Mixmaster cosmological model in theories of gravity with a quadratic Lagrangian

John D. Barrow
Astronomy Centre, University of Sussex, Brighton BN1 9QH, Sussex, England

H. Sirousse-Zia
Laboratoire de Physique Théorique, Institut Henri Poincaré, 11 rue Pierre et Marie Curie, 75231 Paris CEDEX 05, France
(Received 19 December 1988)

We use the method of matched asymptotic expansions to examine the behavior of the vacuum Bianchi type-IX mixmaster universe in a gravity theory derived from a purely quadratic gravitational Lagrangian. The chaotic behavior characteristic of the general-relativistic mixmaster model disappears and the asymptotic behavior is of the monotonic, nonchaotic form found in the exactly soluble Bianchi type-I models of the quadratic theory. The asymptotic behavior far from the singularity is also found to be of monotonic nonchaotic type.

The solution of the Einstein equations for a Bianchi type-IX metric in vacuum has the property that the evolution proceeds towards an initial Weyl curvature singularity via a chaotically unpredictable sequence of oscillations which ergodically pass close to a sequence of Kasner eras.[1] The smooth invariant measure of the associated Poincaré mapping has been found in closed form by Chernoff and Barrow.[2] It has also been shown[3] that under certain conditions the introduction of a cosmological constant does not change the alternation of Kasner eras on approach to the initial singularity. Various detailed investigations have been made into the ergodic theory of this chaotic behavior which have revealed that the spatial dimension of the Bianchi type-IX space-time plays a crucial role in determining whether or not the evolution is chaotic.[4] If the space-time metric has the product manifold structure of Kaluza-Klein type, with internal and external manifolds uncoupled, then chaotic behavior is only possible in universes with three spatial dimensions. If the manifold is of nonproduct type then chaos becomes possible in models with spatial dimension in the range three to nine if off-diagonal terms are included in the type-IX metric tensor. If they are not included, or if they are but the number of spatial dimensions exceeds nine, then the chaotic behavior disappears completely. All of these spatially homogeneous, anisotropic, type-IX cosmological models can be viewed as Hamiltonian dynamical systems in which a "universe point" moves inside a closed time-dependent potential.[5] As the dimensionality of space increases the speed at which the walls expand increases relative to the speed of the "universe point." Eventually, for a sufficiently high dimension the normal component of the velocity of the "universe point" relative to the walls is never sufficient to allow it to catch the wall. No more reflections occur and there can be no chaotic behavior.[6]

On approach to the initial singularity we would expect higher-order curvature corrections to the gravitational Lagrangian to generalize the field equations provided by general relativity[7] and it is therefore of fundamental importance to determine whether the approach to any space-time singularity is chaotic in the most general cases we can analyze. As in general relativity, the most general situation available for complete analysis is that of the spatially homogeneous Bianchi type-IX universe. In this paper we shall describe an investigation of the behavior of the three-dimensional Bianchi type-IX (mixmaster) vacuum universe when the gravitational field equations are derived from a scale-invariant Lagrangian that is purely quadratic in the scalar curvature of space-time:

$$L=R^2 \ . \tag{1}$$

The resulting system of field equations provides a good approximation to the more complicated fourth-order system of equations generated by the more general quadratic Lagrangian $L=R+\alpha R^2$, α constant, in regions of large space-time four-curvature.[7] An analysis of this more complicated case will be presented elsewhere.[8] General features of Lagrangians of general functional form $L(R)$ have been considered in Refs. 7 and 9–11.

We shall examine the field equations derived from (1) for the Bianchi type-IX metric, near its initial singularity. We show that in complete contrast with the situation in general relativity, the nonchaotic Kasner behavior characteristic of the Bianchi type-I universe is asymptotically stable. Finally, we show that for large times the Bianchi type-I asymptote is also stable with exponents related to its behavior near the singularity at early times. We shall analyze directly the field equations coming from the Lagrangian $L=R^2$. The conformal equivalence between these theories and general relativity with an additional scalar field[9,10,12] could also be used as basis for investigating the singularity and the behavior of the solutions.[7,13,14]

The general vacuum field equations derived from the variation of the action formed from (1) with respect to the metric $g_{\alpha\beta}$ are

$$R(R_{\mu\nu}-\tfrac{1}{4}Rg_{\mu\nu})+\Box R g_{\mu\nu}-R_{;\mu;\nu}=0 \ . \tag{2}$$

Taking the trace we have the constraint

$$\Box R = 0 \ , \tag{3}$$

which simplifies (2) to

$$RR_{\mu\nu} - \tfrac{1}{4} R^2 g_{\mu\nu} - R_{;\mu;\nu} = 0 \ . \tag{4}$$

Any solution of the vacuum Einstein equations ($R_{\alpha\beta}=0$) is also a solution of (3) and (4) [see Barrow and Ottewill[7] for more general versions of this correspondence between the solutions of general relativity and an $L(R)$ Lagrangian gravity theory]. For a spatially homogeneous space-time with a metric[1] of Bianchi type-IX,

$$ds^2 = dt^2 - \sum_{i,j} \gamma_{ij}(t)\sigma^i(\mathbf{x})\sigma^j(\mathbf{x}) \ , \quad 1 \le i,j \le 3 \ , \tag{5}$$

where $\sigma^j(\mathbf{x})$ are the SO(3)-invariant differential forms which generate the homogeneous space of Bianchi type-IX, satisfying $d\sigma^i = \epsilon^i{}_{jk}\sigma^j \wedge \sigma^k$, where $\epsilon^i{}_{jk}$ is the completely antisymmetric rank-3 tensor and the time dependence is carried by

$$\gamma_{ij}(t) = \text{diag}\{a^2(t), b^2(t), c^2(t)\} \ . \tag{6}$$

Equation (3) becomes

$$\ddot{R} + 3\frac{\dot{S}}{S}\dot{R} = 0 \ , \tag{7}$$

where $S^3 \equiv abc$ and the overdot denotes a derivative with respect to t. We integrate Eq. (7) and obtain

$$\dot{R} = \frac{1}{S^3} \ . \tag{8}$$

The integration constant is set equal to unity by a change of scale of the x^i ($i=1,2,3$). We introduce a new time coordinate τ defined by

$$d\tau = \frac{dt}{S^3(t)} \ . \tag{9}$$

Hence Eq. (8) can be integrated again:

$$R = \tau \ . \tag{10}$$

Here the integration constant has been set equal to zero without loss of generality by choosing $R=0$ for $\tau=0$.

We introduce the variables α, β, γ, and δ defined as

$$a = e^\alpha, \quad B = e^\beta, \quad c = e^\gamma, \quad \delta = \alpha + \beta + \gamma \ . \tag{11}$$

Using (10) the four essential field equations for the Bianchi type-IX metric, in terms of α, β, γ, δ, and τ, are the cyclic set

$$\tau\alpha'' + \alpha' - \frac{\tau}{2}[(e^{2\beta} - e^{2\gamma})^2 - e^{4\alpha}] + \frac{\tau^2}{4}e^{2\delta} = 0 \ , \tag{12a}$$

$$\tau\beta'' + \beta' - \frac{\tau}{2}[(e^{2\gamma} - e^{2\alpha})^2 - e^{4\beta}] + \frac{\tau^2}{4}e^{2\delta} = 0 \ , \tag{12b}$$

$$\tau\gamma'' + \gamma' - \frac{\tau}{2}[(e^{2\alpha} - e^{2\beta})^2 - e^{4\gamma}] + \frac{\tau^2}{4}e^{2\delta} = 0 \ , \tag{12c}$$

$$\delta'' - 2(\alpha'\beta' + \beta'\gamma' + \gamma'\alpha') + \frac{\tau}{4}e^{2\delta} = 0 \ . \tag{12d}$$

The sum of the first three equations is

$$\tau\delta'' + \delta' - \frac{\tau}{2}[e^{4\alpha} + e^{4\beta} + e^{4\gamma} - 2(e^{2(\alpha+\beta)} + e^{2(\beta+\gamma)} + e^{2(\gamma+\alpha)})] + \frac{3\tau^2}{4}e^{2\delta} = 0 \ . \tag{13}$$

Notice that the field equations are now of second order, contrary to the general field equations which follow from a Lagrangian with quadratic terms.[7] This is due to (8) which is a solution of Eq. (3) in spatially homogeneous space-times. However, the equations contain terms [the second and last in (12a)–(12c)] in addition to those present in the general-relativistic type-IX universe. In order to study the solutions of these equations one can carry through an analysis similar to that made for Bianchi type-IX solutions of the Einstein equations (derived from the linear Lagrangian $L=R$) without the cosmological constant.[1,2] First, we examine the asymptotic behavior as $\tau \to 0$. If we were to neglect terms such as $\exp(4\alpha)$ and $\exp[2(\beta+\gamma)]$ in Eqs. (12a)–(12d), then the field equations reduce *exactly* to those of Bianchi type-I studied by Buchdahl[15] in his analysis of R^2 Bianchi type-I cosmological models:

$$\tau\alpha'' + \alpha' + \frac{\tau^2}{4}e^{2\delta} = 0 \ , \tag{14a}$$

$$\tau\beta'' + \beta' + \frac{\tau^2}{4}e^{2\delta} = 0 \ , \tag{14b}$$

$$\tau\gamma'' + \gamma' + \frac{\tau^2}{4}e^{2\delta} = 0 \ , \tag{14c}$$

$$\delta'' - 2(\alpha'\beta' + \beta'\gamma' + \gamma'\alpha') + \frac{\tau}{4}e^{2\delta} = 0 \ . \tag{14d}$$

These equations can be solved exactly for α, β, γ, and δ to give

$$\exp(-\delta) = (3/4n)^{1/2}\tau^{3/2}(C\tau^n + C^{-1}\tau^{-n}) \ , \tag{15}$$

$$\{a,b,c\} = \exp(\delta/3)\{\tau^{\nu_1}, \tau^{\nu_2}, \tau^{\nu_3}\} \ , \tag{16}$$

where C is a constant and

$$\sum_{i=1}^{3} \nu_i = 0, \quad \sum_{i=1}^{3} \nu_i{}^2 = (4n^2 - 9)/6, \quad n \ge \tfrac{3}{2} \ . \tag{17}$$

For $\tau \to 0$, the scale factors behave as[15]

$$a \sim \left[\frac{4n}{3}\right]^{1/6} C^{1/3}\tau^{p_1} \ , \tag{18a}$$

$$b \sim \left[\frac{4n}{3}\right]^{1/6} C^{1/3}\tau^{p_2} \ , \tag{18b}$$

$$c \sim \left[\frac{4n}{3}\right]^{1/6} C^{1/3}\tau^{p_3} \ , \tag{18c}$$

where

$$p_i = \frac{n}{3} - \frac{1}{2} + \nu_i \quad \text{for } i = 1,2,3 \ . \tag{19}$$

Hence, we have the constraints

$$\sum p_i = n - \tfrac{3}{2} \geq 0 \,, \tag{20}$$

$$\sum p_i{}^2 = (n + \tfrac{1}{2})(n - \tfrac{3}{2}) \geq 0 \,, \tag{21}$$

$$\sum p_i p_j = \tfrac{3}{2} - n \leq 0 \,, \tag{22}$$

where i,j are summed over 1,2,3 in all cases. The last formula shows that at least one of the three p_i must be negative if we exclude the degenerate case $n = \frac{3}{2}$.

Substituting for abc from (18) allows us to integrate (9) to obtain the $\tau(t)$ relation

$$t = \left[\frac{4n}{3} \right]^{1/2} \frac{c}{n - \frac{1}{2}} \tau^{n-1/2} \,. \tag{23}$$

Hence, we see that when τ decreases, t decreases as well. We have chosen a null integration constant in order that $\tau = 0$ corresponds to $t = 0$.

The supplementary terms such as $\exp(4\alpha)$ and $\exp[2(\beta+\gamma)]$ in (12) and (13) can be considered now as perturbations of the Bianchi type-I field equations. We investigate the stability of the Bianchi type-I solution given above as $\tau \to 0$ assuming that at some given time $\tau_0 > 0$ the solution is well approximated by (18)–(21).

If we suppose that the p_i's are ordered as

$$p_1 < 0 < p_2 \leq p_3 \tag{24}$$

then to leading order after neglecting the lower-order terms, Eqs. (12) become

$$\tau\alpha'' + \alpha' + \frac{\tau}{2} e^{4\alpha} = 0 \,, \tag{25a}$$

$$\tau\beta'' + \beta' - \frac{\tau}{2} e^{4\alpha} = 0 \,, \tag{25b}$$

$$\tau\gamma'' + \gamma' - \frac{\tau}{2} e^{4\alpha} = 0 \,. \tag{25c}$$

In order to solve (25a) we set

$$e^{-2\alpha} = X \tag{26}$$

and it becomes

$$\tau X'' X - \tau X'^2 + X' X - \tau = 0 \,. \tag{27}$$

This has the general solution

$$X = D\tau(E\tau^{1/2D} + E^{-1}\tau^{-1/2D}) \,, \tag{28}$$

where D and E are the integration constants and D is positive. For small values of τ we have

$$e^{\alpha} \sim \left[\frac{E}{D} \right]^{1/2} \tau^{1/4D - 1/2} \,. \tag{29}$$

Imposing the Bianchi type-I asymptotic initial conditions (18) at $\tau \equiv \tau_0$ gives

$$\frac{1}{4D} - \frac{1}{2} = p_1 \,, \tag{30}$$

$$\left[\frac{E}{D} \right]^{1/2} = \left[\frac{4n}{3} \right]^{1/6} C^{1/3} \,. \tag{31}$$

Therefore we find that

$$e^{\alpha} \sim \left[\frac{4n}{3} \right]^{1/6} C^{1/3} \tau^{p_1} \,, \tag{32}$$

which has the same behavior as in the initial state (18a). Notice that from (30) and $D > 0$ we obtain

$$p_1 > -\tfrac{1}{2} \,. \tag{33}$$

A solution exists only if p_1 satisfies condition (33). After substituting for $\exp(4\alpha)$ the general solutions of (25b) and (25c) are given by

$$e^{\beta} = H_2 \tau^{F_2} \exp\left[- \left[\frac{4n}{3} \right]^{2/3} \frac{C^{4/3}}{8(2p_1+1)^2} \tau^{2(2p_1+1)} \right] \,, \tag{34a}$$

$$e^{\gamma} = H_3 \tau^{F_3} \exp\left[- \left[\frac{4n}{3} \right]^{2/3} \frac{C^{4/3}}{8(2p_1+1)^2} \tau^{2(2p_1+1)} \right] \,. \tag{34b}$$

Considering the highest-order term (for small τ) and the initial conditions (18) at $\tau \equiv \tau_0$ we obtain

$$F_2 = p_2 \,, \tag{35}$$

$$H_2 = \left[\frac{4n}{3} \right]^{1/6} C^{1/3} \tag{36}$$

and

$$F_3 = p_3 \,, \tag{37}$$

$$H_3 = \left[\frac{4n}{3} \right]^{1/6} C^{1/3} \,. \tag{38}$$

Thus the perturbations do not change the initial behavior of $a(t)$, $b(t)$, and $c(t)$. The monotonic solutions (18a)–(18c), of Bianchi type-I form remain stable. There are no chaotic oscillations. For a discussion of the form of the perturbations which do and do not give rise to chaotic oscillations in a more general context see Ref. 4.

We now examine the evolution of the metric far from the initial singularity. We know that in the absence of the terms such as $\exp(4\alpha)$ and $\exp[2(\beta+\gamma)]$ the solution for $\exp(\delta)$ is given by (15). When τ is very large, we have

$$e^{\delta} \sim \left[\frac{4n}{3} \right]^{1/2} C^{-1} \tau^{-(n+3/2)} \tag{39}$$

and

$$a \sim \left[\frac{4n}{3} \right]^{1/6} C^{-1/3} \tau^{q_1} \,, \tag{40}$$

$$b \sim \left[\frac{4n}{3} \right]^{1/6} C^{-1/3} \tau^{q_2} \,, \tag{41}$$

$$c \sim \left[\frac{4n}{3} \right]^{1/6} C^{-1/3} \tau^{q_3} \tag{42}$$

with

$$q_i = -\frac{n}{3} - \frac{1}{2} - \nu_i \ , \tag{43}$$

$$\sum q_i = -n - \tfrac{3}{2} < 0 \ , \tag{44}$$

$$\sum q_i^2 = (n + \tfrac{3}{2})(n - \tfrac{1}{2}) > 0 \ , \tag{45}$$

$$\sum q_i q_j = 2(n + \tfrac{3}{2}) > 0 \ . \tag{46}$$

This implies that at least one of the q_i is negative. Moreover from (19) and (43) we obtain

$$q_i = p_i - \frac{2n}{3} \ . \tag{47}$$

Therefore $p_i < 0$ implies $q_i < 0$ and $p_1 < p_2 < p_3$ leads to $q_1 < q_2 < q_3$. At this stage we look at the stability of the solution (40)–(42) with respect to the perturbations for increasing times. The higher-order terms in the field equations are τ^{4q_3+1} and τ^{-1-2n}. In order to keep the highest, two cases should be considered:

$$\text{(i)}\ \ q_3 < -\frac{n+1}{2}, \quad \text{(ii)}\ \ q_3 > -\frac{n+1}{2} \ . \tag{48}$$

In case (i) the field equations are identical to Bianchi type-I and the solution is just the solution (40)–(42). In case (ii) the field equations become

$$\tau\alpha'' + \alpha' - \frac{\tau}{2} e^{4\gamma} = 0 \ , \tag{49a}$$

$$\tau\beta'' + \beta' - \frac{\tau}{2} e^{4\gamma} = 0 \ , \tag{49b}$$

$$\tau\gamma'' + \gamma' + \frac{\tau}{2} e^{4\gamma} = 0 \ . \tag{49c}$$

Equation (49c) is similar to (25a). Thus the general solution is of the form (28). For very large τ we have

$$e^{\gamma} \sim \left[\frac{1}{D'E'}\right]^{1/2} \tau^{-1/4D'-1/2} \ , \tag{50}$$

where D' and E' are two integration constants and D' is positive. Setting initial conditions at $\tau = \tau_1 > 0$ requires

$$e^{\gamma} \sim \left[\frac{4n}{3}\right]^{1/6} C^{-1/3} \tau_1^{q_3} \tag{51}$$

and we obtain

$$\frac{1}{4D'} + \frac{1}{2} = -q_3 \ , \tag{52a}$$

$$\frac{1}{D'E'} = \left[\frac{4n}{3}\right]^{1/3} C^{-2/3} \ , \tag{52b}$$

which show that the solution is identical to the initial one. Formula (52a) leads to $q_3 < -\frac{1}{2}$ as a necessary condition for the existence of a solution. The rest of the asymptotic solution to (49) is

$$e^{\alpha} = H_1' \tau^{F_1'} \exp\left[-\left[\frac{4n}{3}\right]^{2/3} \frac{C^{-4/3}}{8(2q_3+1)^2} \tau^{2(2q_3+1)}\right] , \tag{53a}$$

$$e^{\beta} = H_2' \tau_{F_2'} \exp\left[-\left[\frac{4n}{3}\right]^{2/3} \frac{C^{-4/3}}{8(2q_3+1)^2} \tau^{2(2q_3+1)}\right] , \tag{53b}$$

where H_1', F_1', H_2', and F_2' are integration constants. By keeping the highest-order terms (at large τ) and using the initial conditions at $\tau = \tau_1$ given by (40)–(42) we obtain the constraints

$$F_1' = q_1 \ , \tag{54a}$$

$$H_1' = \left[\frac{4n}{3}\right]^{1/6} C^{-1/3} \ , \tag{54b}$$

$$F_2' = q_2 \ , \tag{54c}$$

$$H_2' = \left[\frac{4n}{3}\right]^{1/6} C^{-1/3} \ . \tag{54d}$$

Therefore for large τ also, the perturbations do not change the behavior of a, b, and c and there is no chaotic oscillatory behavior of the sort found in general relativity.[16]

Following Eq. (4) we remarked that any vacuum solution of general relativity is also a solution of the R^2 field equations. In particular this means that the chaotic mixmaster vacuum solution of general relativity is a particular solution of the R^2 vacuum field equations also. What we have shown is that this solution is unstable to the presence of the additional higher-derivative terms that appear in the R^2 field equations. These terms create many more degrees of freedom for the solution. Whereas the most general solution of the Einstein equations in a synchronous coordinate system requires four arbitrary functions of three spatial variables to be prescribed on a Cauchy surface of constant time, the R^2 theory requires more initial data.[17] The field equations (2) and (3) require $R, \dot{R}$ plus $12 = 6 \times 2$ components of the metric and its first derivatives in a synchronous coordinate system (with $g_{\alpha 0} = \delta_{\alpha 0}$), but this total can be reduced to 6 by using the four coordinate covariances and the four Bianchi identities. The most general chaotic vacuum mixmaster model would be determined by four arbitrary constants because it is spatially homogeneous. The additional degrees of freedom present in the R^2 theory render the chaotic particular solution of general relativity unstable to the monotonic Kasner-type behavior. From the Hamiltonian potential viewpoint[5] of the mixmaster model as a particle motion inside a contracting, effectively closed, potential as $t \to 0$, the effect of the R^2 terms is to slow the motion of the moving point relative to that of the walls so that they are never reached by the point. No oscillations occur and the evolution remains characteristic of that with no potential present—that is of the Kasner-type solution.

H.S. would like to thank the members of the Astronomy Centre of the University of Sussex for their hospitality during her stay when this work was begun. Laboratoire de Physique Théorique is Unite Associe No. 769 du CNRS.

[1]C. W. Misner, Phys. Rev. Lett. **22**, 1071 (1969); V. A. Belinskii, I. M. Khalatnikov, and E. M. Lifshitz, Adv. Phys. **19**, 525 (1970).

[2]J. D. Barrow, Phys. Rev. Lett. **46**, 963 (1981); Phys. Rep. **85**, 1 (1982); Gen. Relativ. Gravit. **14**, 523 (1982); D. F. Chernoff and J. D. Barrow, Phys. Rev. Lett. **50**, 134 (1983); J. D. Barrow, in *Classical General Relativity,* edited by W. B. Bonnor, J. N. Islam, and M. A. H. MacCallum (Cambridge University Press, Cambridge, England, 1984), pp. 25–41.

[3]H. Sirousse-Zia, Gen. Relativ. Gravit. **14**, 751 (1982).

[4]J. D. Barrow and J. A. Stein-Schabes, Phys. Rev. D **32**, 1595 (1985); J. Demaret, M. Henneaux, and P. Spindel, Phys. Lett. **164B**, 27 (1985); Y. Elskens and M. Henneaux, Nucl. Phys. **B290**, 111 (1987); J. Demaret, J.-L. Hanquin, M. Henneaux, P. Spindel, and A. Taormina, Phys. Lett. B **175**, 129 (1986); J. Demaret, Y. De Rop, and M. Henneaux, *ibid.* **211**, 37 (1988).

[5]M. Ryan, *Hamiltonian Cosmology* (Springer, Heidelberg, 1972).

[6]J. D. Barrow, in *The Physics of Phase Space,* edited by Y. Kim and W. Zachery (Lecture Notes in Physics Vol. 278) (Springer, New York, 1987), pp. 18–22.

[7]J. D. Barrow and A. C. Ottewill, J. Phys. A **16**, 2757 (1983); A. A. Starobinsky and H.-J. Schmidt, Class. Quantum Gravit. **4**, 695 (1987).

[8]J. D. Barrow and S. Cotsakis (in preparation).

[9]J. D. Barrow and S. Cotsakis, Phys. Lett. B **214**, 515 (1988).

[10]K. Maeda, Tokyo report, 1988 (unpublished).

[11]J. D. Barrow and M. Madsen, Nucl. Phys. B (to be published).

[12]G. V. Bicknell, J. Phys. A **7**, 1061 (1974); B. Whitt, Phys. Lett. **145B**, 176 (1984); H.-J. Schmidt, Class. Quantum Gravit. **5**, 233 (1988).

[13]G. Le Denmat and H. Sirousse-Zia, Phys. Rev. D **35**, 480 (1987).

[14]K. Maeda, Phys. Rev. D **37**, 858 (1988); M. Mijic and J. A. Stein-Schabes, Phys. Lett. B **203**, 353 (1988).

[15]H. A. Buchdahl, J. Phys. A **11**, 871 (1978).

[16]It remains to be shown that chaotic behavior is absent when off-diagonal terms are included in the metric (5).

[17]See Starobinsky and Schmidt (Ref. 7).