_0/da)(\partial J_0/\partial a)]-(dK_0/da)(dK_1/da)J_0+\mathcal{H}J_0=0\ , \qquad (6.1)$$

where we have introduced the Hamiltonian operator for the wave field J_0:

$$\mathcal{H}=-\tfrac{1}{2}a^{-2+12\xi}(\partial^2/\partial\chi^2)+\tfrac{1}{2}v(a,\chi)\ . \qquad (6.2)$$

Let us now introduce the new field

$$\tilde{J}_0=[(dK_0/da)a^{1-6\xi}]^{1/2}J_0\ . \qquad (6.3)$$

Using this new field and Eq. (5.10), Eq. (6.1) becomes

$$i(\partial\tilde{J}_0/\partial\theta)=[(dK_1/d\theta)+\mathcal{H}/a]\tilde{J}_0\ . \qquad (6.4)$$

Now, let us define the ordinary Schrödinger inner product:

$$(\tilde{J}_0,\tilde{K}_0)=t_0\int\tilde{J}_0^*\tilde{K}_0d\chi\ , \qquad (6.5)$$

and contract Eq. (6.3) with J_0 using this product. We obtain

$$(i/2)(\partial/\partial\theta)(\tilde{J}_0,\tilde{J}_0)=(dK_1/d\theta)(\tilde{J}_0,\tilde{J}_0)+(1/a)(\tilde{J}_0,\mathcal{H}\tilde{J}_0)\ . \qquad (6.6)$$

Comparing Eqs. (5.15), (6.3), and (6.4) we see that, up to order σ^2, we have

$$\langle J_0,J_0'\rangle_{(\chi)}=(\tilde{J}_0,\tilde{J}_0')\ . \qquad (6.7)$$

Thus the normalization (4.33) or (5.16) are now

$$(\tilde{J}_0,\tilde{J}_0)=1\ , \qquad (6.8)$$

and Eq. (6.6), with this condition, reads

$$dK_1/d\theta=(-1/a)(\tilde{J}_0,\mathcal{H}\tilde{J}_0)\ . \qquad (6.9)$$

From this equation we can find K_1. Now we can introduce this result in Eq. (6.4) and find the equation for $\tilde{J}_0$:

$$i(\partial\tilde{J}_0/\partial\theta)=(1/a)[\mathcal{H}-(\tilde{J}_0,\mathcal{H}\tilde{J}_0)]\tilde{J}_0\ , \qquad (6.10)$$

a Schrödinger equation modified by the reaction back of the gravitational field.

At this point, we have computed the equations that functions K_0, K_1, and J_0 must satisfy, (5.3), (6.8), and (6.9), and thus we can compute, in principle, the wave function $\psi(a,\chi)$ [Eq. (5.1)] up to order σ^2.

Now we are ready to calculate the equation that $\psi(a,\chi)$ must satisfy, and it will be a parabolic equation, as we shall see. However, if we want to obtain a real Schrödinger equation we must define a new function:

$$\tilde{\psi}=[(dK_0/da)a^{1-6\xi}]^{1/2}\psi\ . \qquad (6.11)$$

In fact, a Schrödinger wave function must be normalized by the Schrödinger product (6.5), while ψ is normlaized by the primitive product (4.33) or (5.16). Both ψ and $\tilde{\psi}$ are normalized in their corresponding products:[24]

$$\langle\psi,\psi\rangle_{(\chi)}=(\tilde{\psi},\tilde{\psi})=1\ , \qquad (6.12)$$

so all the normalizations coincide. As, up to this order,[25]

$$\tilde{\psi}=\{\exp[i(\sigma^{-2}K_0+K_1)]\}\tilde{J}_0\ , \qquad (6.13)$$

using Eqs. (5.3), (6.8), and (6.9) we can compute the derivative of $\tilde{\psi}$ with respect to θ that reads

$$i(\partial\tilde{\psi}/d\theta)=(1/a)[\sigma^{-2}u+\mathcal{H}]\tilde{\psi}\ , \qquad (6.14)$$

i.e., a Schrödinger equation with a self-adjoint operator on the right-hand side (RHS). Also ψ satisfies a parabolic equation but its RHS does not have a self-adjoint operator and it is not related, in the usual way, to the Schrödinger product (6.5).

Therefore, using probabilistic time θ, the field equation for the wave function is parabolic and it is an ordinary Schrödinger equation, if the wave function is properly normalized. (Although this result is not made explicit in this form, it is implicit in Refs. 14 and 15).

Thus we conclude that in the semiclassical level the "usual" normalization (4.33) or (5.16) or (6.12) is really the usual one, because it turns out to be constant in time via the field equation [i.e., the Schrödinger equation (6.14) as in ordinary quantum mechanics].

VII. COMPARISON WITH VILENKIN INTERPRETATION (REF. 1)

We will compare, in this section, our "usual" normalization, with another one, based on a completely different idea, and we will show that they coincide at the semiclassical level. As the Wheeler-DeWitt equation is a hyperbolic one [really Eq. (4.8) is similar to a Klein-Gordon equation in a curved background with variable "mass"] we can introduce the current

$$\mathcal{J}^a=(i/2)G^{ab}(\psi^*\nabla_b\psi-\psi\nabla_b\psi^*)\ , \qquad (7.1)$$

where G^{ab} is the inverse of matrix (4.7). Then it is easy to show that

$$\nabla_a \mathcal{J}^a = 0 \ . \tag{7.2}$$

Let us define, in our minisuperspace, spatial surfaces; as a is the "time" they are the a=const surfaces but, of course, we could define this surface in a more general way, if we wish to consider the minisuperspace as a Riemannian two-manifold with metric G_{ab}. Then, let us define the probability to find the Universe in a point of an element $d\Sigma_a$ of the spatial surface Σ as

$$dP = \mathcal{J}^a d\Sigma_a \ . \tag{7.3}$$

The conservation of probability is ensured by the conservation current (7.2), but these probabilities can be negatives. This is of course an old problem, that this definition of probabilities has, in the Klein-Gordon equation. It cannot be solved in general, but it can be solved at the semiclassical level, because if ψ is given by (5.1) $\mathcal{J}_1$ is just

$$\mathcal{J}_1 = (-i/2)[\psi^*(\partial/\partial a)\psi - \psi(\partial/\partial a)\psi^*]$$
$$= |J(a,\chi)|^2[dK(a)/da] \ , \tag{7.4}$$

that can be always positive choosing the right sign for K.

The normalization corresponding to probability (7.3) is of course

$$\int \mathcal{J}_a n^a d\Sigma = 1 \ , \tag{7.5}$$

where n^a is the unit vector normal to Σ.

Using (7.4) we have

$$(\sqrt{-G}/a)[dK(a)/da] \int |J(a,\chi)|^2 d\chi = 1 \ . \tag{7.6}$$

As K is real at the lowest order, this normalization is just the "usual" one (5.27) if we change $|\psi|^2 \to t_0 |\psi|^2$, i.e., the Klein-Gordon normalization used, by Vilenkin, at the semiclassical level coincides with our normalization, if we introduce the factor t_0 to avoid the divergence problem, as we have explained.

Thus Vilenkin's interpretation coincides with ours, at the semiclassical level, because the normalization is the same and probability dp [cf. Eq. (7.3)] to find the Universe at a time a with a potential χ is proportional to $|\psi(a,\chi)|^2$, as we have shown. In Vilenkin's language a would be a semiclassical variable and χ a quantum variable that corresponds to a "small" subsystem of the Universe, and the coincidence of both interpretations shows also that our interpretation satisfies the correspondence principle, i.e., that one should be able to recover the traditional interpretation and normalization for the quantum variable of the "small" system (this fact was also shown in the last section).

The coincidence of both interpretations at the semiclassical level makes both more reliable. But while Vilenkin's can only be used in the semiclassical domain (to avoid negative probabilities) ours can also be used in the quantum domain. In this sense we consider our interpretation a better one.

VIII. CONCLUSIONS

We propose the following answers to the list of questions in the Introduction.

(a) What is the Hilbert space of states of the theory? The space of states cannot be promoted to a Hilbert space but only to a normed space, using the norm of Eq. (4.33) if we keep the topology fixed.

(b) What is the probability interpretation of the states? The ordinary probability interpretation, with the normalization (4.33), as a consequence of the previous answer. This interpretation coincides with the Klein-Gordon-Vilenkin interpretation at the semiclassical level.

(c) What is the variable that plays the role of time? The probabilistic time, obtained by a natural way from the probability interpretation, and with several good properties: it is obtained in the same way that classical time is "obtained" in quantum mechanics (cf. toy model), it becomes the ordinary proper time in the classical limit, it plays the role of ordinary time in the Schrödinger equation that a rescaled version of the wave function must satisfy, up to order σ^2

All these answers are plausible and will be either confirmed or not by further research.

Let us finish this paper by sketching some future possible line of research and answering some natural questions. We have seen that probabilistic time works up to order σ^2. In the future we shall see how it works in higher orders. For the moment, let us, at least, see that it is also a useful didactical tool in the next order.

Let us consider the conformal case

$$M=0, \quad V=0, \quad \xi=\tfrac{1}{6} \ . \tag{8.1}$$

In this case the wave function, computed by the steepest-descent method, is (cf. Ref. 5), for $a < H^{-1}$,

$$\psi \approx \exp(-\chi^2/2)(-\sigma^2+a^2-H^2a^4)^{-1/4}$$
$$\times \exp[-1/3H^2(1-H^2a^2)^{3/2}] \tag{8.2}$$

and, for $a > H^{-1}$,

$$\psi \approx \exp(-\chi^2/2)(H^2a^4-a^2+\sigma^2)^{-1/4}$$
$$\times \cos[(H^2a^2-1)^{3/2}/3H^2-\pi/4] \ . \tag{8.3}$$

From the last equation using different heuristic arguments, based in the conformal mapping between Einstein and de Sitter universes, etc., it is argued that the quantum solution evolves to de Sitter space in the far future (cf. Refs. 4 and 5). We believe that the real problem with this reasoning is the lack of a clear time parameter. Now, using probabilistic time the results stem more rigorously. In fact, for this case let us repeat the arguments of the last part of Sec. V, and consider a situation where

$$a \gg H^{-1} \quad \text{and } a \gg \sigma \ , \tag{8.4}$$

and compute the probabilistic time.

As the oscillating cosine factor in Eq. (7.3) can be averaged, we obtain

$$d\theta/da \approx (H^2a^2-1)^{-1/2} \ , \tag{8.5}$$

which is again Eq. (5.28). From the solution (5.30) of this equation and from the fact that θ is proper time, in the situation we are dealing with, we obtain, as a result, that the Universe evolves to a de Sitter space-time in the far future. Of course, we can say nothing of the kind for Eq. (8.2), because the exponential factor cannot be averaged. Near the "big bang" quantum gravity does not have an asymptotical regime. We hope that this example, clearer than the one presented at the end of Sec. V, would show how, and how far, the concept of probabilistic time can be used.

We can ask ourselves if the "miraculous" transformation of the Wheeler-DeWitt equation in a Schrödinger equation is an exact feature of probabilistic time or just a property of the approximation we have been working with. But the transformation of a hyperbolic equation in a parabolic one is well-known low-energy phenomenon that happens in a great number of theories. Thus, the elimination of the second derivative is only a feature of the approximation. Even if probabilistic time is a precise quantum gravity concept, defined through Eq. (4.13), it seems that it behaves as a valid and ordinary time in the low-energy approximation only. Additional research must further clarify this point.

Therefore, we have the usual state of affairs at low energies: the usual inner product (6.5) defined by an integration, which turns out to be time invariant, and the usual Schrödinger equation (6.14), related in the usual way.

What happens at high energies? We have the unusual inner product (3.6) or (4.9). It is definitely time independent but it is unusual because it has a t and a χ integration. The time independence is then obvious because this product has a t integration. We also have a hyperbolic equation (4.8), while the inner product is of parabolic type. The inner product and the field equation are not related in the usual way at high energies. Moreover the normalization (4.9) is divergent if $a_0 \to \infty$. But following the example of the toy model we can transform the unusual normalization in a "usual" one (4.25) that turns out to be convergent. This normalization can be used to introduce the probability notion and the field equation to single out the physical states. Why must they be related in the strictly usual way, i.e., hyperbolic product with hyperbolic equation and parabolic product with parabolic equation? Furthermore they behave in the strictly usual way at low energies.

Let us formulate a last question. Is probabilistic time really a quantum notion or is it only a semiclassical notion? From the analogy with the toy model of Sec. IV, we would say that it is really a quantum notion, because its definition is given in an exact way and no approximation is used: on the other hand, most likely the Wheeler-DeWitt equation is only valid at energies lower than Planck's energy, because it does not have good short-distance behavior. Then if we use a ψ computed via the Wheeler-DeWitt equation, in the definition of probabilistic time (4.13), then we only have a low-energy reliable time variable. But the blame must be put on the field equation and not on the probabilistic time definition, in such a way that if we would have a completely satisfactory field equation, for all energies, we would have a similarly satisfactory probabilistic time.

Perhaps this is the solution of the whole interpretation problem, at least in the case of a fixed topology. (For the case of topology fluctuation see suggestions in Refs. 26 and 17.)

A preliminary and incomplete version of this paper can be found in Ref. 27.

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