Renormalized stress-energy tensor near the horizon of a slowly evolving, rotating black hole

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The renormalized expectation value of the stress-energy tensor $\langle T_{\mu\nu}\rangle^{\rm ren}$ of a quantum field in an arbitrary quantum state near the future horizon of a rotating (Kerr) black hole is derived in two very different ways: One derivation (restricted for simplicity to a massless scalar field) makes use of traditional techniques of quantum field theory in curved spacetime, augmented by a variant of the "η formalism" for handling superradiant modes. The other derivation (valid for any quantum field) uses the equivalence principle to infer, from $\langle T_{\mu\nu}\rangle^{\rm ren}$ in flat spacetime, what must be $\langle T_{\mu\nu}\rangle^{\rm ren}$ near the hole's horizon. The two derivations give the same result—a result in accord with a previous conjecture by Zurek and Thorne: $\langle T_{\mu\nu}\rangle^{\rm ren}$, in any quantum state, is equal to that, $\langle T_{\mu\nu}\rangle^{\rm ZAMO}$, which zero-angular-momentum observers (ZAMO's) would compute from their own physical measurements near the horizon, plus a vacuum-polarization contribution $T_{\mu\nu}^{\rm vac\ pol}$, which is the negative of the stress-energy of a rigidly rotating thermal reservoir with angular velocity equal to that of the horizon Ω_H, and (red-shifted) temperature equal to that of the Hawking temperature T_H. A discussion of the conditions of validity for equivalence-principle arguments reveals that curvature-coupling effects (of which the equivalence principle is unaware) should produce fractional corrections of order $\alpha^2 \equiv$(surface gravity of hole)$^2\times$(distance to horizon)^2 to $T_{\mu\nu}^{\rm vac\ pol}$; and since gravitational blue-shifts cause the largest components of $T_{\mu\nu}^{\rm vac\ pol}$ in the proper reference frame of the ZAMO's to be of $O(\alpha^{-2})$, curvature-coupling effects in $T_{\mu\nu}^{\rm vac\ pol}$ and thence in $\langle T_{\mu\nu}\rangle^{\rm ren}$ are of $O(\alpha^0)$ in the ZAMO frame. It is shown, by a quantum-field-theory derivation of the density matrix, that in the Hartle-Hawking vacuum the near-horizon ZAMO's see a thermal reservoir with angular velocity Ω_H and temperature T_H whose thermal stress-energy $\langle T_{\mu\nu}\rangle^{\rm ZAMO}$ gets renormalized away by $T_{\mu\nu}^{\rm vac\ pol}$, annulling the $O(\alpha^{-2})$ and $O(\alpha^{-1})$ pieces of $\langle T_{\mu\nu}\rangle^{\rm ren}$, and leaving only the $O(\alpha^0)$ vacuum-polarization, curvature-coupling contributions. This translates into $\langle T_{ll}\rangle^{\rm ren}=\langle T_{l\phi}\rangle^{\rm ren}=0$ on the future horizon in the Hartle-Hawking vacuum, where l and ϕ denote components on the horizon generator l^μ and on the generator of rotations $\partial/\partial\phi$. In quantum states representing a black hole in the real Universe (with both evaporation and accretion occurring), the fluxes of red-shifted energy and angular momentum across the future horizon, per unit solid angle $\sin\theta\, d\theta\, d\phi$, are shown to equal the corresponding accretion fluxes into the hole's atmosphere from the external universe minus the fluxes evaporated by the hole. As a consequence, the hole's horizon evolves in accord with standard expectations. As an aside it is shown that the Hartle-Hawking vacuum state $|H\rangle$ is singular at and outside the velocity-of-light surface $\mathcal{S}_L$, i.e., at sufficiently large radii that the rest frame of its thermal reservoir is moving at or faster than the speed of light. Its renormalized stress-energy tensor is divergent there, and its Hadamard function does not have the correct behavior. To make $|H\rangle$ be well behaved (and have the properties described above), one must prevent its rotating thermal reservoir from reaching out to $\mathcal{S}_L$, e.g., by placing a perfectly reflecting mirror around the hole just inside $\mathcal{S}_L$.

I. INTRODUCTION AND SUMMARY

Hawking[1] has shown that a rotating black hole formed by gravitational collapse in the distant past must spontaneously emit particles as though it were a rotating thermal reservoir with angular velocity equal to that, Ω_H, of the hole's horizon, and with temperature $T_H=\hbar\kappa/2\pi k_B$, where κ is the hole's "surface gravity," $\hbar$ is Planck's constant, and k_B is Boltzmann's constant. This startling result became more understandable when Unruh[2] and Israel[3] showed that the horizon of a rotating hole is, in fact, surrounded by a rotating thermal atmosphere—an atmosphere held in from immediate escape by the hole's spacetime curvature and by an angular-momentum barrier; the Hawking radiation can be interpreted as a slow leakage of this atmosphere into surrounding space. The atmosphere is completely real, from the viewpoint of static observers who live at constant radius just above the hole's horizon;[2] but freely falling observers near the horizon do not see or feel it.[2] This difference between static and freely falling viewpoints is completely analogous, Unruh[2] has shown, to the fact that in flat, empty spacetime where freely falling observers see and feel no quanta, observers with constant acceleration a see and feel themselves bathed by a thermal reservoir analogous to the hole's atmosphere, with temperature equal to $\hbar a/2\pi k_B$.

The importance of the hole's atmosphere for enforcing the second law of thermodynamics in black-hole processes was made clear by Unruh and Wald[4] using a thought experiment that entailed lowering a box deep into the atmosphere, there opening and emptying or filling it, and then raising it back up. Were it not for the buoyant pressure of the atmosphere and its statistical properties, such a thought experiment could violate the second law. The reason for this was elucidated by Zurek and Thorne,[5,6] who showed that the entropy of a black hole can be interpreted statistical mechanically as the logarithm of the number of ways that the hole's atmosphere, with a given macroscopic structure (renormalized mass and angular momentum as functions of height), could differ microscopically from a perfect thermal reservoir. This interpretation of the entropy permitted Zurek and Thorne[5,6] to show that the second law of thermodynamics for processes involving black holes is nothing but a special case of the ordinary second law of thermodynamics for processes involving a thermal reservoir.

Zurek and Thorne derived these statistical mechanical results not only for nonrotating black holes, but also for rotating, charged black holes that evolve slowly due to exchange of quanta with the external universe (evaporation and accretion). However, their derivation relied on a key property (to be described below) of the renormalized expectation value of the stress-energy tensor $\langle T_{\mu\nu}\rangle^{\rm ren}$ near the hole's horizon—a property that has been proved in the past only for nonrotating black holes.[7] One of the main purposes of this paper is to prove this key property of $\langle T_{\mu\nu}\rangle^{\rm ren}$ for rotating, uncharged holes. The generalization to charged holes should be straightforward but will not be attempted here. A second main purpose is to show, by example, that the equivalence principle if used carefully is just as valid in the domain of quantum field theory in curved spacetime as in the domain of classical physics.[8] A third main purpose is to develop formal techniques for analyzing negative-energy states (such as the Hartle-Hawking vacuum) of superradiant modes of boson fields near a rotating black hole.

The key aspect of $\langle T_{\mu\nu}\rangle^{\rm ren}$, which Zurek and Thorne needed (see above) and we shall prove, is the following. (This property was first conjectured, albeit in less physical language, in the case of nonrotating holes by Christensen and Fulling;[9] and it was proved for the nonrotating case by Candelas.[7]) Consider, just above the horizon of an evaporating, accreting, rotating black hole, the family of zero-angular-momentum observers (ZAMO's; the analog for a rotating hole of a nonrotating hole's static observers).[10] Particle detectors carried by these ZAMO's detect (according to a trivial generalization to Kerr spacetime of Unruh's[2] Schwarzschild argument) "Boulware particles" relative to the "Boulware vacuum" $|B\rangle$.[11] As is well known, the ZAMO's will infer from their detectors' measurements that the hole's atmosphere is nearly, but not precisely, in a thermal state. By those measurements they can determine, in principle, the expectation value of the mean number of (Boulware) particles in each single-particle quantum state of the atmosphere; and from those expectation values they can compute a corresponding expectation value for the stress-energy tensor of the atmosphere $\langle T_{\mu\nu}\rangle^{\rm ZAMO}$. They can also compute the value $T^{\rm th}_{\mu\nu}$ that the stress-energy tensor would have if the atmosphere were precisely thermal, with a temperature equal to the Hawking temperature (appropriately blue-shifted to their location and motion), and with an angular velocity equal to that of the horizon and thus also equal to their own angular velocity (i.e., if the atmosphere were in the "Hartle-Hawking" vacuum state $|H\rangle$). Then the actual, renormalized stress-energy tensor, i.e., the stress-energy which produces spacetime curvature by means of Einstein's equations and which therefore causes the evolution of the hole's horizon, is equal, aside from "curvature-coupling" corrections, to the difference of their measured stress-energy tensor $\langle T_{\mu\nu}\rangle^{\rm ZAMO}$ and the perfectly thermal stress-energy tensor $T^{\rm th}_{\mu\nu}$:

$$\langle T_{\mu\nu}\rangle^{\rm ren}=\langle T_{\mu\nu}\rangle^{\rm ZAMO}+T^{\rm vac\ pol}_{\mu\nu}\ , \tag{1.1a}$$

where

$$T^{\rm vac\ pol}_{\mu\nu}=-T^{\rm th}_{\mu\nu}+(T^{\rm vac\ pol}_{\mu\nu})_{\rm curvature\text{-}coupling\ corrections}\ . \tag{1.1b}$$

We shall regard $T^{\rm vac\ pol}_{\mu\nu}$ as the stress-energy tensor associated with vacuum polarization.

This discussion of the key result of this paper can be translated into more technical and less physical language as follows: Since ZAMO's measure Boulware particles relative to the Boulware vacuum, if the state of the atmosphere is $|\Psi\rangle$, then

$$\langle T_{\mu\nu}\rangle^{\rm ZAMO}=\langle\Psi|\hat{T}_{\mu\nu}|\Psi\rangle-\langle B|\hat{T}_{\mu\nu}|B\rangle\ , \tag{1.2}$$

where $\hat{T}_{\mu\nu}$ is the stress-energy-tensor operator; its expectation values without superscripts are the formal, unrenormalized expectation values; and the difference of two such expectation values in two different states is, of course, finite and well defined.[12] Since an atmosphere that is precisely thermal as seen by ZAMO's has $|\Psi\rangle=|H\rangle$ and $\langle T_{\mu\nu}\rangle^{\rm ZAMO}=T^{\rm th}_{\mu\nu}$ (cf. Appendix C and Sec. IV for proofs and discussions), the stress-energy difference in the Hartle-Hawking and Boulware states must be

$$\langle H|\hat{T}_{\mu\nu}|H\rangle-\langle B|\hat{T}_{\mu\nu}|B\rangle=T^{\rm th}_{\mu\nu}\ . \tag{1.3}$$

The renormalized stress-energy tensor in the Hartle-Hawking state, as inferred from Eqs. (1.1) and $\langle T_{\mu\nu}\rangle^{\rm ZAMO}=T^{\rm th}_{\mu\nu}$, is

$$\langle H \mid \hat{T}_{\mu\nu} \mid H \rangle^{\text{ren}} = (T_{\mu\nu}^{\text{vac pol}})_{\text{curvature-coupling corrections}} ; \tag{1.4}$$

and, correspondingly, the renormalized stress-energy tensor in any other state is

$$\langle \Psi \mid \hat{T}_{\mu\nu} \mid \Psi \rangle^{\text{ren}} = \langle \Psi \mid \hat{T}_{\mu\nu} \mid \Psi \rangle - \langle H \mid \hat{T}_{\mu\nu} \mid H \rangle + (T_{\mu\nu}^{\text{vac pol}})_{\text{curvature-coupling corrections}} . \tag{1.5}$$

In the proper reference frame of the ZAMO's the components of the curvature-coupling corrections remain finite as one approaches the horizon. This has two consequences: (i) the corrections give zero contribution to the fluxes $\langle T_{ll} \rangle^{\text{ren}}$ and $\langle T_{l\phi} \rangle^{\text{ren}}$ of energy and angular momentum across the horizon; and (ii) the corrections are sensitive to spacetime curvature and thus cannot be inferred from the equivalence principle (hence the name "curvature-coupling" corrections). For an evolving hole (one with a nonzero flux of energy and/or angular momentum across the horizon) the difference $\langle T_{\mu\nu} \rangle^{\text{ZAMO}} - T_{\mu\nu}^{\text{th}}$ is divergent, in the proper reference frame of the ZAMO's, as one approaches the horizon. This divergence, which arises from the divergence of the ZAMO acceleration, has two important consequences: (i) it causes the difference $\langle T_{\mu\nu} \rangle^{\text{ZAMO}} - T_{\mu\nu}^{\text{th}}$ to be a local quantity, insensitive to spacetime curvature and thus computable from the equivalence principle; and (ii) it enables that difference to produce the finite horizon fluxes $\langle T_{ll} \rangle^{\text{ren}}$ and $\langle T_{l\phi} \rangle^{\text{ren}}$ of energy and angular momentum which drive the horizon's evolution.

Thus, if all one is interested in are the (ZAMO-frame-) divergent parts of the renormalized stress-energy tensor near the horizon, i.e., the parts that drive the horizon's evolution, one need not compute the curvature-coupling corrections and one can derive the $\langle T_{\mu\nu} \rangle^{\text{ren}}$ of the hole's atmosphere directly from the equivalence principle. This will be the case in the present paper. On the other hand, at distances above the horizon of order 1/(hole's surface gravity κ) the curvature-coupling corrections are of the same magnitude as the thermal contributions to vacuum polarization, so expression (1.5) is useless without a knowledge of them. Very far from the hole, where spacetime is flat and ZAMO's are inertial observers, standard flat-space quantum field theory guarantees that $\langle T_{\mu\nu} \rangle^{\text{ren}} = \langle T_{\mu\nu} \rangle^{\text{ZAMO}}$ without any renormalization. And at any location, if the hole's atmosphere is perfectly thermal as measured by the ZAMO's (i.e., if the hole is in the "Hartle-Hawking vacuum state" $|H\rangle$), the curvature-coupling corrections are all that survive in $\langle T_{\mu\nu} \rangle^{\text{ren}}$ [Eq. (1.4)]. Frolov and Zel'nikov[13] have given a review of computations of the curvature-coupling corrections; see also the more recent calculations reported in Ref. 14.

In Sec. II of this paper we shall use the equivalence principle to derive, from well-known properties of $\langle T_{\mu\nu} \rangle^{\text{ren}}$ in flat spacetime, the form (1.1) of the renormalized stress-energy tensor near the horizon of a black hole that is rotating, evaporating, and accreting. Then from Eq. (1.1) we shall derive expressions for the finite values of the energy flux $\langle T_{ll} \rangle^{\text{ren}}$ and angular momentum flux $\langle T_{l\phi} \rangle^{\text{ren}}$ on the future horizon. These expressions (when multiplied by radius squared to convert them to flux per unit solid angle) will coincide, as functions of angular position θ, with the corresponding expressions derived by Hawking[1] for the energy and angular momentum flux into the atmosphere of a rotating, evaporating black hole—augmented by obvious contributions from accretion. From this we shall infer, via the semiclassical Einstein equations applied at the horizon, that the horizon of an evaporating black hole evolves in accord with standard expectations[1] (which are a direct consequence of $\langle T^{\mu\nu} \rangle^{\text{ren}}{}_{;\nu} = 0$).

In Sec. III we shall give an alternative derivation of Eq. (1.1) based on quantum field theory in curved spacetime. Regrettably, there will be one gap in our derivation: We shall have to assume, without proof, that in the Hartle-Hawking vacuum state the renormalized stress-energy tensor is regular on the hole's future horizon. For simplicity the derivation will be confined to massless scalar quanta.

It may increase the reader's confidence in the equivalence-principle derivation of $\langle T_{\mu\nu} \rangle^{\text{ren}}$ to know that it was carried out before the quantum-field-theory derivation and gave the correct result [Eq. (1.1) and, in more explicit form, Eqs. (2.50) below].

In our formal, quantum-field theory derivation of $\langle T_{\mu\nu} \rangle^{\text{ren}}$ special attention must be paid to superradiant modes of the scalar field. The proper manner of handling these modes has been, until now, somewhat murky and controversial:[15] Because distant observers and near-horizon ZAMO's measure opposite signs of frequency for a superradiant mode, one must make a (not totally obvious) choice of convention for the sign of the frequency when quantizing the mode. Moreover, because the total ZAMO-measured energy in the mode is negative in some states of interest and positive in others, one is forced, in one case or the other, to deal with a Fock space that has negative numbers of quanta and/or an indefinite metric. In Appendix A we sort out these complexities with the aid of a variant of the "η formalism" for handling spaces of indefinite metric.

In Sec. IV we use the results of our formal quantum-field-theory calculations to elucidate several important properties of the Hartle-Hawking state $|H\rangle$: We show unequivocally that the mean rest frame of $|H\rangle$'s thermal quanta rotates rigidly with the angular velocity of the horizon, and not (as has sometimes been suggested) differentially with the angular velocity of the Carter tetrad[16] or differentially with the angular velocity of the ZAMO's.[17] We then go on to elucidate the nature of a singular behavior of $|H\rangle$ that has been found by Kay and Wald:[18] For ease of calculation we specialize to a hole that rotates arbitrarily slowly, so the surface $\mathcal{S}_L$ outside which $|H\rangle$'s thermal reservoir moves faster than the velocity of light (the "velocity-of-light surface"), is far from the horizon where spacetime is flat. For such a hole we show that the renormalized stress-energy tensor (as computed from our formalism) diverges as one approaches $\mathscr{S}_L$ from its inside, and is infinite outside $\mathscr{S}_L$; and we show that the Hadamard elementary function derived from our formalism is also ill behaved everywhere outside $\mathscr{S}_L$. It seems reasonable to expect that, similarly, for a rapidly rotating hole the stress-energy tensor and Hadamard function of $|H\rangle$ will be singular at and outside the velocity-of-light surface $\mathscr{S}_L$.

These results cast a pall over our formal derivations of the properties of $|H\rangle$. However, we go on in Sec. IV to argue that, if one places a perfectly reflecting mirror around the hole, just inside $\mathscr{S}_L$, thereby preventing $|H\rangle$'s thermal reservoir from reaching out to $\mathscr{S}_L$, the state $|H\rangle$ will become well behaved. Moreover, we show that this "modified," nonsingular state, which we denote $|H_M\rangle$ in Sec. IV, has the same properties as we derived formally for $|H\rangle$: it appears perfectly thermal to ZAMO's and its renormalized stress-energy tensor vanishes aside from "curvature coupling corrections" (which, however, are influenced by the presence of the mirror).

These conclusions dictate that in the equivalence-principle analysis of Sec. II we should think of the Hartle-Hawking state of a rotating hole as actually the state $|H_M\rangle$, made nonsingular by surrounding the hole with a mirror.

Throughout this paper we shall use the notation and sign conventions of Misner, Thorne, and Wheeler[19] (MTW), except that in addition to setting to unity the speed of light c and Newton's gravitation constant G, we shall also set to unity Planck's constant $\hbar$ and Boltzmann's constant k_B. In large measure our formal quantum-field-theory calculations will follow the notation and conventions of Ref. 20, while our physical descriptions of the results and our equivalence principle arguments will be couched in the language of Ref. 6.

II. DERIVATION BASED ON THE EQUIVALENCE PRINCIPLE

In this section we shall use an equivalence-principle argument to derive expression (1.1) for the renormalized stress-energy tensor of an arbitrary field Ψ near the horizon of a rotating (Kerr) black hole. The first step (Sec. II A) will be to express the renormalized stress-energy tensor for the field Ψ in flat spacetime in terms of measurements made by a family of uniformly accelerating observers ("Rindler observers"). The next step (Sec. II B 1) will be to argue (by the equivalence principle) that, if the field near the hole's horizon is in "the same physical state" as the field in flat spacetime, then its renormalized stress-energy tensor must be expressible in terms of ZAMO measurements by the same formula as is used, with Rindler measurements, in flat spacetime. The meaning of "the same physical state" will be clarified in Sec. II B 2. The final step (Sec. II B 3) will be to discuss, in order of magnitude, the curvature-coupling corrections to the resulting renormalized stress-energy tensor $\langle T_{\mu\nu}\rangle^{\text{ren}}$. Having thereby derived the rather formal expression (1.1) for $\langle T_{\mu\nu}\rangle^{\text{ren}}$, we will deduce from it, in Sec. II C, the more explicit expressions (2.44) for the fluxes of renormalized energy and angular momentum across the hole's future horizon. In those expressions, by virtue of our equivalence-principle derivation, we will use for the field's superradiant *in* modes (superradiant modes that originate at past null infinity) the unusual viewpoint of negative frequency as measured at infinity. In Sec. II D we will transform the superradiant *in*-mode contributions into the more usual viewpoint of positive frequency as measured at infinity, thereby obtaining our final formula—Eqs. (2.50) and (2.51)—for the fluxes of renormalized energy and angular momentum across the future horizon. In Sec. II E we will show that these horizon fluxes are equal to the fluxes of energy and angular momentum into the hole's atmosphere from the external universe (including the negative contribution from evaporation), and that consequently the horizon evolves in accord with standard expectations.[1]

A. Renormalized stress-energy tensor in flat spacetime

Let T, Z, x, y be the Lorentz coordinates of an inertial reference frame in flat spacetime. In these coordinates the metric has the standard form

$$ds^2=-dT^2+dZ^2+dx^2+dy^2 \ . \qquad (2.1)$$

A family of uniformly accelerated observers (Rindler observers) moves through this flat spacetime along world lines

$$Z=z\cosh\kappa t \ , \quad T=z\sinh\kappa t \ . \qquad (2.2)$$

Here z is a new spatial coordinate which is constant along the Rindler world lines; t is a new time coordinate which varies along them; and κ is an arbitrary constant (the "surface gravity of the Rindler horizon") which can be altered by altering the normalization of t. In terms of these new coordinates the flat metric (2.1) takes the standard Rindler form[21]

$$ds^2=-\alpha^2dt^2+dz^2+dx^2+dy^2 \ . \qquad (2.3)$$

Here the "lapse function" (also called the "red-shift function") α is

$$\alpha=\kappa z \qquad (2.4)$$

and the Rindler horizon is located at $\alpha=z=0$.

The Rindler observers are at rest in the Rindler coordinate system, and the basis vectors of their proper reference frames are

$$\mathbf{e}_{\hat{0}}=\frac{1}{\alpha}\frac{\partial}{\partial t} \ , \quad \mathbf{e}_{\hat{j}}=\frac{\partial}{\partial x^j} \ , \qquad (2.5)$$

where the latin index j runs over the spatial coordinates x, y, z. Correspondingly, a Rindler observer measures a particle with four-momentum $\mathbf{p}$ to have energy ("locally measured energy")

$$\epsilon_{\text{loc}}=-\mathbf{p}\cdot\mathbf{e}_{\hat{0}} \ , \qquad (2.6)$$

which is related to the particle's "red-shifted energy" $\mathscr{E}$ (a quantity conserved along the world line of a freely falling particle) by

$$\tilde{\epsilon} \equiv -\mathbf{p}\cdot\frac{\partial}{\partial t} = \alpha\epsilon_{\rm loc} \ . \tag{2.7}$$

Rindler observers, when quantizing any field Ψ, find it natural to choose basis states that are eigenfunctions of red-shifted energy with eigenvalues $\tilde{\epsilon}$, and of x and y components of momentum with eigenvalues k_x and k_y. If the field's quanta have rest mass μ, then the z component of their momenta will be $k_z = \pm(\tilde{\epsilon}^2/\alpha^2 - \mu^2 - k_x{}^2 - k_y{}^2)^{1/2}$. In our equivalence-principle arguments, when dealing with a mode of given μ, $\tilde{\epsilon}$, k_x, k_y we shall restrict attention to heights so small that $\epsilon_{\rm loc} = \tilde{\epsilon}/\alpha \gg \{\mu, k_x, k_y\}$; and correspondingly we shall be able to regard the quanta, when measured by the Rindler observers, as essentially massless and as propagating very nearly vertically. More specifically, we shall restrict attention to a box with upper face at $z = z_2$ such that

$$\kappa z_2 \ll \{1, \ \tilde{\epsilon}/\mu, \ \tilde{\epsilon}/k_x, \ \tilde{\epsilon}/k_y\} \ , \tag{2.8a}$$

with lower face at $z = z_1$ arbitrarily close to the horizon so that

$$z_1/z_2 \ll 1 \ , \tag{2.8b}$$

and with arbitrarily large lateral dimensions; and inside that box the modes of fixed $\tilde{\epsilon}$, k_x, k_y which propagate upward ($\uparrow$ mode) and downward ($\downarrow$ mode) will have the form

$$\begin{aligned} \Psi_I^{\uparrow} &= \text{const} \times e^{ik_x x} e^{ik_y y} e^{i(\tilde{\epsilon}/\kappa)\ln(\kappa z)} e^{-i\tilde{\epsilon}t} \ , \\ \Psi_I^{\downarrow} &= \text{const} \times e^{ik_x x} e^{ik_y y} e^{-i(\tilde{\epsilon}/\kappa)\ln(\kappa z)} e^{-i\tilde{\epsilon}t} \ . \end{aligned} \tag{2.9}$$

Here $I \equiv \{\tilde{\epsilon}, k_x, k_y\}$, and the z dependence is such as to produce $k_z = \partial(\text{phase})/\partial z = \pm\tilde{\epsilon}/\alpha$.

For pedagogical reasons (to be encountered in subsequent sections), we shall restrict attention to wave-packet modes that have the form (2.9) everywhere inside the box at some arbitrary time t_0, but that are cut off ($\Psi \to 0$ at time t_0) just outside the box, at $z > z_2$ and at $z < z_1$. We shall use these wave-packet modes to quantize the field inside the box, for times t near t_0; and accordingly we shall impose on $\tilde{\epsilon}$ the usual periodic boundary condition [cf. Eq. (2.9)]

$$e^{\pm i(\tilde{\epsilon}/\kappa)\ln(\kappa z_1)} = e^{\pm i(\tilde{\epsilon}/\kappa)\ln(\kappa z_2)} \ , \tag{2.10}$$

i.e.,

$$\tilde{\epsilon} = 2\pi N \frac{\kappa}{\ln(z_2/z_1)} \ , \tag{2.11}$$

where N is an integer. Although our analysis will be restricted to the interior of the box and to times t near t_0, because t_0 is arbitrary and because the bottom face of the box is arbitrarily near the horizon, our analysis will permit us to discuss all aspects of the field in any region of spacetime near the horizon.

Suppose, now, that the field Ψ is in its Minkowski vacuum state $|M\rangle$. This has several consequences (derivable by standard flat-spacetime quantum-field-theory techniques[22,2,3]): (i) freely falling (inertial) observers measure no quanta at all; (ii) the renormalized stress-energy tensor vanishes

$$\langle M \mid T_{\mu\nu} \mid M \rangle^{\rm ren} = 0 \ ; \tag{2.12}$$

and (iii) the Rindler observers measure the field to be precisely thermally excited,[2] with locally measured temperature $T_{\rm loc}$ equal to the Rindler observers' acceleration $a = \kappa/\alpha$ divided by 2π—corresponding to a red-shifted temperature

$$T_H = \alpha T_{\rm loc} = \alpha\frac{a}{2\pi} = \frac{\kappa}{2\pi} \ , \tag{2.13}$$

which is the same as the Hawking temperature of a black hole with surface gravity κ. Thermal excitation means that the two ($\uparrow$ and $\downarrow$) modes with quantum numbers $I = \{\tilde{\epsilon}, k_x, k_y\}$, from the Rindler observers' viewpoint, are in mixed states with a probability of containing n quanta

$$p_n^{\rm th} = (1 \mp e^{-\tilde{\epsilon}/T_H})^{\pm 1} e^{-n\tilde{\epsilon}/T_H} \tag{2.14}$$

(where the upper and lower signs correspond to boson and fermion fields), and with density matrix for $\uparrow$ and $\downarrow$ modes,[2,3]

$$\begin{aligned} \hat{\rho}^{\uparrow}_{(M)I} &= \sum_{n_I^{\uparrow}} |n_I^{\uparrow}\rangle p^{\rm th}_{n_I^{\uparrow}} \langle n_I^{\uparrow}| \ , \\ \hat{\rho}^{\downarrow}_{(M)I} &= \sum_{n_I^{\downarrow}} |n_I^{\downarrow}\rangle p^{\rm th}_{n_I^{\downarrow}} \langle n_I^{\downarrow}| \ . \end{aligned} \tag{2.15}$$

(Here $|n_I^{\uparrow}\rangle$ is the state of mode $I\uparrow$ containing $n_I^{\uparrow}$ quanta, and similarly for $|n_I^{\downarrow}\rangle$.) The mean number of quanta in the modes is, correspondingly,

$$\begin{aligned} \langle M \mid \hat{n}_I^{\uparrow} \mid M\rangle &= \text{tr}(\hat{\rho}^{\uparrow}_{(M)I}\hat{n}_I^{\uparrow}) = \langle M \mid \hat{n}_I^{\downarrow} \mid M\rangle \\ &= \text{tr}(\hat{\rho}^{\downarrow}_{(M)I}\hat{n}_I^{\downarrow}) = n_I^{\rm th} \equiv \frac{1}{e^{\tilde{\epsilon}/T_H} \mp 1} \ . \end{aligned} \tag{2.16}$$

From the wave functions (2.9) for modes I, the Rindler observers can compute the stress-energy tensor $T^{I\uparrow}_{\mu\nu}$ and $T^{I\downarrow}_{\mu\nu}$ which those modes would produce if they each contained precisely one quantum and if there were no corrections due to vacuum polarization:

$$T^{I(\uparrow\text{or}\downarrow)}_{\hat{\mu}\hat{\nu}} = \frac{1}{A_B\kappa^{-1}\ln(z_2/z_1)}\frac{k_{\hat{\mu}}k_{\hat{\nu}}}{\tilde{\epsilon}} \ . \tag{2.17a}$$

Here A_B is the area of the box's base and top, $\kappa^{-1}\ln(z_2/z_1)$ is the time required for a quantum to pass through the box, from top to bottom or bottom to top, and $k_{\hat{\mu}}$ are the components of the four-momentum of one quantum in the Rindler observers' rest frame:

$$\begin{aligned} &k_{\hat{0}} = -\epsilon_{\rm loc} = -\tilde{\epsilon}/\alpha \ , \quad k_{\hat{1}} = k_x \ , \quad k_{\hat{2}} = k_y \ , \\ &k_{\hat{3}} = k_z \simeq \pm k_{\hat{0}} \ . \end{aligned} \tag{2.17b}$$

Correspondingly, if the modes I are in the Minkowski vacuum state so their mean number of quanta are $n_I^{\rm th}$, then the Rindler observers will infer for them, before renormalization, an expectation value for the stress-energy tensor

$$\langle M \mid \hat{T}^{I}_{\mu\nu} \mid M \rangle = T^{I\uparrow}_{\mu\nu} n^{\rm th}_I + T^{I\downarrow}_{\mu\nu} n^{\rm th}_I \ . \quad (2.18)$$

In order that the renormalized stress-energy tensor vanish in the Minkowski vacuum, it must be that vacuum polarization, from the Rindler observers' viewpoint, contributes to the stress-energy tensor an amount equal to the negative of (2.18), i.e., the negative of that in a perfect thermal bath:

$$T^{I\,\rm vac\,pol}_{\mu\nu} = -(T^{I\uparrow}_{\mu\nu} n^{\rm th}_I + T^{I\downarrow}_{\mu\nu} n^{\rm th}_I) \ . \quad (2.19)$$

Next let the fields in flat spacetime be in an arbitrary state. Denote by $\langle n_I^{\uparrow} \rangle$ and $\langle n_I^{\downarrow} \rangle$ the expectation value, in this state, for the number of quanta in the modes I as measured by the Rindler observers, and correspondingly denote by

$$\langle T_{\mu\nu} \rangle^{\rm RO}_I = T^{I\uparrow}_{\mu\nu} \langle n_I^{\uparrow} \rangle + T^{I\downarrow}_{\mu\nu} \langle n_I^{\downarrow} \rangle \quad (2.20)$$

the Rindler-observer-measured stress-energy tensor for the modes I. Since the contribution of vacuum polarization to the stress-energy tensor is independent of the state of the fields, $T^{I\,\rm vac\,pol}_{\mu\nu}$ must still be the negative of that of a perfect thermal bath [Eq. (2.19)]; and correspondingly, the renormalized stress-energy tensor associated with the modes I must have the form

$$\begin{aligned}\langle T_{\mu\nu} \rangle^{\rm ren}_I &= \langle T_{\mu\nu} \rangle^{\rm RO}_I + T^{I\,\rm vac\,pol}_{\mu\nu} \\ &= T^{I\uparrow}_{\mu\nu}(\langle n_I^{\uparrow} \rangle - n^{\rm th}_I) + T^{I\downarrow}_{\mu\nu}(\langle n_I^{\downarrow} \rangle - n^{\rm th}_I) \ . \end{aligned} \quad (2.21)$$

That the renormalized stress-energy tensor actually does have this form has been proved directly,[23] via formal quantum-field-theory calculations, for massless fields of spins 0, $\frac{1}{2}$, 1, and $\frac{3}{2}$ in the "Fulling[22] vacuum state" $|F\rangle$ (the state where Rindler observers see no quanta at all). [We note in passing that for the Fulling vacuum and other states that are singular on the horizon, modes with $k_x \sim \tilde{\epsilon}/\alpha$ and/or $k_y \sim \tilde{\epsilon}/\alpha$ contribute significantly to $\langle T_{\mu\nu} \rangle^{\rm ren} = \sum_I \langle T_{\mu\nu} \rangle^{\rm ren}_I$ at every height z, even arbitrarily small z; and correspondingly one must use in evaluating $\langle T_{\mu\nu} \rangle^{\rm ren}$ more accurate (spin-dependent) expressions than (2.17) for the stress-energy carried by individual quanta.[22,23] By contrast, for states that are regular on the future or past horizon (the kinds of states that will be of concern in this paper), $\langle T_{\mu\nu} \rangle^{\rm ren}$ will be kept finite (in well-behaved, e.g., inertial, reference frames) by the fact that modes of sufficiently high k_x or k_y will be seen by Rindler observers as perfectly thermalized, and thus will not contribute to $\langle T_{\mu\nu} \rangle^{\rm ren}$. This means that, for regular states, when one is sufficiently close to the horizon all the modes that contribute to $\langle T_{\mu\nu} \rangle^{\rm ren}$ are well approximated by expressions (2.17).]

B. Renormalized stress-energy tensor just outside the horizon of a rotating black hole

1. Equivalence-principle analysis

Turn, now, to the spacetime around a slowly evolving, rotating, uncharged (Kerr) black hole with mass M and angular momentum $J = Ma$. The Kerr metric for such a hole takes the form

$$ds^2 = -\alpha^2 dt^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \varpi^2 (d\phi - \omega\, dt)^2 \ , \quad (2.22)$$

where α, ρ, Δ, ϖ, and ω are functions of r and θ which are given in standard references, e.g., Eqs. (33.2) and (33.3) of MTW[19] and Eqs. (3.5) and (3.6) of Ref. 6 (cited henceforth as BHMP, which stands for "Black Holes: The Membrane Paradigm"). Far from the black hole,

$$\alpha \to 1 \ , \quad \rho^2/\Delta \to 1 \ , \quad \rho \to r \ , \quad \varpi \to r \sin\theta \ , \quad \omega \to 0 \ , \quad (2.23)$$

so the metric becomes that of flat spacetime in spherical, inertial coordinates. The horizon of the black hole is located at $\alpha \equiv$(lapse function)$=0$. As the hole slowly evolves due to evaporation and accretion, the mass M and angular momentum $J = Ma$, which appear in the metric functions, slowly change. One objective of this paper is to obtain from $\langle T_{\mu\nu} \rangle^{\rm ren}$ expressions for those slow changes [Eqs. (2.55) and (2.50) below].

Our analysis will rely heavily on the Rindler approximation to the Kerr metric. We shall sketch only briefly how the Rindler approximation comes about; for greater detail see Sec. VI C 1 of BHMP.[6] The Rindler approximation is valid only very close to the horizon, i.e., at values of the lapse function $\alpha \ll 1$. There we can replace the radial coordinate r by proper radial distance z above the horizon. When we do so, the lapse function assumes the standard Rindler form (2.4):

$$\alpha = \kappa z + O(\kappa^3 z^3) \ , \quad (2.24)$$

where κ is the hole's surface gravity (denoted g_H in BHMP), and the value of κ is fixed uniquely by the demand that t become proper time far from the hole [Eqs. (2.22) and (2.23)]. Similarly, the relevant metric coefficients take the form

$$\begin{aligned}&\omega = \Omega_H + O(\alpha^2) \ , \quad \rho = \rho_H(\theta) + O(\alpha^2) \ , \\ &\varpi = \varpi_H(\theta) + O(\alpha^2) \ .\end{aligned} \quad (2.25)$$

The quantity Ω_H is the angular velocity of the horizon and is independent of θ. In coordinates that rotate with the horizon,

$$\bar{\phi} = \phi - \Omega_H t \ , \quad (2.26)$$

the Kerr metric (2.22) then takes on the simple form

$$ds^2 = -\alpha^2 dt^2 + dz^2 + {\rho_H}^2 d\theta^2 + {\varpi_H}^2 d\bar{\phi}^2 \ , \quad (2.27)$$

aside from fractional corrections of order α^2, which we shall ignore.

In the neighborhood of some angular location (θ_0, ϕ_0) we can introduce local Cartesian coordinates

$$x = (\theta - \theta_0)\rho_H \ , \quad y = (\bar{\phi} - \bar{\phi}_0)\varpi_H \quad (2.28)$$

and can rewrite the metric (2.27) in the Rindler form

$$ds^2 = -\alpha^2 dt^2 + dz^2 + dx^2 + dy^2 \ . \quad (2.29)$$

The zero-angular-momentum observers (ZAMO's;[10] called FIDO's in BHMP) reside at fixed r and θ, but they move with angular velocity $d\phi/dt=\omega$, where ω is the metric function appearing in Eq. (2.22). By virtue of Eq. (2.25), very near the horizon the ZAMO's orbit rigidly with the horizon, aside from fractional errors of order α^2 which we ignore. In other words, the near-horizon ZAMO's reside at constant r, θ, $\bar{\phi}$ and correspondingly, in the Rindler approximation, at constant z, x, y. Thus, the ZAMO's become uniformly accelerated, Rindler observers in the Rindler approximation.

The equivalence principle states that, aside from certain delicacies to be discussed below, the laws of physics in any locally uniform gravitational field in curved spacetime should take on the same form as they do in a uniformly accelerated reference frame in flat spacetime. Specialized to the present situation, this implies that the laws of physics as studied by ZAMO's just above the horizon of a rotating black hole should be the same as the laws of physics as studied by Rindler observers in flat spacetime. To the extent that the delicacies discussed below do not interfere, this means in particular that the renormalized stress-energy tensor of the modes I just above the horizon of a black hole should be expressible in the same form (2.21) as in flat spacetime:

$$\langle T_{\mu\nu}\rangle_I^{\rm ren}=\langle T_{\mu\nu}\rangle_I^{\rm ZAMO}+T_{\mu\nu}^{I\ \rm vac\ pol} \\ =T_{\mu\nu}^{I\uparrow}(\langle n_I^{\uparrow}\rangle-n_I^{\rm th})+T_{\mu\nu}^{I\downarrow}(\langle n_I^{\downarrow}\rangle-n_I^{\rm th}) \ . \quad (2.30)$$

Here, as in flat spacetime, the mode I is to be a wave-packet mode, confined at time $t=t_0$ to the interior of a box of the form (2.8) with periodic boundary conditions (2.10) and wave function (2.9), and with $T_{\mu\nu}^{I\uparrow}$ and $T_{\mu\nu}^{I\downarrow}$ given by Eqs. (2.17). Moreover, as in flat spacetime, the mode is an eigenstate of red-shifted energy:

$$\mathcal{E}\equiv-\mathbf{p}\cdot(\partial/\partial t)_{r,\theta,\bar{\phi}}=\alpha\times(\text{locally measured energy } \epsilon_{\rm loc}) \ ; \quad (2.31)$$

$\langle n_I^{\uparrow}\rangle$ and $\langle n_I^{\downarrow}\rangle$ are the expectation values for the number of quanta in modes $I\uparrow$ and $I\downarrow$ as measured by the ZAMO's, and $n_I^{\rm th}$ is the mean number of quanta that modes $I\uparrow$ and $I\downarrow$ would have if they were perfectly thermalized [Eq. (2.16)]. Correspondingly, as in flat spacetime so also near the horizon of a rotating hole, in a state where the ZAMO's measure all fields to be perfectly thermalized (a state that turns out[2,3,24] to be the Hartle-Hawking vacuum[25] $|H\rangle$; see Appendix C), $\langle T_{\mu\nu}\rangle_I^{\rm ren}$ will vanish—at least to the order of accuracy of our equivalence-principle analysis.

There are two delicacies that could invalidate or cause errors in our equivalence-principle analysis: "same-state" delicacies and "curvature-coupling" delicacies.

2. "Same-state" delicacies

By "same-state" delicacies we mean the following: It is essential, when applying the equivalence principle,[8] to be sure that the physical systems being studied—one in curved spacetime and the other in flat spacetime—are in "the same physical state." This is as true in classical physics as in quantum. In classical physics, for example, if we study the motions of two freely falling particles, one in a locally uniform gravitational field in curved spacetime and the other in an accelerated reference frame in flat spacetime, we will see the same motions only if both particles are free of electric charge—or, in the charged case, only if the electric and magnetic fields felt by the particles are the same.

A well-known example of violation of the "same-state" restriction in quantum physics is the fact that an observer on the surface of an isolated, zero-temperature neutron star will not detect any quanta, whereas a uniformly accelerated observer in flat, empty spacetime (Minkowski vacuum) will detect a precise thermal bath of quanta. The reason the equivalence principle fails is that the states of the fields are different in the two cases: the "Minkowski vacuum" $|M\rangle$ in flat spacetime is not the same as (does not correspond, for equivalence principle purposes, to) the "Boulware vacuum"[11] $|B\rangle$ around an isolated, cold neutron star.

Perhaps the best way to identify the "same states" in quantum-mechanical equivalence-principle arguments is by first identifying corresponding modes I of the relevant fields (as we have done above by introducing identical boxes and quantizing inside them in the same manner), and by then demanding that the accelerated observers in curved spacetime and in flat spacetime see the same numbers of quanta in corresponding modes. This method of handling the "same-state" delicacies is embedded in our equivalence-principle derivation of $\langle T_{\mu\nu}\rangle_I^{\rm ren}$ (above), since our answer is expressed in terms of the mean number of quanta $\langle n_I^{\uparrow}\rangle$, $\langle n_I^{\downarrow}\rangle$ that the accelerated observers measure in the corresponding modes $I\uparrow$, $I\downarrow$.

As one can infer from the work of Unruh,[2] Israel,[3] and Gibbons and Perry[24] and as we shall show explicitly in Appendix C, near the horizon of a rotating black hole it is the Hartle-Hawking vacuum state[25] $|H\rangle$ that is "the same as" the Minkowski vacuum state $|M\rangle$ of flat spacetime. In each of these states the near-horizon accelerated observers (ZAMO's and Rindler observers) measure all modes to be perfectly thermalized with the same, Hawking temperature; and correspondingly $|H\rangle$ and $|M\rangle$ are characterized by the same, thermal, density operators (2.15). [Wald has pointed out to us an alternative way to see that $|H\rangle$ is "the same state" as $|M\rangle$, aside from curvature coupling effects. This way relies on the Kay-Wald theorem[18] that in any globally hyperbolic spacetime which has a Killing field with a bifurcate Killing horizon, there can be at most one quasifree (i.e., "generalized vacuum") state which is invariant under the isometry generated by that Killing field and is regular everywhere, including the entire past and future horizons. Since all such spacetimes are geometrically identical near their horizons, except for curvature effects, their unique, regular quasifree states must be identical near their horizons, except for curvature-coupling effects. Kerr (with the interiors of the inner Cauchy horizons removed, and with a perfectly reflecting mirror inserted around the outer horizons so as to make $|H\rangle$ regular, cf. the end of Sec. IV) presumably is such a spacetime, as is Minkowski; and their unique, regular quasifree states are $|H\rangle$ and $|M\rangle$. In this sense, $|H\rangle$ and $|M\rangle$ are the "same states" up to curvature-coupling effects.]

3. "Curvature-coupling" delicacies

"Curvature coupling" is a more difficult and subtle issue than "same state." For an extensive classical discussion see, e.g., Chapter 16 of MTW.[19] Curvature coupling leads to fractional errors of order (size of system being studied)^2/(radius of curvature of spacetime)^2. In our equivalence-principle derivation of $\langle T_{\mu\nu}\rangle_I^{\rm ren}$ the size of the system is the vertical size of the "box" in which we resolve the fields into normal modes, which is of order the height z above the horizon at which $\langle T_{\mu\nu}\rangle^{\rm ren}$ is being evaluated. For comparison, the radius of curvature of spacetime is of order the horizon radius r_H, which in turn (if we ignore for pedagogical simplicity the physically unrealistic limit of an "extreme Kerr" hole where $\kappa\to 0$) is of order κ^{-1}. Thus, curvature coupling effects should produce fractional corrections of order $(z/\kappa^{-1})^2=\alpha^2$ to the stress-energy tensor associated with vacuum polarization $T_{\mu\nu}^{I\ \rm vac\ pol}$. Equations (2.19) and (2.17) show that, in the proper reference frame of the ZAMO's, the dominant part of $T_{\hat\mu\hat\nu}^{I\ \rm vac\ pol}$ (the vertical energy flux) diverges as α^{-2} (one factor of α^{-1} for the gravitational blue-shift of energy; the other α^{-1} for the blue-shift of the "per unit proper time" in the flux). Correspondingly, the curvature-coupling corrections to $T_{\hat\mu\hat\nu}^{I\ \rm vac\ pol}$ are of order $\alpha^2\times\alpha^{-2}=\alpha^0$ in the ZAMO's reference frame

$$(T_{\hat\mu\hat\nu}^{I\ \rm vac\ pol})_{\rm curvature\text{-}coupling\ corrections}=O(\alpha^0)\sim 1\ . \quad (2.32)$$

[For a proof of this from the full formalism of quantum field theory see Eq. (3.49) and associated discussion.]

Near the horizon these curvature-coupling corrections are unimportant, except in special states where the ZAMO-measured stress-energy $\langle T_{\hat\mu\hat\nu}\rangle_I^{\rm ZAMO}$ is perfectly thermal and thus is precisely canceled by the $O(\alpha^{-2})$ and $O(\alpha^{-1})$ parts of the vacuum polarization. [Most notable among these special states is the Hartle-Hawking vacuum state $|H\rangle$, for which all modes are perfectly thermal and hence $\langle T_{\mu\nu}\rangle=(T_{\mu\nu}^{I\ \rm vac\ pol})_{\rm curvature\text{-}coupling\ corrections}$.] Farther from the horizon, where $\alpha\sim 1$, the curvature-coupling effects are always (in any state) of the same order as the equivalence-principle effects; so the equivalence principle becomes useless. For summaries of extensive calculations of the renormalized stress-energy tensor in the Hartle-Hawking vacuum, $\langle T_{\mu\nu}\rangle^{\rm ren}=(T_{\mu\nu}^{\rm vac\ pol})_{\rm curvature\text{-}coupling\ corrections}$, at a variety of distances from a black hole see Refs. 13 and 14.

In the above discussion we have glossed over an embarrassing aspect of the curvature-coupling issue: In many realistic situations it is only modes with slow angular variations, i.e., with just a few nodes going all the way around the hole ($|k_x|\lesssim r_H^{-1}$, $|k_y|\lesssim r_H^{-1}$ where $r_H\sim\kappa^{-1}$ is the horizon radius) that deviate significantly from perfect thermality and that thus contribute significantly to $\langle T_{\mu\nu}\rangle^{\rm ren}$ near the horizon. The reason is that quanta with many transverse nodes and with $\tilde\epsilon\sim T_H\sim\kappa$ have great difficulty penetrating the "angular momentum barrier" that surrounds the black hole.[26] The lateral wavelengths of these modes ($2\pi/k_x\sim 2\pi/k_y\sim r_H$) remain unchanged as they near the horizon, by contrast with their radial (vertical) wavelengths which become shorter and shorter, $\propto\alpha$. In order to treat such modes, the box in which we do the quantization that underlies our equivalence-principle argument must have a transverse size comparable to the size of the hole.

At first sight one might expect this to make the curvature-coupling corrections to $T_{\mu\nu}^{I\ \rm vac\ pol}$ be of order the equivalence-principle contributions (2.30) and thereby invalidate our equivalence-principle argument. That our argument almost certainly remains valid can be seen from the fact that the dominant, equivalence-principle-derived contributions to $T_{\hat\mu\hat\nu}^{I\ \rm vac\ pol}$ diverge as $O(\alpha^{-2})$ and $O(\alpha^{-1})$ precisely because of the compression (blue-shift) of vertical wavelengths near the horizon. Lateral wavelengths are irrelevant to that divergence, so there is no reason to expect lateral effects to promote $(T_{\hat\mu\hat\nu}^{I\ \rm vac\ pol})_{\rm curvature\text{-}coupling\ corrections}$ from $O(\alpha^0)$ to $O(\alpha^{-1})$ or $O(\alpha^{-2})$.

In the next section we shall bring the $\langle T_{\mu\nu}\rangle_I^{\rm ren}$ of Eq. (2.30) into a more explicit and useful form. As an aid in doing so—and with the justification of the above discussion—we shall expand our quantization box laterally so it completely encircles the hole; i.e., so it is a spheroidal shell at $z_1\le z\le z_2$.

C. Explicit form of $\langle T_{\mu\nu}\rangle^{\rm ren}$ for massless fields in near-horizon viewpoint

Every massless field (scalar, neutrino, electromagnetic, gravitational) near a Kerr black hole can be described by a Teukolsky function Ψ.[27] When quantizing the field globally (not just in our quantization box) it is conventional to break Ψ into normal modes characterized by $I=\{l, m, h, \epsilon\}$, where l and m are spheroidal harmonic indices, $h=\pm s$ is a helicity index (with s the field's spin), and ϵ is the angular frequency measured by observers at rest far from the hole. Since observers far from the hole are at rest with respect to the ϕ angular coordinate while near-horizon ZAMO's are at rest with respect to $\tilde\phi=\phi-\Omega_H t$ [Eq. (2.26)], the relation

$$\Psi\propto e^{-i\epsilon t}e^{im\phi}=e^{-i\tilde\epsilon t}e^{im\tilde\phi} \quad (2.33)$$

tells us that the angular frequency ϵ measured far from the hole is related to that $\tilde\epsilon$ measured near the horizon by

$$\epsilon=\tilde\epsilon+m\Omega_H\ . \quad (2.34)$$

One has a choice of sign convention (henceforth called *viewpoint*; cf. Appendix A) for the modes I: One can insist that $\epsilon>0$ (quantization from viewpoint of distant observers), in which case for modes with $0<\epsilon<m\Omega_H$ ("superradiant modes") $\tilde\epsilon$ is negative. Alternatively, one can insist that $\tilde\epsilon>0$ (quantization from viewpoint of near-horizon observers), in which case for modes with $0<\tilde\epsilon<-m\Omega_H$ (the superradiant modes in the near-horizon viewpoint) ϵ is negative. Classically the two viewpoints are related, for superradiant modes, by $\Psi^I\to\bar\Psi^I$ (where the overbar denotes complex conjugation); and for nonsuperradiant modes, by $\Psi^I\to\Psi^I$ (no change). The corresponding quantum-mechanical relationship is discussed in Sec. II D below, and in Appendix A. Our equivalence-principle analysis (quantization in near-horizon box; last section and this one) is based on the near-horizon viewpoint, $\tilde\epsilon>0$ for all modes. In Sec.

II D below we shall translate our results into the distant-observer viewpoint, $\epsilon > 0$. (Note: for terminological simplicity we use the phrase "superradiant modes" whether Ψ is bosonic or fermionic—even though in the fermionic case the "superradiant modes" do not exhibit superradiance.)

For each $I = \{l, m, h, \tilde{\epsilon}\}$ there are two very special, orthogonal, global modes of Ψ^I;[28] see Fig. 1, Sec. III A, and Appendix B: the *up* mode (denoted q_I in Sec. III and Appendix B), which propagates up from the horizon and, at height $z \sim$(size of hole), is partially transmitted to future null infinity and partially reflected back down to the horizon; and the *in* mode (denoted v_I in Sec. III and Appendix B), which propagates in from past null infinity and is partially transmitted to the horizon and partially reflected back to future null infinity. If, as we shall in this section, one uses the same viewpoint (here $\tilde{\epsilon} > 0$) for all modes, then the probability $|A_I|^2$ of reflection for the *up* mode with quantum numbers I is the same as the probability of reflection for the *in* mode with the same quantum numbers (a consequence of the conservation of the Wronskian of the modes, as in elementary quantum-mechanical scattering theory).[27]

When we quantize the field Ψ inside our near-horizon box, each mode will be characterized by the same quantum numbers $I = \{l, m, h, \tilde{\epsilon}\}$ as for global modes. (The quantum numbers l, m replace the k_x, k_y that we used when the quantization box was flat rather than a spheroidal shell; and h was suppressed in our previous discussion.) For each I there are two wave-packet modes $\uparrow$ and $\downarrow$, which are confined to the box's interior at time t_0, entering and leaving it at earlier and later times. These modes are constructable as linear superpositions of *up* and *in* modes with a range of values of $\tilde{\epsilon}$ that is very sharply peaked about the value for the $\uparrow$ or $\downarrow$ mode ($\Delta\tilde{\epsilon} \sim$(time Δt that packet spends in box)$^{-1}$ $\sim \kappa[\ln(z_2/z_1)]^{-1} \ll \kappa$). Figure 1 shows the propagation of the $\downarrow$ and $\uparrow$ modes and the *in* and *up* modes in a Penrose conformal diagram. From that diagram it is clear that the $\uparrow$ modes must be composed solely of *up* modes, while the $\downarrow$ modes must be a linear combination of *up* modes and *in* modes. (Figure 1 makes it clear that the set of all $\uparrow$ and $\downarrow$ modes, including those that are in the box at time t_0 and also those in the box at times $t_0 - \Delta t$, $t_0 - 2\Delta t, \ldots$; $t_0 + \Delta t$, $t_0 + 2\Delta t, \ldots$ cannot form a complete, globally orthogonal set of modes in terms of which to do global quantization. For example, the $\{I\uparrow, t_0\}$ mode will have nonzero scalar product with the $\{I\downarrow, t_0 + n\Delta t\}$ mode, where $n\Delta t$ is the time required for the $\{I\uparrow, t_0\}$ mode to rise out of the box, backscatter off the spacetime curvature and centrifugal barrier, and reenter the box. Thus, the $\uparrow$ and $\downarrow$ modes, while powerful for quantization inside our box at and near an arbitrary fixed time t_0, and hence anywhere near the horizon, are not powerful for global quantization.)

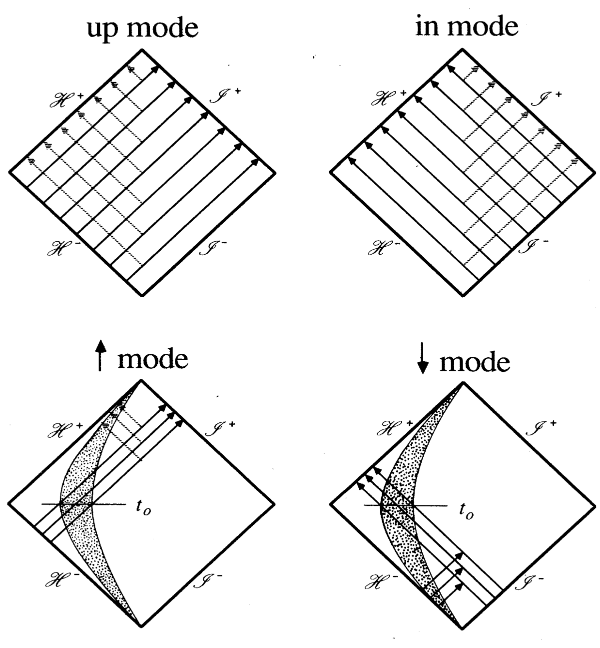

FIG. 1. Penrose spacetime diagrams depicting the propagation and backscatter of the *up* and *in* modes of a field outside a black hole, and also the $\uparrow$ and $\downarrow$ wave-packet modes that are constructable from them. The stippled region is the world tube of the interior of the box which is used in defining the $\uparrow$ and $\downarrow$ modes. The horizontal line marked t_0 identifies the interior of that box at time $t = t_0$. $\mathscr{I}^-$ denotes past null infinity, $\mathscr{I}^+$ future null infinity, $\mathscr{H}^-$ the hole's past horizon, and $\mathscr{H}^+$ its future horizon.

For a real, astrophysical black hole (one formed in the past by gravitational collapse) the *up* modes are precisely thermally populated,[1,2] $\langle n_I \rangle = n_I^{\text{th}}$, while the mean number of quanta in the *in* modes, $\langle n_I^{in} \rangle$ (accreting quanta) depends on the hole's astrophysical environment and thus may be regarded as arbitrary.

Since the $\uparrow$ modes are composed solely of *up* modes, they like *up* must be perfectly thermally populated. This guarantees that they contribute nothing to $\langle T_{\mu\nu} \rangle^{\text{ren}}$ inside our box [Eq. (2.30)], and thus can be ignored. By contrast, since the $\downarrow$ modes are a superposition of *up* modes and *in* modes, and since the probability of a quantum in the I *in* mode being transmitted and thereby entering an $I\downarrow$ mode is $1 - |A_I|^2$, and the probability of a quantum in the I *up* mode being reflected and thereby entering an $I\downarrow$ mode is $|A_I|^2$, the mean number of quanta in the locally downward propagating mode $I\downarrow$ is

$$\langle n_I^{\downarrow} \rangle = \langle n_I^{in} \rangle (1 - |A_I|^2) + n_I^{\text{th}} |A_I|^2 . \tag{2.35}$$

The renormalized number of quanta in an $I\downarrow$ mode is thus

$$\langle n_I^{\downarrow} \rangle - n_I^{\text{th}} = (\langle n_I^{\text{in}} \rangle - n_I^{\text{th}})(1 - |A_I|^2) . \tag{2.36}$$

This relation, together with $\langle n_I^{\uparrow} \rangle = n_I^{\text{th}}$ and Eq. (2.30) implies that the renormalized stress-energy tensor associated with modes I inside our quantization box is

$$\langle T_{\mu\nu} \rangle_I^{\text{ren}} = T_{\mu\nu}^{I\downarrow} (\langle n_I^{in} \rangle - n_I^{\text{th}})(1 - |A_I|^2) . \tag{2.37}$$

When our box was flat, the stress-energy tensor associated with one quantum $T_{\mu\nu}^{I(\uparrow \text{or} \downarrow)}$ had the form [Eq. (2.17)]

$$T_{\mu\nu}^{I(\uparrow\text{or}\downarrow)}=\frac{1}{A_B\kappa^{-1}\ln(z_2/z_1)}\frac{k_\mu k_\nu}{\tilde{\epsilon}} ,$$

where k_μ was the wave vector of the mode—or, equivalently, the four-momentum of one quantum. By switching to a spheroidal box we induce two changes: (i) the probability of finding a quantum in a horizontal area dA is changed from dA/A_B to $|{}_hS_{lm}(\theta,a\epsilon)|^2dA/A_H$, with A_H the horizon area and ${}_hS_{lm}(\theta,a\epsilon)$ the "spin-weighted spheroidal harmonic" that carries the angular dependence of Ψ^I;[27] and (ii) the modes I are changed from eigenfunctions of $\hat{p}_x$ and $\hat{p}_y$ with eigenvalues k_x and k_y to eigenfunctions of $\hat{p}_\phi=$(angular momentum) with eigenvalue $k_\phi=m$. Correspondingly the $T_{\mu\nu}^{I(\uparrow\text{or}\downarrow)}$ of Eq. (2.17) is changed to

$$T_{\mu\nu}^{I(\uparrow\text{or}\downarrow)}=\frac{|{}_hS_{lm}(\theta,a\epsilon)|^2}{A_H\kappa^{-1}\ln(z_2/z_1)}\frac{k_\mu k_\nu}{\tilde{\epsilon}} , \tag{2.38a}$$

$$k_{\hat{0}}=-\tilde{\epsilon}/\alpha , \quad k_{\hat{z}}=\tilde{\epsilon}/\alpha \text{ for } \uparrow \text{ mode},$$
$$k_{\hat{z}}=-\epsilon/\alpha \text{ for } \downarrow \text{ mode}, \quad k_\phi=m . \tag{2.38b}$$

(Here and below for simplicity we shall forego any attempt to discuss θ components of the stress-energy tensor; they are relatively uninteresting since they cannot influence the evolution of the hole's mass or angular momentum.)

To get the total stress-energy tensor near the future horizon, we must sum over all modes in the box, $\sum_I$, with $I=\{\tilde{\epsilon}, h, l, m\}$. The sum on $\tilde{\epsilon}$ can be reexpressed as an integral by noting that the vertical periodic boundary conditions (2.11) imply that there are

$$dN=\frac{\ln(z_2/z_1)}{2\pi\kappa}d\tilde{\epsilon} \tag{2.39}$$

values of $\tilde{\epsilon}$ in the frequency interval $d\tilde{\epsilon}$. Correspondingly, the total stress-energy tensor takes the form

$$\langle T_{\mu\nu}\rangle^{\text{ren}}=\sum_{l,m,h}\int_0^\infty T_{\mu\nu}^{I\,in}(\langle n_I^{in}\rangle-n_I^{\text{th}})d\tilde{\epsilon} , \tag{2.40}$$

where $T_{\mu\nu}^{I\,in}$ denotes the stress-energy tensor associated with one quantum injected at infinity ($\mathcal{J}^-$) into the I in state, which for $\{\mu, \nu\}=\{\hat{0}, \hat{z}, \phi\}$ has the form [Eqs. (2.37), (2.38), and (2.40)]

$$T_{\mu\nu}^{I\,in}=\frac{|{}_hS_{lm}(\theta,a\epsilon)|^2}{2\pi A_H}\frac{k_\mu k_\nu}{\tilde{\epsilon}}(1-|A_I|^2) , \tag{2.41a}$$

$$k_{\hat{0}}=-\tilde{\epsilon}/\alpha , \quad k_{\hat{z}}=-\tilde{\epsilon}/\alpha , \quad k_\phi=m \quad \text{for } in \text{ modes} . \tag{2.41b}$$

Because the modes I are not eigenfunctions of $\partial/\partial\theta$, the θ components of $T_{\mu\nu}^{I\,in}$ do not take the simple form (2.41a). However, from the actual form of the modes [Eq. (3.2) below for a scalar field] and the general classical expression for the stress-energy tensor in terms of the field [Eq. (3.48) below for a scalar field] one can show that

$$T_{\hat{0}\theta}^{I\,in}=T_{\phi\theta}^{I\,in}=0 , \quad T_{\hat{z}\theta}^{I\,in}=O(\alpha) , \quad T_{\theta\theta}^{I\,in}\sim 1 . \tag{2.41c}$$

By taking the limit of expression (2.40) as $\alpha\to 0$ we obtain the fluxes of energy and angular momentum across the future horizon. Taking the limit is facilitated by noting that, as $\alpha\to 0$,

$$\frac{\partial}{\partial t}=\alpha \mathbf{e}_{\hat{0}}\to \mathbf{l}, \quad \alpha\mathbf{e}_{\hat{z}}\to \mathbf{l} , \tag{2.42}$$

where $\mathbf{l}$ is the generator of the future horizon, and that consequently the components $\langle T_{ll}\rangle^{\text{ren}}$ and $\langle T_{l\phi}\rangle^{\text{ren}}$ which drive the evolution of the horizon's mass and angular momentum are

$$\langle T_{ll}\rangle^{\text{ren}}=\lim_{\alpha\to 0}(\alpha^2\langle T_{\hat{0}\hat{0}}\rangle^{\text{ren}}) , \tag{2.43a}$$

$$\langle T_{l\phi}\rangle^{\text{ren}}=\lim_{\alpha\to 0}(\alpha\langle T_{\hat{0}\phi}\rangle^{\text{ren}}) , \tag{2.43b}$$

and similarly

$$\langle T_{l\theta}\rangle^{\text{ren}}=\lim_{\alpha\to 0}(\alpha\langle T_{\hat{0}\theta}\rangle^{\text{ren}})=0 . \tag{2.43c}$$

Correspondingly, Eq. (2.40) yields, on the future horizon,

$$\langle T_{ll}\rangle^{\text{ren}}_{\mathcal{H}^+}=\sum_{l,m,h}\int_0^\infty d\tilde{\epsilon}\frac{|{}_hS_{lm}(\theta,a\epsilon)|^2}{A_H}(\langle n_I^{in}\rangle-n_I^{\text{th}})\times(1-|A_I|^2)\frac{\tilde{\epsilon}}{2\pi} , \tag{2.44a}$$

$$\langle T_{l\phi}\rangle^{\text{ren}}_{\mathcal{H}^+}=-\sum_{l,m,h}\int_0^\infty d\tilde{\epsilon}\frac{|{}_hS_{lm}(\theta,a\epsilon)|^2}{A_H}(\langle n_I^{in}\rangle-n_I^{\text{th}})\times(1-|A_I|^2)\frac{m}{2\pi} , \tag{2.44b}$$

$$\langle T_{l\theta}\rangle^{\text{ren}}_{\mathcal{H}^+}=0 . \tag{2.44c}$$

D. Conversion of $\langle T_{\mu\nu}\rangle^{\text{ren}}$ to viewpoint of distant observers

Expressions (2.40) and (2.44) are written from the viewpoint of a near-horizon ZAMO ($\tilde{\epsilon}>0$ for all modes; $\epsilon<0$ for superradiant modes and $\epsilon>0$ for nonsuperradiant modes). Below we shall call this the "old" viewpoint. Since it is the in modes that contribute to $\langle T_{\mu\nu}\rangle^{\text{ren}}$ and those modes originate far from the hole (at $\mathcal{J}^-$), it is desirable to rewrite (2.44) from the viewpoint of distant observers ($\epsilon>0$ for all modes)—a "new" viewpoint. For superradiant modes this entails changes that can be inferred as follows (cf. Appendix A).

Focus attention on a specific, superradiant mode. In the "old" (near-horizon) viewpoint the mode is characterized by quantum numbers $I_{\text{old}}=(l, m, h, \tilde{\epsilon})_{\text{old}}$ with $0<\tilde{\epsilon}_{\text{old}}<-m_{\text{old}}\Omega_H$. In the "new" (distant-observer) viewpoint it is characterized by $I_{\text{new}}=(l, m, h, \tilde{\epsilon})_{\text{new}}$ with

$$\tilde{\epsilon}_{\text{new}}=-\tilde{\epsilon}_{\text{old}} , \quad m_{\text{new}}=-m_{\text{old}} ,$$
$$l_{\text{new}}=+l_{\text{old}} , \quad h_{\text{new}}=-h_{\text{old}} . \tag{2.45a}$$

[This transformation follows from Eq. (2.33), from the symmetries[29] of ${}_hS_{lm}(\theta,a\epsilon)$, and from the relation

$$\Psi_{\text{new}}^{I_{\text{new}}}=\bar{\Psi}_{\text{old}}^{I_{\text{old}}} \tag{2.45b}$$

between the Teukolsky functions for this mode in the new and old viewpoints; cf. the paragraph following Eq. (2.34).] Note that Eqs. (2.45a) and (2.34) imply

$$\epsilon_{\rm new} \equiv \tilde{\epsilon}_{\rm new} + m_{\rm new}\Omega_H = -(\tilde{\epsilon}_{\rm old} + m_{\rm old}\Omega_H) = -\epsilon_{\rm old} > 0 \;. \qquad (2.45c)$$

Restrict attention, in this paragraph, to one of the superradiant modes $I\downarrow$ or $I\uparrow$ in our near-horizon box. This mode is characterized in the old, near-horizon viewpoint, by creation and annihilation operators $\hat{a}^\dagger_{\rm old}$, $\hat{a}_{\rm old}$ with the conventional commutation relations, Hamiltonian, and number operator

$$[\hat{a}_{\rm old}, \hat{a}^\dagger_{\rm old}]_\pm = 1 \;, \quad [\hat{a}_{\rm old}, \hat{a}_{\rm old}]_\pm = 0 \;,$$
$$\hat{H}_{\rm old} = (\hat{n}_{\rm old} \pm \tfrac{1}{2})\tilde{\epsilon}_{\rm old} \;, \quad \hat{n}_{\rm old} = \hat{a}^\dagger_{\rm old}\hat{a}_{\rm old} \;. \qquad (2.46a)$$

Here and in the remainder of this paragraph we suppress the index I; the upper signs refer to the bosonic case and the lower signs to the fermionic; and $[\;,\;]_+$ denotes a commutator, while $[\;,\;]_-$ denotes an anticommutator. The eigenstates of $\hat{H}_{\rm old}$ are $\{|0\rangle_{\rm old}, |1\rangle_{\rm old}, \ldots\}$ in the bosonic case and $\{|0\rangle_{\rm old}, |1\rangle_{\rm old}\}$ in the fermionic, with

$$\hat{a}_{\rm old}|0\rangle_{\rm old} = 0, \quad |n\rangle_{\rm old} = \frac{1}{\sqrt{n!}}(\hat{a}^\dagger_{\rm old})^n|0\rangle_{\rm old} \;, \quad n \geq 0 \;. \qquad (2.46b)$$

Since the creation of a positive-energy quantum, $\tilde{\epsilon}_{\rm old} > 0$, must correspond to annihilation of a negative-energy quantum, $\tilde{\epsilon}_{\rm new} = -\tilde{\epsilon}_{\rm old} < 0$ (total energy change the same in both viewpoints), the annihilation and creation operators of the new (negative-energy) viewpoint must be

$$\hat{a}_{\rm new} = \hat{a}^\dagger_{\rm old} \;, \quad \hat{a}^\dagger_{\rm new} = \hat{a}_{\rm old} \;. \qquad (2.47a)$$

[This can also be inferred mathematically from Eq. (2.45b) and the expressions $\hat{\Psi} = \hat{a}_{\rm old}\Psi_{\rm old} + \hat{a}^\dagger_{\rm old}\overline{\Psi}_{\rm old} = \hat{a}_{\rm new}\Psi_{\rm new} + \hat{a}^\dagger_{\rm new}\overline{\Psi}_{\rm new}$ for the contribution of this mode to the field operator $\hat{\Psi}$.] Equations (2.47a) and (2.46a) imply the commutation relations

$$[\hat{a}_{\rm new}, \hat{a}^\dagger_{\rm new}]_\pm = \mp 1 \;, \quad [\hat{a}_{\rm new}, \hat{a}_{\rm new}]_\pm = 0 \qquad (2.47b)$$

(which are unusual in the bosonic case but standard in the fermionic). For any classical system with well-defined total red-shifted energy E and total angular momentum L, when one changes from an "old" frame to a "new" frame that rotates with angular velocity $-\Omega_H$ relative to the old, the total energy transforms as

$$E_{\rm new} = E_{\rm old} + \Omega_H L_{\rm old} = E_{\rm old}(1 + \Omega_H L_{\rm old}/E_{\rm old})$$

[cf. Eq. (2.34)]. Correspondingly, the quantum-mechanical Hamiltonian must transform as

$$\hat{H}_{\rm new} = \hat{H}_{\rm old}(1 + \Omega_H L_{\rm old}/E_{\rm old}) \;.$$

For our mode, since $L_{\rm old}/E_{\rm old} = m_{\rm old}/\tilde{\epsilon}_{\rm old}$, and since $\tilde{\epsilon}_{\rm old} + \Omega_H m_{\rm old} = -\epsilon_{\rm new}$, this implies

$$\hat{H}_{\rm new}/\epsilon_{\rm new} = -\hat{H}_{\rm old}/\tilde{\epsilon}_{\rm old} \;. \qquad (2.47c)$$

Writing each Hamiltonian in terms of its number operator,

$$\hat{H}_{\rm new} = (\hat{n}_{\rm new} \pm \tfrac{1}{2})\epsilon_{\rm new} \;, \quad \hat{H}_{\rm old} = (\hat{n}_{\rm old} \pm \tfrac{1}{2})\tilde{\epsilon}_{\rm old} \;, \qquad (2.47d)$$

and comparing with Eqs. (2.46a), (2.47a) and (2.47c), we infer the relation

$$\hat{n}_{\rm new} = \mp \hat{a}^\dagger_{\rm new}\hat{a}_{\rm new} = -(\hat{n}_{\rm old} \pm 1) \qquad (2.47e)$$

between the number operators for the old viewpoint and the new viewpoint. [An alternative derivation of this relation is given for the bosonic case in Eqs. (8.12) of BHMP.[6]] Correspondingly, the eigenstates of $\hat{n}_{\rm new}$ are $|-1\rangle_{\rm new}$, $|-2\rangle_{\rm new}, \ldots$ in the bosonic case and $|0\rangle_{\rm new}$, $|1\rangle_{\rm new}$ in the fermionic, with

$$\hat{a}^\dagger_{\rm new}|-1\rangle_{\rm new} = 0 \;, \quad |n\rangle_{\rm new} = \frac{\hat{a}_{\rm new}^{|n+1|}}{\sqrt{|n+1|!}}|-1\rangle_{\rm new} \;,$$
$$\hat{n}_{\rm new}|n\rangle_{\rm new} = n|n\rangle_{\rm new} \;, \quad \text{bosonic} \;;$$
$$a_{\rm new}|0\rangle_{\rm new} = 0 \;, \quad |1\rangle_{\rm new} = a^\dagger_{\rm new}|0\rangle_{\rm new} \;,$$
$$\hat{n}_{\rm new}|n\rangle_{\rm new} = n|n\rangle_{\rm new} \;, \quad \text{fermionic}; \qquad (2.47f)$$

and with the precise correspondences

$$|0\rangle_{\rm old} \leftrightarrow |-1\rangle_{\rm new} \;, \quad |1\rangle_{\rm old} \leftrightarrow |-2\rangle_{\rm new}, \;\ldots, \quad \text{bosonic},$$
$$|0\rangle_{\rm old} \leftrightarrow |1\rangle_{\rm new} \;, \quad |1\rangle_{\rm new} \leftrightarrow |0\rangle_{\rm old} \;, \quad \text{fermionic}. \qquad (2.47g)$$

One can readily verify that the normalizations ${}_{\rm new}\langle -1|-1\rangle_{\rm new} = 1$ in the bosonic case and ${}_{\rm new}\langle 0|0\rangle_{\rm new} = 1$ in the fermionic produce, as a result of (2.47e) and (2.47f) and the commutation relations (2.47b),

$${}_{\rm new}\langle n|n'\rangle_{\rm new} = \delta_{nn'} \;. \qquad (2.47h)$$

Thus, the Fock spaces constructed by Eqs. (2.47f) are properly normalized. [Note: Equations (2.46) and (2.47) are completely analogous in the bosonic case to the transformation of a Cherenkov-radiation calculation from the rest frame of the medium, where the emitted quanta carry positive energy, to the rest frame of the emitting particle, where the emitted quanta carry negative energy.]

The key result that we shall need from this analysis is the relationship (2.47e) between the number operators of the new and old viewpoints. This relationship, with the suppressed indices $I\uparrow$ or $I\downarrow$ restored, says

$$(\hat{n}_I^\uparrow)_{\rm new} = -[(\hat{n}_I^\uparrow)_{\rm old} \pm 1] \;,$$
$$(\hat{n}_I^\downarrow)_{\rm new} = -[(\hat{n}_I^\downarrow)_{\rm old} \pm 1] \;. \qquad (2.48a)$$

The superradiant $I\,in$ mode can be analyzed in a similar fashion, but with these differences: since the *in* mode originates at past null infinity, $\mathscr{I}^-$, the usual *conventions* and *viewpoint* for its quantization are those of a distant observer (cf. Appendix A): $\epsilon_{\rm new} > 0$, $[\hat{a}_{\rm new}, \hat{a}^\dagger_{\rm new}]_\pm = 1$, and $\hat{n}_{\rm new} = \hat{a}^\dagger_{\rm new}\hat{a}_{\rm new}$. By arguments completely analogous to the above we then conclude that $\hat{a}_{\rm old} = \hat{a}^\dagger_{\rm new}$, $[\hat{a}_{\rm old}, \hat{a}^\dagger_{\rm old}]_\mp = \mp 1$, and $\hat{n}_{\rm old} = \mp \hat{a}^\dagger_{\rm old}\hat{a}_{\rm old}$. Correspondingly, $\hat{n}_{\rm old} = -(\hat{n}_{\rm new} \pm 1)$, which is a symmetric relation between new and old: $\hat{n}_{\rm new} = -(\hat{n}_{\rm old} \pm 1)$. Rewritten with the indices $I\,in$ restored this says

$$(\hat{n}_I^{in})_{\rm new} = -[(\hat{n}_I^{in})_{\rm old} \pm 1] \ . \tag{2.48b}$$

Note that Eqs. (2.48a) and (2.48b) for the number operators of the two viewpoints imply analogous relationships between the mean number of quanta in any state

$$\langle n_I^{\uparrow} \rangle_{\rm new} = -(\langle n_I^{\uparrow} \rangle_{\rm old} \pm 1) \ ,$$
$$\langle n_I^{\downarrow} \rangle_{\rm new} = -(\langle n_I^{\downarrow} \rangle_{\rm old} \pm 1) \ , \tag{2.48c}$$
$$\langle n_I^{in} \rangle_{\rm new} = -(\langle n_I^{in} \rangle_{\rm old} \pm 1) \ ; \tag{2.48d}$$

and, in particular, imply that the mean number in a perfectly thermalized state of the black-hole atmosphere is [cf. the old-viewpoint relation (2.16)]

$$(n_I^{\rm th})_{\rm new} = -[(n_I^{\rm th})_{\rm old} \pm 1] = -\left[\frac{1}{e^{\tilde{\epsilon}_{\rm old}/T_H} \mp 1} \pm 1 \right]$$
$$= \mp \left[\frac{1}{1 \mp e^{-\tilde{\epsilon}_{\rm old}/T_H}} \right] = \frac{1}{e^{\tilde{\epsilon}_{\rm new}/T_H} \mp 1} \ . \tag{2.48e}$$

Notice that, although $(n_I^{\rm th})_{\rm new}$ is given in terms of $\tilde{\epsilon}_{\rm new}$ by the same standard Bose-Einstein mathematical expression as $(n_I^{\rm th})_{\rm old}$ in terms of $\tilde{\epsilon}_{\rm old}$ [cf. Eqs. (2.47c) and (2.16)], the numerical value of $(n_I^{\rm th})_{\rm new}$ is negative in the bosonic case and $> \frac{1}{2}$ in the fermionic (because $\tilde{\epsilon}_{\rm new} < 0$), while the numerical value of $(n_I^{\rm th})_{\rm old}$ is positive in the bosonic case and $< \frac{1}{2}$ in the fermionic (because $\tilde{\epsilon}_{\rm old} > 0$).

With the relationship between the old, near-horizon viewpoint and the new, distant-observer viewpoint now understood, we can return to the expectation value of the renormalized stress-energy tensor $\langle T_{\mu\nu} \rangle^{\rm ren}$ on the future horizon $\mathcal{H}^+$. In the old viewpoint that $\langle T_{\mu\nu} \rangle^{\rm ren}$ is given by expression (2.44). By (i) leaving the nonsuperradiant modes unchanged in (2.44), (ii) transforming the superradiant modes to the new viewpoint using Eqs. (2.45a), (2.45c), (2.48d), and (2.48e), and (iii) noting that[29] $|{}_hS_{lm}(\theta,a\epsilon)|^2 = |{}_{-h}S_{l-m}(\theta,-a\epsilon)|^2$, we bring Eqs. (2.40) and (2.44) into the *new-viewpoint forms*

$$\langle T_{\mu\nu} \rangle^{\rm ren} = \sum_{l,m,h} \int_0^\infty d\epsilon \, T_{\mu\nu}^{I\,in} (\langle n_I^{in} \rangle - n_I^{\rm th}) \ , \tag{2.49}$$

$$\langle T_{ll} \rangle_{\mathcal{H}^+}^{\rm ren} = \sum_{l,m,h} \int_0^\infty d\epsilon \frac{|{}_hS_{lm}(\theta,a\epsilon)|^2}{A_H} (\langle n_I^{in} \rangle - n_I^{\rm th}) \times (1 - |A_I|^2) \frac{\tilde{\epsilon}}{2\pi} \ , \tag{2.50a}$$

$$\langle T_{l\phi} \rangle_{\mathcal{H}^+}^{\rm ren} = -\sum_{l,m,h} \int_0^\infty d\epsilon \frac{|{}_hS_{lm}(\theta,a\epsilon)|^2}{A_H} (\langle n_I^{in} \rangle - n_I^{\rm th}) \times (1 - |A_I|^2) \frac{m}{2\pi} \ . \tag{2.50b}$$

Here, in keeping with the new, distant-observer viewpoint,

$$\tilde{\epsilon} = \epsilon - m\Omega_H \ , \quad \epsilon > 0 \ \text{ for all modes}, \tag{2.51a}$$

$$n_I^{\rm th} = \frac{1}{e^{\tilde{\epsilon}/T_H} \mp 1} = \frac{1}{e^{(\epsilon - m\Omega_H)/T_H} \mp 1} \ , \tag{2.51b}$$

which in the bosonic case (upper sign) is positive for nonsuperradiant modes but negative for superradiant; and $\langle n_I^{in} \rangle \ge 0$ is the mean number of positive-energy, $\epsilon > 0$, quanta injected into the mode $I\,in$ by the external universe. Also, in Eq. (2.49) $T_{\mu\nu}^{I\,in}$ has the same *form* (2.41) as previously. However, for superradiant modes, because the signs of $\tilde{\epsilon}$ and m have been reversed in going from the near-horizon viewpoint to the distant-observer viewpoint, the sign of $T_{\mu\nu}^{I\,in}$ [Eq. (2.41)] has reversed, as has the sign of $(\langle n_I^{in} \rangle - n_I^{\rm th})$, leaving the value of $\langle T_{\mu\nu} \rangle^{\rm ren}$ unchanged.

E. Global conservation of energy and angular momentum

Hawking[1] has shown that, when a rotating black hole is evaporating into an external vacuum (i.e., when the fields around the hole are in the Unruh[2] vacuum state $|U\rangle$), static external observers measure the evaporated radiation to have a radial flux of energy and angular momentum given, in terms of the distant-observer viewpoint, by

$$T^{r0} = \sum_{l,m,h} \int_0^\infty d\epsilon \frac{|{}_hS_{lm}(\theta,a\epsilon)|^2}{4\pi r^2} n_I^{\rm th} (1 - |A_I|^2) \frac{\epsilon}{2\pi} \ , \tag{2.52a}$$

$$T^r{}_\phi = \sum_{l,m,h} \int_0^\infty d\epsilon \frac{|{}_hS_{lm}(\theta,a\epsilon)|^2}{4\pi r^2} n_I^{\rm th} (1 - |A_I|^2) \frac{m}{2\pi} \ , \tag{2.52b}$$

where $n_I^{\rm th}$ is given by Eq. (2.51b). (There is no necessity to renormalize since far from the hole where these fluxes are measured spacetime is flat, the observers are inertial, and the renormalized stress-energy tensor is equal to the stress-energy tensor $T_{\mu\nu}$ measured by the observers.) For nonsuperradiant bosonic modes $n_I^{\rm th}$ [Eq. (2.51b)] is positive, $1 - |A_I|^2$ is positive, and thus the energy flux T^{r0} is positive. For superradiant bosonic modes $n_I^{\rm th}$ is negative, $1 - |A_I|^2$ is negative, and thus the energy flux T^{r0} is again positive. For all fermionic modes $n_I^{\rm th}$ is positive, $1 - |A_I|^2$ is positive (no superradiance even for "superradiant modes"), and thus T^{r0} is positive. (We note in passing that, although the mean occupation number $n_I^{\rm th}$ appearing in the Hawking flux (2.52) has often been labeled, for superradiant modes, as "nonthermal," it in fact is precisely thermal. It has an unfamiliar form [Eq. (2.51b) with a negative quantity in the exponent] only because it is a near-horizon thermal occupation number [Eq. (2.16)] translated into a distant-observer's viewpoint.)

If the hole is accreting at the same time as it evaporates, with $\langle n_I^{in} \rangle$ quanta in the mode $I\,in$, then standard flat-spacetime quantum field theory (together with the probability $|A_I|^2$ for an accreting quantum in mode $I\,in$ to be reflected by the hole's spacetime curvature and centrifugal barrier) implies that Eqs. (2.52) will be modified to read

$$T^{r0} = \sum_{l,m,h} \int_0^\infty d\epsilon \frac{|{}_hS_{lm}(\theta,a\epsilon)|^2}{4\pi r^2} (n_I^{\rm th} - \langle n_I^{in} \rangle) \times (1 - |A_I|^2) \frac{\epsilon}{2\pi} \ , \tag{2.53a}$$

$$T^r{}_\phi = \sum_{l,m,h} \int_0^\infty d\epsilon \frac{|{}_hS_{lm}(\theta,a\epsilon)|^2}{4\pi r^2}(n_I^{\rm th} - \langle n_I^{in}\rangle) \times (1-|A_I|^2)\frac{m}{2\pi} . \quad (2.53b)$$

The rate at which the mass M and angular momentum J of the hole change due to this evaporation-plus-accretion are

$$\frac{dM}{dt} = -\int_{\mathcal{S}} T^{r0} r^2 \sin\theta \, d\theta \, d\phi , \quad (2.54a)$$

$$\frac{dJ}{dt} = -\int_{\mathcal{S}} T^r{}_\phi r^2 \sin\theta \, d\theta \, d\phi , \quad (2.54b)$$

where the integrals are over a sphere $\mathcal{S}$ far from the hole.

We can also compute the evolution of the hole's mass and angular momentum using flux integrals of $\langle T_{\mu\nu}\rangle^{\rm ren}$ over the future horizon $\mathcal{H}^+$. (It is $\langle T_{\mu\nu}\rangle^{\rm ren}$ that goes on the right-hand side of the Einstein field equations and thus that produces evolution of the hole.) Standard analyses [e.g., Eqs. (6.56) and (6.58) of BHMP[6] together with $\lim_{\alpha\to 0}\alpha \mathbf{n} = \mathbf{l}$] reveal that

$$\frac{dM}{dt} = \int_{\mathcal{H}^+} (\langle T_{ll}\rangle^{\rm ren} - \Omega_H \langle T_{l\phi}\rangle^{\rm ren}) dA_H , \quad (2.55a)$$

$$\frac{dJ}{dt} = \int_{\mathcal{H}^+} (-\langle T_{l\phi}\rangle^{\rm ren}) dA_H , \quad (2.55b)$$

where the area integration element is

$$dA_H = (A_H/4\pi)\sin\theta \, d\theta \, d\phi \quad (2.55c)$$

[cf. Eqs. (3.74) and (3.75) of BHMP[6]]. It is straightforward to verify that, not only do the rates of evolution, dM/dt and dJ/dt as computed on the horizon [Eqs. (2.55)] and far from the hole [Eqs. (2.54)], agree; their integrands, the fluxes of energy and angular momentum, also agree as functions of (θ,ϕ) when the integration elements are expressed in terms of solid angle $\sin\theta \, d\theta \, d\phi$:

$$[(\langle T_{ll}\rangle^{\rm ren} - \Omega_H \langle T_{l\phi}\rangle^{\rm ren})(A_H/4\pi)]_{\mathcal{H}^+} = (-T^{r0}r^2)_{\mathcal{S}} , \quad (2.56a)$$

$$[-\langle T_{l\phi}\rangle^{\rm ren}(A_H/4\pi)]_{\mathcal{H}^+} = (-T^r{}_\phi r^2)_{\mathcal{S}} ; \quad (2.56b)$$

cf. Eqs. (2.50), (2.53), and (2.34). Thus (as is also guaranteed[1] by the relation[12] $\langle T^{\mu\nu}\rangle_{;\nu}=0$), the mass and angular momentum which characterize the hole's horizon evolve in accord with the rates at which distant observers see energy and angular momentum enter and leave the hole's vicinity.

III. DERIVATION BASED ON QUANTUM FIELD THEORY IN CURVED SPACETIME

We now turn to a quantum-field-theory derivation of Eq. (1.1) for the renormalized stress-energy tensor near the horizon of a rotating, uncharged black hole. For simplicity we shall confine attention to a conformal massless scalar field Φ.

Some prior analyses of quantum fields around rotating black holes have been bedeviled by problems with the superradiant modes. Especially troublesome is the fact that the total energy in such a mode has one sign for the Unruh vacuum state $|U\rangle$ and the opposite sign for the Hartle-Hawking vacuum state $|H\rangle$. These troubles will be dealt with smoothly in our derivation by the use of a variant of the "η formalism." We derive that formalism from standard quantum field theory in Appendix A.

In Sec. III A and Appendix B we lay out, very carefully, the mathematical foundations for our derivation of Eq. (1.1). Specifically, in Sec. III A we introduce the wave functions for *up* modes and *in* modes of our scalar field in the region outside the hole's horizon; we express the field operator in terms of them, their creation and annihilation operators, and the operator $\hat{\eta}$; we define the Hartle-Hawking state $|H\rangle$, the Unruh state $|U\rangle$, and a state $|UN\rangle$ that describes a realistic, accreting and evaporating black hole; and we write down the key formulas for the renormalized stress-energy tensor using point-splitting regularization. In Appendix B we quantize the scalar field on the complete, analytically extended Kerr spacetime (including η formalism contributions); and from that quantization we derive the Hadamard elementary function that enters into the point-splitting expressions of Section III A for the renormalized stress-energy tensor.

In Appendix C we derive the density operators that describe the states $|H\rangle$, $|U\rangle$, and $|UN\rangle$ in the spacetime region outside a black-hole horizon; and from those density operators we infer the physical descriptions of these states in terms of ZAMO measurements.

In Sec. III B we use the tools of Sec. III A and Appendix B to derive the renormalized stress-energy tensor for the Hartle-Hawking vacuum in the vicinity of the black-hole horizon. This, then, becomes a foundation for the derivation in Sec. III C, of the renormalized stress-energy tensor for the evaporating, accreting state $|UN\rangle$. The result of that derivation is in perfect accord with Eq. (1.1).

Throughout Sec. III and the Appendixes we carry out our formal manipulations of the Hartle-Hawking state $|H\rangle$ without surrounding the hole by a mirror (and thus without removing its singular behavior); and we do so as though $|H\rangle$ were a regular, well-behaved state. Thus, our formal manipulations of $|H\rangle$ are not soundly based. Nevertheless, as we shall argue in Sec. IV, the results of our formal calculations (the propagator, Hadamard function, and renormalized stress-energy tensor) are arbitrarily close to the correct results with mirror present, for events arbitrarily close to the horizon and arbitrarily close together. Moreover (cf. end of Sec. IV A), nowhere do our formally derived results for the states $|U\rangle$ and $|UN\rangle$ rely on the regularity of $|H\rangle$.

A. Mathematical preliminaries: *In* and *up* modes; states of interest; $\langle T_{\mu\nu}\rangle^{\rm ren}$ computed by point splitting

In this section we shall study the properties of a conformal massless scalar field Φ (special case of the field Ψ in Sec. II, with $s=0$ and thus with helicity h irrelevant). We shall denote by $v_I(t,r,\theta,\phi)$ that solution of the vacuum scalar wave equation

$$\Box\Phi-\tfrac{1}{6}R\,\Phi=0 \text{ with } R=(\text{scalar curvature}), \tag{3.1}$$

which describes an *in* mode[28] (also sometimes called "past-null-infinity mode"); and we shall adapt for it the distant-observer viewpoint, $\epsilon>0$ (cf. Appendix A). More specifically, v_I is the unique solution of the scalar wave equation with the asymptotic forms

$$v_I\sim\frac{S_{lm}}{\sqrt{16\pi^2\epsilon}}\frac{1}{\sqrt{r^2+a^2}}\times\begin{cases}e^{-i\epsilon v+im\phi} & \text{at } \mathcal{J}^-,\\ A_I^+e^{-i\epsilon u+im\phi} & \text{at } \mathcal{J}^+,\\ 0 & \text{at } \mathcal{H}^-,\\ B_I^+e^{-i\tilde\epsilon v+im\bar\phi} & \text{at } \mathcal{H}^+.\end{cases} \tag{3.2}$$

Here $I\equiv\{\epsilon,l,m\}$, $\tilde\epsilon=\epsilon-m\Omega_H$, $\bar\phi$ is given by Eq. (2.26), and u is retarded and v advanced time

$$u=t-r^*\ ,\quad v=t+r^*\ ,\quad dr^*=\frac{r^2+a^2}{\Delta}dr \tag{3.3}$$

[notation of Eqs. (3.5) and Sec. VIII C 3 of BHMP[6]]. Similarly we shall denote by $q_I(t,r,\theta,\phi)$ the solution of (3.1) for an *up* mode[28] (also sometimes called "past-horizon mode"); and we shall adapt for it the near-horizon viewpoint, $\tilde\epsilon>0$. It is the unique solution with asymptotic form

$$q_I\sim\frac{S_{lm}}{\sqrt{16\pi^2\tilde\epsilon}}\frac{1}{\sqrt{r^2+a^2}}\times\begin{cases}0 & \text{at } \mathcal{J}^-\ ,\\ B_I^-e^{-i\epsilon u+im\phi} & \text{at } \mathcal{J}^+\ ,\\ e^{-i\tilde\epsilon u+im\bar\phi} & \text{at } \mathcal{H}^-\ ,\\ A_I^-e^{-i\tilde\epsilon v+im\bar\phi} & \text{at } \mathcal{H}^+\ .\end{cases} \tag{3.4}$$

In these solutions $S_{lm}=S_{lm}(\theta,a\epsilon)$ are the spheroidal harmonics (specialization of ${}_hS_{lm}$ of the last section to the case of zero spin and helicity), which are real and are normalized so that

$$\tfrac{1}{2}\int_0^\pi S_{lm}S_{l'm}\sin\theta\,d\theta=\int\frac{S_{lm}S_{l'm}}{A_H}dA_H=\delta_{ll'}\ . \tag{3.5}$$

The "reflection coefficients" $A_I^\pm$ and "transmission coefficients" $B_I^\pm$ appearing in the asymptotic forms (3.2) and (3.4) satisfy the following relations, which follow from conservation of the Wronskian of the solutions, and which appear unusually complicated because we have chosen to use different viewpoints for the *in* and *up* superradiant modes (distant observer, $\epsilon>0$, for *in*; near horizon, $\tilde\epsilon>0$, for *up*):

$$1-|A_I^+|^2=\frac{\tilde\epsilon}{\epsilon}|B_I^+|^2 \quad\text{for } \epsilon>0,$$

$$1-|A_I^-|^2=\frac{\epsilon}{\tilde\epsilon}|B_I^-|^2 \quad\text{for } \tilde\epsilon>0\ , \tag{3.6a}$$

$$\tilde\epsilon B_I^+\bar A_I^-=-\epsilon\bar B_I^-A_I^+\ ,\quad \tilde\epsilon B_I^+=\epsilon B_I^- \quad\text{for } \tilde\epsilon>0,\ \epsilon>0\ , \tag{3.6b}$$

$$\tilde\epsilon B_I^+A_{I_1}^-=-\epsilon B_{I_1}^-A_I^+\ ,\quad \tilde\epsilon B_I^+=\epsilon\bar B_{I_1}^- \quad\text{for } \epsilon>0,\ \tilde\epsilon<0\ , \tag{3.6c}$$

$$\tilde\epsilon B_{I_1}^+A_I^-=-\epsilon B_I^-A_{I_1}^+\ ,\quad \tilde\epsilon B_{I_1}^+=\epsilon\bar B_I^- \quad\text{for } \epsilon<0,\ \tilde\epsilon>0\ . \tag{3.6d}$$

Here $I=\{\epsilon,l,m\}$ and $I_1=\{-\epsilon,l,-m\}$. (Note that switching viewpoints on superradiant modes, from $\epsilon>0$ and $\tilde\epsilon<0$ to $\epsilon<0$ and $\tilde\epsilon>0$ or conversely, complex conjugates and converts the indices I into the indices I_1.) From these relations it follows that

$$A_I^-=-\bar A_I^+\frac{B_I^+}{\bar B_I^+} \quad\text{and}\quad |A_I^-|^2=|A_I^+|^2$$
$$\text{for } \epsilon>0,\ \tilde\epsilon>0\ , \tag{3.6e}$$

$$A_{I_1}^-=-A_I^+\frac{\bar B_I^+}{B_I^+} \quad\text{and}\quad |A_{I_1}^-|^2=|A_I^+|^2$$
$$\text{for } \epsilon>0,\ \tilde\epsilon<0\ , \tag{3.6f}$$

$$A_{I_1}^+=-A_I^-\frac{\bar B_I^-}{B_I^-} \quad\text{and}\quad |A_{I_1}^+|^2=|A_I^-|^2$$
$$\text{for } \epsilon<0,\ \tilde\epsilon>0\ . \tag{3.6g}$$

Note that because, in Sec. II, we adopted near-horizon conventions for all modes, the reflection probability denoted $|A_I|^2$ in that section is denoted here $|A_I^-|^2=(|A_I^+|^2$ if $\epsilon>0,\ \tilde\epsilon>0)=(|A_{I_1}^+|^2$ if $\epsilon<0,\ \tilde\epsilon>0)$.

When quantizing the scalar field Φ we shall adopt for each superradiant mode the same convention as viewpoint (cf. Appendixes A and B): distant-observer convention and viewpoint for the *in* mode v_I, and near-horizon convention and viewpoint for the *up* mode q_I. Correspondingly, we shall expand the field operator $\hat\Phi$ in the form [Eq. (B6) or (B16) specialized to the exterior of the black hole, region I of Fig. 2]

$$\hat\Phi(x)=\sum_{\epsilon>0}[v_I(x)\hat a_I+\bar v_I(x)\hat\eta\hat a_I^\dagger\hat\eta]+\sum_{\tilde\epsilon>0}[q_I(x)\hat b_I+\bar q_I(x)\hat\eta\hat b_I^\dagger\hat\eta]\ , \tag{3.7}$$

with creation and annihilation operators that satisfy the standard commutation relations [Eq. (B7)]

$$[\hat a_{I'},\hat a_I^\dagger]=\delta_{I'I}\ ,\quad [\hat b_{I'},\hat b_I^\dagger]=\delta_{I'I}\ ,\quad \text{others vanish}, \tag{3.8}$$

and with number operators of the standard form [Eq. (B8)]

$$\hat n_I^{in}=\hat a_I^\dagger\hat a_I\ ,\quad \hat n_I^{up}=\hat b_I^\dagger\hat b_I\ . \tag{3.9}$$

In Eq. (3.7) and below we use the notations

$$\sum_{\epsilon>0}\equiv\sum_{lm}\int_0^\infty d\epsilon\ ,\quad \sum_{\tilde\epsilon>0}\equiv\sum_{lm}\int_0^\infty d\tilde\epsilon=\sum_{lm}\int_{\Omega_Hm}^\infty d\epsilon\ ; \tag{3.10}$$

and $\hat\eta$ is an operator introduced in Appendixes A and B [especially Eqs. (B1)] to permit us to deal with the "Hartle-Hawking state" $|H\rangle$ without changing our conventions:

$$\hat\eta=\prod_I\hat\eta_I^{in}\hat\eta_I^{up}\ , \tag{3.11a}$$

$$\hat\eta_I^{in\text{ or }up}=1 \tag{3.11b}$$

when I is a nonsuperradiant mode,

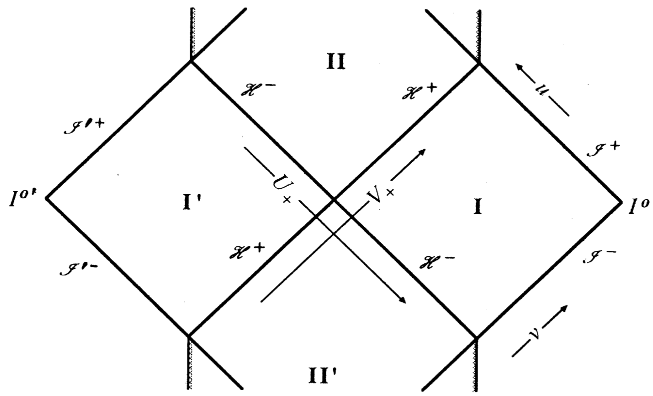

FIG. 2. Penrose spacetime diagram for the complete, analytically extended Kerr spacetime (Ref. 11). Region I is identical to the spacetime outside a black hole; region I′ is a "mirror" universe, isometric to region I. The directions of increase of the null coordinates u and v at null infinity are depicted, as are the directions of increase of the null coordinates U_{+} and V_{+} on the horizons $\mathscr{H}^{-}$ and $\mathscr{H}^{+}$.

$$\hat{\eta}_I^{in \text{ or } up}=1 \tag{3.11c}$$

when acting on a state with non-negative numbers of quanta,

$$\hat{\eta}_I^{in}=(-1)^{\hat{n}_I^{in}+1}\;, \quad \hat{\eta}_I^{up}=(-1)^{\hat{n}_I^{up}+1} \tag{3.11d}$$

when acting on states of superradiant modes with negative numbers of quanta, $n_I^{in \text{ or } up}\leq -1$. Note that $\hat{\eta}$ is unitary, $\hat{\eta}^2=1$.

Three states of the quantum field $\hat{\Phi}$ will be of special interest in our analysis: the "Hartle-Hawking vacuum state" $|H\rangle$, the "Unruh vacuum state" $|U\rangle$, and the state $|UN\rangle$ obtained by adding

$$N\equiv\{n_{I_1}^{in}, n_{I_2}^{in}, \ldots\} \tag{3.12}$$

quanta to the *in* modes I of the Unruh state $|U\rangle$, with the distant-observer viewpoint being taken on the added quanta ($n_I^{in}\geq 0$, since they are in the *in* modes). We denote the Green's functions (Feynman propagators) corresponding to these states by

$$\begin{aligned} G_H(x,x')&=i\langle H\,|\,\hat{\eta}T\hat{\Phi}(x)\hat{\eta}\hat{\Phi}(x')\hat{\eta}\,|\,H\rangle\;, \\ G_U(x,x')&=i\langle U\,|\,T\hat{\Phi}(x)\hat{\Phi}(x')\,|\,U\rangle\;, \\ G_{UN}(x,x')&=i\langle UN\,|\,T\hat{\Phi}(x)\hat{\Phi}(x')\,|\,UN\rangle\;, \end{aligned} \tag{3.13}$$

where T denotes time ordering of the product, and where the factors $\hat{\eta}$ must be present in G_H because the state $|H\rangle$ involves negative numbers of quanta, but it is absent in G_U and G_{UN} because $|U\rangle$ and $|UN\rangle$ involve only positive numbers of quanta; see Eq. (A9g). [The first $\hat{\eta}$ in G_H is the standard one that appears in all expectation values; the second and third $\hat{\eta}$'s are dictated by those in the commutation relation (B4) for the field operators.] When, as here, the scalar curvature of the background spacetime vanishes, these Green's functions satisfy the equation

$$\Box G_{\bullet}(x,x')=-(-g)^{-1/2}\delta(x,x')\;. \tag{3.14}$$

There is a one-to-one correspondence between the choice of the state $|\bullet\rangle$ and the choice of the boundary conditions that single out the unique solution $G_{\bullet}(x,x')$ of Eq. (3.14). We show this more specifically in the next two paragraphs.

In this paper we shall define the Hartle-Hawking[25] vacuum state $|H\rangle$ as that state with unit norm, $\langle H\,|\,\hat{\eta}\,|\,H\rangle=+1$, for which the Green's function $G_H(x,x')$ [Eq. (3.13)] has the following properties: (i) for x on the future horizon $\mathscr{H}^{+}$ and x' outside the horizon, when Fourier analyzed in terms of a future-increasing affine parameter V on $\mathscr{H}^{+}$, G_H contains only positive-frequency components ($e^{-i\sigma V}$ with $\sigma>0$); and (ii) for x on the past horizon $\mathscr{H}^{-}$ and x' outside the horizon, when Fourier analyzed in terms of a future-increasing affine parameter U on $\mathscr{H}^{-}$, G_H contains only negative-frequency components. Similarly, we shall define the Unruh vacuum state $|U\rangle$ as that state with unit norm for which the Green's function $G_U(x,x')$ has the following properties: (i) for x on the past horizon $\mathscr{H}^{-}$ and x' outside the horizon, when Fourier analyzed in terms of an affine parameter U on $\mathscr{H}^{-}$, G_U contains only negative-frequency components; and (ii) for x at $\mathscr{I}^{-}$ and x' outside the horizon, when Fourier analyzed in terms of the affine advanced time v on $\mathscr{I}^{-}$, G_U contains only negative-frequency components. Finally, the state $|UN\rangle$ is obtained from the state $|U\rangle$ by the creation of $N=\{n_{I_1}, n_{I_2}, \ldots\}$ particles in the *in* modes $I_1, I_2, \ldots$,

$$|UN\rangle=\prod_{\epsilon>0}(n_I^{in}!)^{-1/2}(\hat{a}_{I_i}^{\dagger})^{n_I^{in}}\,|\,U\rangle\;, \tag{3.15}$$

and thus, like $|U\rangle$, has unit norm; and the corresponding Green's function $G_{UN}(x,x')$ differs from $G_U(x,x')$ by a homogeneous solution of Eq. (3.1) and can be written in the form

$$G_{UN}(x,x')=G_U(x,x')+i\sum_{\epsilon>0}n_I^{in}v_I(x,x')\;, \tag{3.16}$$

where

$$v_I(x,x')=v_I(x)\bar{v}_I(x')+\bar{v}_I(x)v_I(x')\;. \tag{3.17}$$

(For more information about these states see Appendixes B and C.)

If $\bullet$ is one of the states introduced above and $G_{\bullet}(x,x')$ is the corresponding Green's function, then one can write

$$G_{\bullet}(x,x')=i[S_{\bullet}(x,x')\theta(x,x')+S_{\bullet}(x',x)\theta(x',x)]\;, \tag{3.18a}$$

where $S_{\bullet}(x,x')$ is the so-called positive-frequency function

$$S_{\bullet}(x,x')=\langle\bullet\,|\,\hat{\eta}\hat{\Phi}(x)\hat{\eta}\hat{\Phi}(x')\hat{\eta}\,|\,\bullet\rangle \tag{3.18b}$$

(with $\hat{\eta}$ replaceable by unity except when $|\bullet\rangle=|H\rangle$), and $\theta(x,x')\equiv\theta(t-t')$ is $+1$ if x is to the future of $x'(t>t')$, and 0 if x is to the past of x'. In what follows it is also useful to introduce the so-called Hadamard function, defined by

$$G^{(1)}_{\bullet}(x,x')\equiv S_{\bullet}(x,x')+S_{\bullet}(x',x) . \quad (3.19)$$

It is evident that when the separation between x and x' is spacelike then

$$G_{\bullet}(x,x')=\frac{i}{2}G^{(1)}_{\bullet}(x,x') . \quad (3.20)$$

It can be shown from Eqs. (3.19), (B29), (B30), (B18)–(B20), (3.15), (3.18b), and (3.7) (with $p=w=0$ in our black-hole spacetime) that the Hadamard functions for the states $|H\rangle$, $|U\rangle$, and $|UN\rangle$ can be written as follows:

$$G^{(1)}_{H}(x,x')=\sum_{\epsilon>0}\coth(\pi\tilde{\epsilon}/\kappa)v_I(x,x') + \sum_{\tilde{\epsilon}>0}\coth(\pi\tilde{\epsilon}/\kappa)q_I(x,x') , \quad (3.21)$$

$$G^{(1)}_{U}(x,x')=\sum_{\epsilon>0}v_I(x,x')+\sum_{\tilde{\epsilon}>0}\coth(\pi\tilde{\epsilon}/\kappa)q_I(x,x') , \quad (3.22)$$

$$G^{(1)}_{UN}(x,x')=G^{(1)}_{U}(x,x')+2\sum_{\epsilon>0}n_I^{in}v_I(x,x') , \quad (3.23)$$

where $v_I(x,x')$ is given by Eq. (3.17) and

$$q_I(x,x')=q_I(x)\bar{q}_I(x')+\bar{q}_I(x)q_I(x') . \quad (3.24)$$

[Note that the coefficient of v_I in $G^{(1)}_H$ is $\coth(\pi\tilde{\epsilon}/\kappa)$, *not* $\coth(\pi\epsilon/\kappa)$ as one would infer from Eq. (3.4) of Ref. 16. Reference 16 is in error because of a too cavalier treatment of the *in* modes.]

The renormalized expectation value $T^{\bullet}_{\mu\nu}(x)$ of the stress-energy tensor for a conformal massless scalar field in a given state $|\bullet\rangle$,

$$T^{\bullet}_{\mu\nu}(x)\equiv\langle\bullet|\hat{T}_{\mu\nu}(x)|\bullet\rangle^{\text{ren}} , \quad (3.25)$$

may be computed by point-splitting techniques as follows:

$$T^{\bullet}_{\mu\nu}(x)=\lim_{x'\to x}[T^{\bullet}_{\mu\nu}(x,x')-T^{\text{div}}_{\mu\nu}(x,x')] \quad (3.26)$$

where

$$T^{\bullet}_{\mu\nu}(x,x')\equiv\hat{D}_{\mu\nu}(x,x')G^{(1)}_{\bullet}(x,x')$$
$$\equiv\tfrac{1}{6}(g_{\mu}{}^{\mu'}G^{(1)}_{\bullet}{}_{;\mu'\nu}+g_{\nu}{}^{\nu'}G^{(1)}_{\bullet}{}_{;\mu\nu'})-\tfrac{1}{12}g_{\mu\nu}g^{\rho\rho'}G^{(1)}_{\bullet}{}_{;\rho\rho'}-\tfrac{1}{12}(G^{(1)}_{\bullet}{}_{;\mu\nu}+g_{\mu}{}^{\mu'}g_{\nu}{}^{\nu'}G^{(1)}_{\bullet}{}_{;\mu'\nu'})+\tfrac{1}{48}g_{\mu\nu}(G^{(1)}_{\bullet}{}_{;\rho}{}^{;\rho}+G^{(1)}_{\bullet}{}_{;\rho'}{}^{;\rho'}) . \quad (3.27)$$

Here $g_{\mu}{}^{\mu'}(x,x')$ is the bivector of parallel transport along the geodesic connecting x and x'. The explicit expression for $T^{\text{div}}_{\mu\nu}(x,x')$ can be found in Eqs. (5.5)–(5.8) of Ref. 30. We shall not write this expression here; but we stress for future reference that, in our case where $R_{\mu\nu}=0$ [and thus where (i) the $\Box R$ issues discussed in Sec. VI of Ref. 30 are irrelevant, and (ii) the functions $H^{(2)\mu\nu}$ and $H^{(1)\mu\nu}$ of Ref. 30 vanish, cf. Eqs. (6.54) and (6.55) of Ref. 31], $T^{\text{div}}_{\mu\nu}(x,x')$ is a linear combination of terms constructed from the metric $g_{\mu\nu}$, the Riemann curvature $R_{\mu\nu\alpha\beta}$ and its covariant derivatives, and a product $\sigma_{;\mu}\sigma_{;\nu}$ of the bivector $\sigma_{;\mu}$ with itself [where $\sigma(x,x')=\frac{1}{2}s^2(x,x')$ is the biscalar geodesic interval between x and x']. It must be stressed also that each of the terms which enters $T^{\text{div}}_{\mu\nu}$ contains an even number of $\sigma_{;\mu}$.

B. Renormalized stress-energy tensor for the Hartle-Hawking vacuum

In preparation for applying the above point-splitting analysis of $T^{\bullet}_{\mu\nu}$ to a realistic black hole that is evaporating and accreting (i.e., to the quantum state $|UN\rangle$), we shall first apply it to the Hartle-Hawking vacuum state $|H\rangle$.

For the Hartle-Hawking state it is convenient to consider a special choice of the separated points x and x' for which the quantities T^{div}_{rt} and $T^{\text{div}}_{r\phi}$ vanish. Because of the symmetries of the Kerr spacetime we can put $x=(t=0, r^{*\prime}, \theta, \phi=0)$ without loss of generality, and we can then make the special choice $x'=(t=0, r^{*\prime}, \theta', \phi=0)$. It should be noted that the two-dimensional surface $\Sigma\equiv\{t=0, \phi=0\}$ is invariant under the symmetry transformation $t\to-t$, $\phi\to-\phi$; and hence the unique geodesic connecting x and x' (for x' close to x) must lie in this surface. If we denote the indices r^*, θ by A and the indices t, ϕ by X, then we have

$$\sigma^{;\mu}=\delta_A{}^{\mu}\sigma^{;A} , \quad (3.28)$$

$$g_{\mu}{}^{\mu'}=\delta_{A'}{}^{\mu'}\delta_{\mu}{}^{A}g_{A}{}^{A'}+\delta_{X'}{}^{\mu'}\delta_{\mu}{}^{X}g_{X}{}^{X'} . \quad (3.29)$$

Equation (3.28) states that the tangent vector to the geodesic connecting x and x' lies in the plane tangent to the surface Σ; Eq. (3.29) states that any tensor lying in the plane tangent to Σ will remain tangent to Σ after parallel transport from x to x' along the geodesic, and any tensor orthogonal to Σ will remain orthogonal.

Because of the invariance of the Kerr metric under the simultaneous inversion $t\to-t$, $\phi\to-\phi$, those components of the metric, the Riemann curvature tensor, and the curvature tensor's covariant derivatives which contain an odd number of X indices must vanish. This property and the fact that only an even number of $\sigma^{;\mu}$ enters the expression for $T^{\text{div}}_{\mu\nu}$ imply, after using Eq. (3.28), that

$$T^{\rm div}_{AX}=0. \tag{3.30}$$

Consequently, for our chosen point splitting we have

$$T^{\bullet}_{AX}(x)=T^{\bullet}_{AX}(x,x). \tag{3.31}$$

Next we show that for the special case of the Hartle-Hawking vacuum the tensor $T^{H}_{AX}(x,x)$ [Eq. (3.27)] vanishes:

$$T^{H}_{AX}(x)=T^{H}_{AX}(x,x)=0\ . \tag{3.32}$$

For this purpose we consider the symmetry transformation

$$x\equiv(t,\ r^*,\ \theta,\ \phi)\rightarrow x_1\equiv(-t,\ r^*,\ \theta,\ -\phi) \tag{3.33}$$

and remark that the Hadamard function $G^{(1)}_{H}(x,x')$ [Eq. (3.21)] is invariant under this transformation

$$G^{(1)}_{H}(x_1,x_1{}')=G^{(1)}_{H}(x,x')\ . \tag{3.34}$$

This invariance is a consequence of the following relations connecting the functions $v_I(x)$, $q_I(x)$ and $v^1_I(x)\equiv\bar v_I(x_1)$, $q^1_I(x)\equiv\bar q_I(x_1)$:

$$v^1_I(x)=\overline{A}{}^{+}_{I}v_I(x)+\left|\frac{\tilde\epsilon}{\epsilon}\right|^{1/2}\overline{B}{}^{+}_{I}[\theta(\tilde\epsilon)q_I(x)+\theta(-\tilde\epsilon)\bar q_{I_1}(x)] \quad \text{for } \epsilon>0\ ,$$

$$q^1_I(x)=\overline{A}{}^{-}_{I}q_I(x)+\left|\frac{\epsilon}{\tilde\epsilon}\right|^{1/2}\overline{B}{}^{-}_{I}[\theta(\epsilon)v_I(x)+\theta(-\epsilon)\bar v_{I_1}(x)] \quad \text{for } \tilde\epsilon>0\ , \tag{3.35}$$

where $I\equiv\{\epsilon,\ l,\ m\}$ and $I_1\equiv\{-\epsilon,\ l,\ -m\}$. In order to prove these relations one need only compare the asymptotics of v^1_I and q^1_I with Eqs. (3.2) and (3.4), and note that v_I, q_I, $\bar v_{I_1}$, $\bar q_{I_1}$ all have, at all radii r, the same t, θ, ϕ dependences and all satisfy the same radial differential equation. In our proof of Eq. (3.32) we shall need (3.34) rewritten in the equivalent form

$$\begin{aligned}G^{(1)}_{H}(x,x')&=G^{(1)}_{H}(t-t',\phi-\phi',r^*,r^{*'},\theta,\theta')\\&=G^{(1)}_{H}(t'-t,\phi'-\phi,r^*,r^{*'},\theta,\theta')\ .\end{aligned} \tag{3.36}$$

Next we note that the metric (2.22) is of the form

$$ds^2=g_{AB}(x^C)dx^A dx^B+g_{XY}(x^C)dx^X dx^Y \tag{3.37}$$

and hence $\Gamma^{B}_{AX}=0$. Equations (3.36) and (3.37) imply that, for x and x' lying in the two-dimensional surface Σ,

$$\begin{aligned}&G^{(1)}_{H;X}=G^{(1)}_{H}{}_{;X'}=0\ ,\\&G^{(1)}_{H}{}_{;AX}=G^{(1)}_{H}{}_{;AX'}=G^{(1)}_{H}{}_{;A'X}=G^{(1)}_{H}{}_{;A'X'}=0\ .\end{aligned} \tag{3.38}$$

Equations (3.27), (3.29), and (3.38) then imply, for x and x' lying in Σ,

$$T^{H}_{AX}(x,x')=0 \tag{3.39}$$

and hence the relation (3.32) is satisfied at all points x outside the horizon.

Equations (3.32) and (3.31) imply, in accord with an assertion by Zannias and Israel,[17] that in the Hartle-Hawking vacuum, and at any point x outside the horizon,

$$\langle H\mid \hat T_{AX}(x)\mid H\rangle^{\rm ren}\equiv T^{H}_{AX}(x)=0\ , \tag{3.40}$$

where A runs over r^*, θ and X runs over t, ϕ. Stated in words: *In the Hartle-Hawking vacuum the fluxes of redshifted energy, $-T^{H}_{At}$, and angular momentum, $T^{H}_{A\phi}$, in the $A=r^*$ and $A=\theta$ directions vanish everywhere outside the horizon.*

In order to proceed further we must make an unproved assumption—which, however, we (like Candelas, Chrzanowski, and Howard[16]) are convinced must be true: *We must assume that $T^{H}_{\mu\nu}\equiv\langle H\mid\hat T_{\mu\nu}\mid H\rangle^{\rm ren}$ is regular in the vicinity of the future horizon $\mathcal{H}^+$*; i.e., that all its components are finite and continuous in any regular basis (i.e., in any basis whose basis vectors $\mathbf{e}_\mu$ are continuous, finite, and linearly independent throughout a neighborhood of $\mathcal{H}^+$). We see no way this assumption can fail, given the definition of the state $|H\rangle$ [paragraph following Eq. (3.14)]; and the assumption is known to be correct in the case of a nonrotating, Schwarzschild hole.[32] Moreover, a theorem due to Kay and Wald [18] tells us that for Kerr (or, more relevantly, for Kerr with a wall inserted around the horizons to hold the field Φ away from the "velocity of light surface"; cf. the end of Sec. IV), if there is any quasifree (i.e., "generalized vacuum") state which is invariant under the isometry generated by $(\partial/\partial t)_{r,\theta,\tilde\phi}$ and which is regular everywhere including the horizons $\mathcal{H}^+$ and $\mathcal{H}^-$, then that state is unique and is equal to $|H\rangle$ as we have defined it [paragraph following Eq. (3.14)]. In a sense this theorem is the converse of our unproved assumption: regularity on the horizons implies our definition of $|H\rangle$, instead of our definition implying regularity.

Accept, then, our assumption of regularity on the future horizon $\mathcal{H}^+$. As one approaches $\mathcal{H}^+$, the time and radial basis vectors of the ZAMO proper reference frame both asymptote to $\alpha^{-1}\mathbf{l}$, where $\mathbf{l}$ is the horizon generator. Correspondingly, near $\mathcal{H}^+$, the ZAMO basis takes the form

$$\begin{aligned}&\mathbf{e}_{\hat 0}=\left[\frac{1}{\alpha}\frac{\partial}{\partial t}\right]_{r,\,\theta,\,\tilde\phi}=\frac{1}{\alpha}\mathbf{l}-\alpha\mathbf{n}\ ,\\&\mathbf{e}_{\hat r}=\frac{\Delta^{1/2}}{\rho}\frac{\partial}{\partial r}=\frac{1}{\alpha}\mathbf{l}+\alpha\mathbf{n}\ ,\\&\mathbf{e}_{\hat\theta}=\frac{1}{\rho}\frac{\partial}{\partial\theta}\ ,\quad \mathbf{e}_{\hat\phi}=\frac{1}{\varpi}\frac{\partial}{\partial\tilde\phi}\ ,\end{aligned} \tag{3.41}$$

where $\{\mathbf{l},\ \mathbf{n},\ \mathbf{e}_{\hat\theta},\ \mathbf{e}_{\hat\phi}\}$ is a regular basis and α is the lapse function [Eq. (2.22)]. This, together with Eq. (3.40) and our assumption that $T^{H}_{\mu\nu}$ is regular, implies that the components of $T^{H}_{\mu\nu}$ on the ZAMO basis have the magnitudes

$$\begin{aligned}&T^{H}_{\hat0\hat0}=-T^{H}_{\hat r\hat r}\sim1\ ,\quad T^{H}_{\hat0\hat r}=0\ ,\quad T^{H}_{\hat0\hat\phi}=O(\alpha)\ ,\\&T^{H}_{\hat r\hat\phi}=0\ ,\quad T^{H}_{\hat0\hat\theta}=0\ ,\quad T^{H}_{\hat r\hat\theta}=O(\alpha)\ ,\\&T^{H}_{\hat\phi\hat\phi}\sim1\ ,\quad T^{H}_{\hat\theta\hat\theta}\sim1\ ,\quad T^{H}_{\hat\phi\hat\theta}=0\ ,\end{aligned} \tag{3.42}$$

which are all less than or of order the curvature coupling corrections to the vacuum polarization [Eq. (2.32)]. This, together with the fact (proved in Appendix C and Sec. IV B) that near-horizon ZAMO's measure the field $\hat{\Phi}$, in the Hartle-Hawking state $|H\rangle$, to be precisely thermal, guarantees that Eq. (1.1) correctly describes the renormalized stress-energy tensor of the Hartle-Hawking state. Moreover, Eqs. (3.42) and (2.43) guarantee that the flux of energy and angular momentum across the stretched horizon vanish in the Hartle-Hawking state,

$$\langle H | T_{ll} | H \rangle^{\rm ren}_{\mathcal{H}^+} = \langle H | T_{l\phi} | H \rangle^{\rm ren}_{\mathcal{H}^+} = 0 \ , \qquad (3.43)$$

and the horizon therefore does not evolve. Finally, Eq. (3.42) implies that not only is $T^H_{\mu\nu}$ regular on the future horizon $\mathcal{H}^+$, it also (as one should expect) is regular on the past horizon $\mathcal{H}^-$ and on their bifurcation 2-surface $\mathcal{H}^+ \cap \mathcal{H}^-$.

C. Renormalized stress-energy tensor for the state $|UN\rangle$

We turn, finally, to a proof that in the state $|UN\rangle$, where the *in* modes contain N quanta, the renormalized stress-energy tensor has the standard form (1.1).

First we remark that Eqs. (3.21)—(3.23) allow us to write $G^{(1)}_{UN}(x,x')$ in the form

$$G^{(1)}_{UN}(x,x') = 2 \sum_{\epsilon>0} (n_I^{in} - n_I^{\rm th}) v_I(x,x') + G^{(1)}_H(x,x') \ , \qquad (3.44)$$

where n_I^{in} is the number of quanta "accreting" onto the black hole via the I *in* mode, and

$$n_I^{\rm th} \equiv \frac{1}{e^{2\pi\tilde{\epsilon}/\kappa} - 1} \qquad (3.45)$$

is the number of quanta the I *in* mode would have if it were perfectly thermalized [Eqs. (2.51b) and (C23)]. Equation (3.44), together with the relation

$$T^{UN}_{\mu\nu}(x) - T^H_{\mu\nu}(x) = \lim_{x' \to x} \hat{D}_{\mu\nu}(x,x')[G^{(1)}_{UN}(x,x') - G^{(1)}_H(x,x')] \qquad (3.46)$$

[which follows from Eqs. (3.26) and (3.27)], implies that

$$T^{UN}_{\mu\nu}(x) = \sum_{\epsilon>0} (n_I^{in} - n_I^{\rm th}) T^{I\,in}_{\mu\nu}(x) + T^H_{\mu\nu}(x) \ , \qquad (3.47)$$

where

$$T^{I\,in}_{\mu\nu}(x) = v_{I,\mu}\bar{v}_{I,\nu} + v_{I,\nu}\bar{v}_{I,\mu} - \tfrac{1}{3}(v_I\bar{v}_I)_{;\mu\nu} - \tfrac{1}{3} g_{\mu\nu} v_{I,\rho}\bar{v}_{I,\rho} \qquad (3.48)$$

is the stress-energy tensor associated with each I *in* quantum.

Equations (3.47) and (3.48) are exact, and are valid at all locations outside the horizon. Moreover, they agree with—and constitute a proof of—the general near-zone expressions (1.1), if we identify $T^H_{\mu\nu}$ with the curvature-coupling corrections to the stress-energy of vacuum polarization:

$$(T^{I\ \rm vac\ pol}_{\mu\nu})_{\rm curvature\text{-}coupling\ corrections} = T^H_{\mu\nu} \ . \qquad (3.49)$$

This identification, together with Eqs. (3.42), furnishes a proof that the magnitudes of the curvature-coupling corrections are as given in Eq. (2.32).

In the vicinity of the horizon, where v_I has the asymptotic form (3.2), expression (3.48) for $T^{I\,in}_{\mu\nu}$ assumes the form (2.41a) and (2.41b), which we derived in Sec. II from the equivalence principle, and expression (3.47) for $T^{UN}_{\mu\nu}$ takes the equivalence-principle-derived form (2.49), aside from the curvature-coupling corrections. Correspondingly, by taking the limit $\alpha \to 0$ of expression (3.47) we obtain for the fluxes of energy and angular momentum across the future horizon [Eqs. (2.50)]

$$T^{UN}_{ll} = \sum_{\epsilon>0} (n_I^{in} - n_I^{\rm th}) \frac{\tilde{\epsilon}}{2\pi} (1 - |A_I^+|^2) \frac{|S_{lm}|^2}{A_H} \ , \qquad (3.50)$$

$$T^{UN}_{l\phi}{}' = - \sum_{\epsilon>0} (n_I^{in} - n_I^{\rm th}) \frac{m}{2\pi} (1 - |A_I^+|^2) \frac{|S_{lm}|^2}{A_H} \ . \qquad (3.51)$$

IV. SINGULAR NATURE OF THE HARTLE-HAWKING VACUUM STATE AND ITS REPAIR

Throughout this paper, until now, we have performed formal manipulations of the Hartle-Hawking vacuum state $|H\rangle$ as though it were perfectly well-behaved. However (as Fredenhagen[33] has conjectured and Kay and Wald have shown, see below), for a rotating black hole $|H\rangle$ is so ill behaved that, strictly speaking, it does not really exist. In Section IV A we shall explore the bad behavior of $|H\rangle$ and shall show that it has no influence at all on this paper's derivations of the properties of the states $|U\rangle$ and $|UN\rangle$. Then in Section IV B we shall discuss methods of removing the bad behavior from $|H\rangle$ and turning it into a "modified," perfectly respectable quantum state $|H_M\rangle$ with all the good properties that this paper has attributed to $|H\rangle$.

A. Singular properties of $|H\rangle$

Consider a rotating black hole surrounded by a conformal scalar field Φ that is in the state $|H\rangle$; and, by contrast with Sec. III and the Appendixes, adopt (temporarily) the near-horizon viewpoint on all superradiant modes. Then, as is shown in Appendix C, the field Φ is characterized by a thermal probability distribution for ZAMO-measured quanta: For any I *up* or I *in* mode the probability of containing n ZAMO-measured quanta is given by Eq. (C19b), (C20b), or (C21b) [with the change to near-horizon viewpoint, $\tilde{\epsilon} \to -\tilde{\epsilon}$, $n \to -(n+1)$]:

$$p_n = p_n^{\rm th} = (1 - e^{-\tilde{\epsilon}/T_H}) e^{-n\tilde{\epsilon}/T_H} \quad \text{for } n \ge 0, \ \ \tilde{\epsilon} \equiv \epsilon - m\Omega_H > 0 \ . \qquad (4.1)$$

Since any mode of Φ that is an eigenstate or near-eigenstate of $\partial/\partial t$ and $\partial/\partial\phi$ with eigenvalues ϵ and m will be expandable in terms of *in* and *up* modes with that same ϵ and m, any such mode will have its quanta distributed in accord with the thermal probability distribution

(4.1). Each of these quanta, when measured by an observer who orbits the hole with angular velocity Ω at a radius and latitude where the lapse function is α, will exhibit a locally measured energy

$$\epsilon_{\rm loc}=(\gamma/\alpha)(\epsilon-m\Omega) \ . \tag{4.2a}$$

Here γ is the time-dilation factor associated with the observer's velocity relative to the ZAMO's:

$$\gamma=(1-v^2)^{-1/2}\ , \quad v=\alpha^{-1}(\Omega-\omega)\varpi \ ; \tag{4.2b}$$

cf. Eq. (2.22). Comparison of Eqs. (4.1) and (4.2a) shows that the observer will see the field's quanta to be anisotropically distributed (i.e., will see p_n to depend not only on the locally measured energy $\epsilon_{\rm loc}$ of the quanta but also on their angular momentum), unless the observer has angular velocity Ω equal to that of the horizon, Ω_H. Moreover, when $\Omega=\Omega_H$, the probability distribution's form

$$\begin{aligned} &p_n=(1-e^{\epsilon_{\rm loc}/T_{\rm loc}})e^{n\epsilon_{\rm loc}/T_{\rm loc}}\ , \quad T_{\rm loc}=(\gamma/\alpha)T_H\ ,\\ &\gamma=(1-v^2)^{-1/2}\ , \quad v=\alpha^{-1}(\Omega_H-\omega)\varpi\ , \end{aligned} \tag{4.3}$$

is that of an isotropic, thermal reservoir with locally measured temperature $T_{\rm loc}$. Thus, *in the reference frame of an observer who orbits the hole with the same angular velocity Ω_H as the horizon, the state $|H\rangle$ displays the isotropic, perfectly thermal distribution of quanta (4.3) with locally measured temperature $T_{\rm loc}=(\gamma/\alpha)T_H$.* This is a precise version of the statement that "the quanta in the state $|H\rangle$ constitute a perfect thermal bath that rotates rigidly with and is in thermodynamic equilibrium with the horizon."

This description of the quanta in $|H\rangle$ suggests (in accord with a conjecture by Fredenhagen[33]) that something pathological must happen at and outside the hole's "velocity-of-light surface"—i.e., at radii where, in order to corotate with the horizon, an observer must have $v\geq 1$ and thus must move on a spacelike world line. In the remainder of this section we shall investigate that pathology for a black hole which rotates arbitrarily slowly. Slow rotation will simplify our analysis since it places the velocity-of-light surface at arbitrarily large radii where (i) spacetime is arbitrarily flat, and (ii) the *up* modes make an arbitrarily small contribution to the field Φ and thus can be ignored.

To simplify further our investigation we shall convert the only modes that contribute (the *in* modes) from spheroidal to cylindrical coordinates, where they take the form

$$v_I=\frac{1}{\sqrt{8\pi^2}}J_m[(\epsilon^2-k^2)^{1/2}\varpi]e^{ikz}e^{im\phi}e^{-i\epsilon t}\ , \tag{4.4a}$$

and we shall switch back to the distant-observer viewpoint so that $I=\{\epsilon,k,m\}$ runs over the ranges

$$\begin{aligned} &0<\epsilon<+\infty \quad \text{(continuous)}\ ,\\ &-\epsilon<k<+\epsilon \quad \text{(continuous)}\ ,\\ &-\infty<m<+\infty \quad \text{(integer)}\ . \end{aligned} \tag{4.4b}$$

Because we are studying the field Φ in a region where spacetime is arbitrarily flat, the renormalization of the stress-energy tensor takes its standard flat-spacetime form: we must simply remove from each mode the contributions of the one-half quantum of zero-point energy. Within our formalism this can be verified, and the renormalized stress-energy tensor can be derived by noting that the Boulware vacuum state $|B\rangle$ coincides with the Minkowski vacuum far from the hole and thus has vanishing renormalized stress-energy tensor; and, therefore,

$$T^H_{\mu\nu}=T^H_{\mu\nu}-T^B_{\mu\nu}=\langle H\,|\,\hat T_{\mu\nu}\,|\,H\rangle-\langle B\,|\,\hat T_{\mu\nu}\,|\,B\rangle\ ,$$

which can be expressed as follows [cf. Eqs. (3.26), (3.27), (3.21), (3.19), and (B28)]:

$$T^H_{\mu\nu}=\sum_I T^{I\,in}_{\mu\nu}(x)n_I^{\rm th}\ , \tag{4.5a}$$

where the sum is over the values of I in the range (4.4b), $n_I^{\rm th}$ is the mean thermal occupation number in the distant-observer viewpoint

$$n_I^{\rm th}=\frac{1}{e^{(\epsilon-m\Omega_H)/T_H}-1}\ , \tag{4.5b}$$

and $T^{I\,in}_{\mu\nu}(x)$ is given by Eq. (3.48).

Inside the velocity-of-light surface, i.e., at ϖ less than $1/\Omega_H$ so the velocity of the reservoir's mean rest frame $v=\varpi\Omega_H$ is less than unity, one can verify that modes with $k^2+(m/\varpi)^2<\epsilon^2$ (which are forbidden in the classical, geometric optics limit) contribute negligibly to the renormalized stress-energy tensor: their contributions are strongly suppressed by the Bessel function J_m in v_I. The superradiant modes all lie in this suppressed regime and thus contribute negligibly. As a result, inside the velocity-of-light surface the renormalized stress-energy tensor (4.5), as measured in the hole's asymptotic rest frame, is the standard one for a thermal reservoir with temperature $T=\gamma T_H$ [Eq. (4.3)] and with velocity v:

$$\begin{aligned} &T^H_{00}=\frac{\pi^2}{30}(1+\tfrac{1}{3}v^2)\gamma^2(\gamma T_H)^4\ ,\\ &T^H_{0\hat\phi}=-\frac{4}{3}\frac{\pi^2}{30}v\gamma^2(\gamma T_H)^4\ ,\\ &T^H_{\hat\phi\hat\phi}=\frac{\pi^2}{30}(\tfrac{1}{3}+v^2)\gamma^2(\gamma T_H)^4\ ,\\ &T^H_{zz}=T^H_{\varpi\varpi}=\frac{1}{3}\frac{\pi^2}{30}(1+v^2)\gamma^2(\gamma T_H)^4\ . \end{aligned} \tag{4.6}$$

Note that as one moves outward toward the velocity-of-light surface, this stress-energy tensor diverges as γ^6.

Outside the velocity-of-light surface, where $v>1$, the contributions of the superradiant modes are not negligible. In fact, in the infinite region of phase space

$$\left[k^2+\frac{(m\Omega_H)^2}{v^2}\right]^{1/2}<\epsilon<m\Omega_H \tag{4.7}$$

the geometric optics approximation is valid (because of the first inequality), the modes are all superradiant (because of the second inequality), and $n_I^{\rm th}$ is ≤ -1 [Eq. (4.5b)]; and, as a result, this region of phase space carries an infinite, negative, renormalized energy density T^H_{00} [Eq. (4.5)]. That this conclusion makes no physical sense is an indication that, not only does the state $|H\rangle$ become singular as one approaches the velocity of light surface from inside; at least as described by our formalism, it remains singular everywhere outside that surface.

Kay and Wald[18] recently have used algebraic quantum field theory to prove (with a higher level of rigor than that to which we aspire in this paper) that in the globally hyperbolic region I∪I′∪II∪II′ of Kerr spacetime there is at most one quasifree (i.e., "generalized vacuum") state which is invariant under the isometry generated by $(\partial/\partial t)_{r,\theta,\bar\phi}$ and which is regular everywhere, in the sense that its Hadamard function is well-behaved; and, moreover, that if this state exists, it is the state $|H\rangle$ as we have defined it [paragraph following Eq. (3.14)]. Kay and Wald have gone on to show that, in fact, Kerr spacetime possesses no such quasifree states.

The failure of the state $|H\rangle$, as we define and analyze it, to satisfy the properties demanded by Kay and Wald is caused by a pathology of its Hadamard function $G_H^{(1)}(x,x')$ at and outside the velocity-of-light surface. At least this is so for a slowly rotating Kerr spacetime. One can verify this by noting that (i) all well-behaved Hadamard functions must have the same standard singularity structure (one confined to points x and x' that lie on each others' light cones);[34] (ii) the Boulware vacuum $|B\rangle$ far from the horizon, being coincident with the Minkowski vacuum, must have a well-behaved Hadamard function; (iii) therefore, $G_H^{(1)}(x,x')$ will be well-behaved if and only if [cf. Eqs. (3.21), (3.18a), (B28), and (3.17)]

$$G_H^{(1)}(x,x')-G_B^{(1)}(x,x')=2\sum_I n_I^{\text{th}} v_I(x,x') \tag{4.8}$$

is nonsingular for all x, x'; (iv) for x and x' inside the velocity-of-light surface expression (4.8) is nonsingular because the Bessel function in v_I suppresses all superradiant contributions and the exponent in n_I^{th} suppresses all large-momentum contributions; (v) but for x and x' outside the velocity-of-light circle the same superradiant modes which produce the negatively infinite renormalized energy density also produce singular behaviors in Eq. (4.8).

One might worry that these pathologies of the Hartle-Hawking state will invalidate our derivations of the properties of the states $|U\rangle$ and $|UN\rangle$ (Sec. III and Appendixes B and C). Not so. In those derivations there is only one apparent reliance on the state $|H\rangle$, and that reliance is illusory: In Eq. (3.44) $G_{UN}^{(1)}(x,x')$ is expressed in terms of $G_H^{(1)}(x,x')$; this expression is then used to derive $T_{\mu\nu}^{UN}(x)$ in terms of $T_{\mu\nu}^{H}(x)$ [Eq. (3.47)]; and the vanishing of the relevant components of $T_{\mu\nu}^{H}(x)$ on the horizon is then invoked. In fact, each of these steps is valid if one merely *defines* $G_H^{(1)}(x,x')$ by Eq. (3.21) (which is nonsingular in the vicinity of the horizon), and defines $T_{\mu\nu}^{H}$ in terms of this $G_H^{(1)}$ by expressions (3.26) and (3.27). It is not necessary to invoke any connection between the singular state $|H\rangle$ and these $G_H^{(1)}$ and $T_{\mu\nu}^{H}$. Thus, our analyses of $|U\rangle$ and $|UN\rangle$ are immune to the defects of $|H\rangle$.

B. The modified Hartle-Hawking state $|H_M\rangle$

Although the state $|H\rangle$ is singular, one can modify it (in a variety of ways) to make it well behaved, while retaining its key defining property of having a Feynman propagator with positive (affine-parameter) frequencies on $\mathcal{H}^+$ and negative frequencies on $\mathcal{H}^-$, and while retaining its key equivalence-principle property of appearing perfectly thermal to near-horizon ZAMO's. In this section we shall explore such modified states $|H_M\rangle$.

The first step in the modification of $|H\rangle$ is to modify the spacetime in which the quantum field $\hat\Phi$ lives in such a way that (a) the modification is bounded away from the horizon, i.e., near the horizon the modified spacetime remains precisely Kerr, and (b) the ZAMO-measured velocity v of the thermal atmosphere is everywhere less than the speed of light.

The nicest such modification, conceptually, is to place a stationary, axisymmetric, perfectly reflecting "mirror" around the black hole, somewhere inside the velocity-of-light surface. Then, so far as the field $\hat\Phi$ is concerned, the modified spacetime terminates at that mirror (with a boundary condition $\Phi=0$ there). Equally satisfactory in principle is a modification of the metric coefficients outside some radius r_0 so as to keep $|v| = |\alpha^{-1}(\Omega-\omega)\varpi| < 1$ everywhere. For example, the functions α, ρ, Δ, and ϖ appearing in the metric (2.22) might be kept in precisely Kerr form, while ω might be altered outside r_0, including $\omega\to\Omega$ as $r\to\infty$ so that "infinity" rotates with the same angular velocity as the horizon and there is no superradiance.

For any such modified spacetime we define the Hartle-Hawking state $|H_M\rangle$ in precisely the same way as $|H\rangle$ was defined for Kerr: It is the state whose Green's function (Feynman propagator) $G_{H_M}(x,x')$ in the modified spacetime has the standard positive- and negative-frequency behavior on the future and past horizons [paragraph following Eq. (3.14)]. The Kay-Wald uniqueness theorem tells us there is at most one such state $|H_M\rangle$; and our condition (b) that $|v|<1$ everywhere, plus the discussion of the last subsection, strongly suggests that there will be precisely one. We shall assume so.

Because the spacetime was modified only in a region bounded away from the horizon, for events x and x' that are arbitrarily close to the horizon and arbitrarily close to each other, the propagator $G_{H_M}(x,x')$ and corresponding Hadamard function $G_{H_M}^{(1)}(x,x')$ are arbitrarily insensitive to the detailed method of modification. More specifically, they are the same, aside from tiny differences that should be no larger than curvature-coupling effects, whether the modification is achieved by means of a mirror, or by changing ω outside the radius r_0, or by any other method satisfying conditions (a) and (b) above. Moreover, these G_{H_M} and $G_{H_M}^{(1)}$ can differ only by curvature-coupling-magnitude terms from the propagator $G_H(x,x')$ and Hadamard function $G_H^{(1)}(x,x')$ of Eqs. (B30), (3.18), and (3.21), which we derived by formal manipulations of the unmodified, singular state $|H\rangle$. This is because that $G_H(x,x')$ satisfies the same boundary conditions as $G_{H_M}(x,x')$ on the horizon, and it satisfies the same differential equation (3.14) throughout the region that is common to the modified and unmodified spacetimes. This guarantees that the renormalized stress-energy tensor in the state $|H_M\rangle$, like that computed from our formally derived $G_H^{(1)}$, must vanish aside from curvature-coupling terms.

That $|H_M\rangle$ appears perfectly thermal to near-horizon ZAMO's (and thus can replace $|H\rangle$ in the equivalence-principle analysis of Sec. II) one can see by constructing its density matrix ρ_{H_M} in the manner of Appendix C. For this purpose introduce into the modified spacetime a complete, orthonormal basis $\{u_I^M(x)\}$ of solutions to $\Box\Phi=0$. For simplicity adopt the near-horizon conventions and viewpoint. Insist that $u_I^M(x)$ be an eigenstate of $(\partial/\partial t)_{\bar\phi}$ with eigenfrequency $\tilde\epsilon$ and an eigenstate of $\partial/\partial\bar\phi$ with eigenvalue m so that $I=\{\tilde\epsilon, m, \ldots\}$. As in unmodified Kerr, so also here, because the ZAMO's see these modes to oscillate sinusoidally with respect to their own proper time, the quanta measured by the ZAMO's particle detectors are these modes' quanta; cf. the fourth paragraph of Sec. I. Denoting by $\hat a_I^M$ the annihilation operator associated with the mode u_I^M and by $\hat n_I^M=\hat a_I^{M\dagger}\hat a_I^M$ the number operator, we obtain, by a calculation completely analogous to that in Appendix C,

$$\hat\rho_{H_M}=\prod_{\tilde\epsilon>0}(1-e^{-\tilde\epsilon/T_H})\exp(-\hat n_I^M\tilde\epsilon/T_H)\ . \qquad (4.9)$$

As in Appendix C, so also here, this density operator has the interpretation that near-horizon ZAMO's measure all modes to be perfectly thermally populated.

In summary, arbitrarily near the horizon and for events x and x' that are arbitrarily close to each other, the modified state $|H_M\rangle$ is an arbitrarily good surrogate for the singular state $|H\rangle$. Elsewhere (Secs. III.1.2 and III.1.3 of Ref. 20) one of the authors constructs the state $|H_M\rangle$ explicitly and studies its properties, for the case of a Kerr hole surrounded by a perfectly reflecting mirror at a θ- and ϕ-independent radius $r=r_0$.

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APPENDIX A: CONVENTIONS AND VIEWPOINTS ON SUPERRADIANT MODES

In quantum-field-theory calculations around a Kerr black hole, the superradiant (SR) modes often cause computational headaches. In this appendix we attempt to elucidate those headaches and we indicate methods to circumvent or avoid them. We do this by introducing with some care the concepts of *conventions* and *viewpoints*, which one can adopt when setting up the computational formalism, and by distinguishing two *types* of quantum states in which a SR mode can find itself.

Throughout our discussion we shall restrict attention to a specific mode of a massless scalar field—i.e., a mode characterized by specific values of the quantum numbers l, $|m|$, $|\epsilon|$ and by specific boundary conditions such as "*up* in region I of the extended Kerr spacetime" (Fig. 2) (in which case the mode is q_I) or "*out* in region I′" (in which case it is w_I) or "↑ in box of Sec. II." (Only the moduli, not the signs, of ϵ and m enter into the choice of the mode because the signs—which are the same for ϵ and m since the mode is SR—depend on one's choice of *viewpoint*; see below.)

We shall define the two *types* of quantum states for our chosen SR mode as follows: For an energy eigenstate the type is determined by the sign of the state's total "energy-at-infinity," $E_\infty=(n+\frac{1}{2})\epsilon$ (where n is the state's number of quanta). Specifically, the state is said to be a

$$\text{positive-}E_\infty \text{ state } (\mathcal{J}\text{ state}) \text{ if } E_\infty>0\ , \qquad (\text{A1a})$$

$$\text{negative-}E_\infty \text{ state } (\mathcal{H}\text{ state}) \text{ if } E_\infty<0\ . \qquad (\text{A1b})$$

(The shorthand phrases "$\mathcal{J}$ state" and "$\mathcal{H}$ state" are motivated by the fact that classically a positive-E_∞ wave packet made of SR modes can exist near $\mathcal{J}^+$ and $\mathcal{J}^-$ but not near $\mathcal{H}^+$ and $\mathcal{H}^-$, while a negative-E_∞ SR wave packet can exist near $\mathcal{H}^+$ and $\mathcal{H}^-$ but not near $\mathcal{J}^+$ and $\mathcal{J}^-$.) Any pure state which is a superposition of $\mathcal{J}$ states ($\mathcal{H}$ states) will be called a $\mathcal{J}$ state ($\mathcal{H}$ state). Similarly, any mixed state which is a mixture of $\mathcal{J}$ states ($\mathcal{H}$ states) will be called a $\mathcal{J}$ state ($\mathcal{H}$ state). As an example, for any SR *in* mode $|UN\rangle$ is a $\mathcal{J}$ state (positive-E_∞ state), while $|H\rangle$ is a $\mathcal{H}$ state (negative-E_∞ state); see Appendix C for proofs, and see the paragraphs following Eq. (3.11d) for the definitions of $|UN\rangle$ and $|H\rangle$.

When setting up the quantum-field-theory formalism for our chosen mode, we can adopt one of two *viewpoints*.

distant-observer viewpoint ($\mathcal{J}$ viewpoint):

$$\epsilon>0 \ \ (\text{and thus } \tilde\epsilon<0)\ ; \qquad (\text{A2a})$$

near-horizon viewpoint ($\mathcal{H}$ viewpoint):

$$\tilde\epsilon>0 \ \ (\text{and thus } \epsilon<0)\ . \qquad (\text{A2b})$$

As examples, in Sec. III and Appendixes A and B we adopt the $\mathcal{J}$ viewpoint for the *in* mode v_I and the $\mathcal{H}$ viewpoint for the *up* mode q_I; and in Sec. II we adopt the $\mathcal{H}$ viewpoint for the ↑ and ↓ modes in our near-horizon box—and then in Sec. II D we transform those modes to the $\mathcal{J}$ viewpoint.

As we have seen in Sec. II D, the number operator $\hat n$ for a SR mode is not always expressed in terms of the creation and annihilation operators $\hat f^\dagger$, $\hat f$ in the familiar way $\hat n=\hat f^\dagger\hat f$. There may be a sign reversal, $\hat n=\pm\hat f^\dagger\hat f$, which goes hand-in-hand with a sign reversal in the commutation relations for $\hat f$ and $\hat f^\dagger$. Correspondingly, we identify two different choices of *convention* that can be made when setting up the quantum-field-theory formalism for a SR mode:

distant-observer convention ($\mathscr{I}$ convention):

$$[\hat{f},\hat{f}^{\dagger}]=\mathrm{sgn}(\epsilon), \quad \hat{n}=\mathrm{sgn}(\epsilon)\hat{f}^{\dagger}\hat{f} \; ; \quad \text{(A3a)}$$

near-horizon convention ($\mathscr{H}$ convention):

$$[\hat{f},\hat{f}^{\dagger}]=\mathrm{sgn}(\tilde{\epsilon}), \quad \hat{n}=\mathrm{sgn}(\tilde{\epsilon})\hat{f}^{\dagger}\hat{f} \; . \quad \text{(A3b)}$$

Because, for any SR mode, ϵ and $\tilde{\epsilon}$ have opposite signs, these two conventions are the opposite of each other. As an example, in Sec. III and Appendixes B and C we adopt the $\mathscr{I}$ convention (distant-observer convention) for the *in* modes v_I, and the $\mathscr{H}$ convention (near-horizon convention) for the *up* modes q_I; cf. Eq. (3.8) and associated discussion. As another example, in Sec. II we adopt the $\mathscr{H}$ convention for the $\uparrow$ and the $\downarrow$ modes.

When the type of state being studied, the viewpoint, and the convention all agree (all $\mathscr{I}$ or all $\mathscr{H}$), quantum field theory takes on its standard form. However, it occasionally is useful to work with states, viewpoints, and conventions that disagree; examples are encountered in Sec. II D and Appendix C of this paper. In the remainder of this appendix we shall derive the transformation laws that take quantum field theory from its standard form for agreeing states, viewpoints, and conventions to its nonstandard forms appropriate for states, viewpoints, and/or conventions that disagree. First we shall study the transformation of viewpoint with convention and state-type held fixed. Then we shall discuss the peculiarities that arise when the state-type and convention disagree. Finally, motivated by this discussion, we shall study the transformation of convention with viewpoint and state-type held fixed.

The *transformation of viewpoint with convention and state-type held fixed* is straightforward. In Sec. II D we derived that transformation for a specific case: $\mathscr{H}$ viewpoint transformed to $\mathscr{I}$ viewpoint holding the convention and state-type fixed as $\mathscr{H}$. The derivation in the general case (arbitrary, but fixed convention, arbitrary but fixed state-type, arbitrary initial viewpoint) is essentially identical to the derivation in Sec. II D, and it produces essentially the same conclusions: Use the subscript "old" to denote quantities defined in the "old" viewpoint and "new" to denote those in the "new" viewpoint. Then define new quantum numbers for the chosen SR mode by

$$\epsilon_{\mathrm{new}}\equiv-\epsilon_{\mathrm{old}} \; , \quad l_{\mathrm{new}}\equiv+l_{\mathrm{old}} \; , \quad m_{\mathrm{new}}\equiv-m_{\mathrm{old}} \; . \quad \text{(A4a)}$$

Further define new annihilation and creation operators by

$$f_{\mathrm{new}}\equiv f^{\dagger}_{\mathrm{old}} \; , \quad f^{\dagger}_{\mathrm{new}}\equiv f_{\mathrm{old}} \; ; \quad \text{(A4b)}$$

this definition produces a reversal of the sign of the commutation relation

$$[\hat{f}_{\mathrm{new}}, \hat{f}^{\dagger}_{\mathrm{new}}]=-[\hat{f}_{\mathrm{old}}, \hat{f}^{\dagger}_{\mathrm{old}}] \quad \text{(A4c)}$$

in accord with Eqs. (A3); and it goes hand in hand with the definition of a new number operator [Eqs. (A3)] which is related to the old by

$$\hat{n}_{\mathrm{new}}=-(\hat{n}_{\mathrm{old}}+1) \; . \quad \text{(A4d)}$$

Equation (A4d) in turn goes hand-in-hand with the new notation for the mode's Fock space

$$|\, n \,\rangle_{\mathrm{new}}\equiv|\, -n-1 \,\rangle_{\mathrm{old}} \quad \text{for integers } n \; ; \quad \text{(A4e)}$$

if n_{old} runs over the nonnegative integers 0, 1, 2, ... with $\hat{f}_{\mathrm{old}}\,|\,0\,\rangle_{\mathrm{old}}=0$, then n_{new} runs over the negative integers $-1, -2, -3, \ldots$ with $\hat{f}^{\dagger}_{\mathrm{new}}\,|\,-1\,\rangle=0$, and conversely. These definitions maintain the commutation relations for $\hat{f}$ and $\hat{f}^{\dagger}$ with $\hat{n}$:

$$[\hat{f},\hat{n}]=\hat{f} \; , \quad [\hat{f}^{\dagger},\hat{n}]=-\hat{f}^{\dagger} \quad \text{in both new and old} \quad \text{(A4f)}$$

[cf. Eqs. (A3)]; they maintain the eigenrelations

$$\hat{n}\,|\,n\,\rangle=n\,|\,n\,\rangle \quad \text{in both new and old;} \quad \text{(A4g)}$$

and they maintain the standard forms for the Hamiltonians of the mode

$$\hat{H}_{\infty}=(\hat{n}+\tfrac{1}{2})\epsilon, \quad \hat{H}_{\mathscr{H}}=(\hat{n}+\tfrac{1}{2})\tilde{\epsilon} \quad \text{in both new and old.} \quad \text{(A4h)}$$

Here $\hat{H}_{\infty}$ is the Hamiltonian in the reference frame of a distant observer (which has eigenvalues E_{∞}), and $\hat{H}_{\mathscr{H}}$ is that in the reference frame of a near-horizon observer (which has eigenvalues $\tilde{E}$).

We turn next to a *comparison of states and conventions*: For a Fock space of positive-E_{∞} states ($\mathscr{I}$ states), each state $|\,n\,\rangle$ has $E_{\infty}=(n+\frac{1}{2})\epsilon>0$, and thus $n\epsilon\geq 0$. By contrast, for a Fock space of $\mathscr{H}$ states, each state $|\,n\,\rangle$ has $\tilde{E}=(n+\frac{1}{2})\tilde{\epsilon}>0$ [i.e., $(n+\frac{1}{2})\epsilon<0$] and thus $n\tilde{\epsilon}>0$. If we regard the sign of zero to be positive, we can summarize this by

$$\mathrm{sgn}(n)=\mathrm{sgn}(\epsilon) \quad \text{for } \mathscr{I} \text{ states,} \quad \text{(A5a)}$$

$$\mathrm{sgn}(n)=\mathrm{sgn}(\tilde{\epsilon}) \quad \text{for } \mathscr{H} \text{ states,} \quad \text{(A5b)}$$

and correspondingly

$$\mathrm{sgn}(n)=+1 \quad \text{(A6a)}$$

if the states and viewpoint agree,

$$\mathrm{sgn}(n)=-1 \quad \text{(A6b)}$$

if the states and viewpoint disagree. Combining relations (A5) with expressions (A3) for $\hat{n}$ we see that

$$\hat{n}=\mathrm{sgn}(n)\hat{f}^{\dagger}\hat{f} \quad \text{(A7a)}$$

if the states and convention agree, i.e., if both are $\mathscr{I}$ or both are $\mathscr{H}$, and

$$\hat{n}=-\mathrm{sgn}(n)\hat{f}^{\dagger}\hat{f} \quad \text{(A7b)}$$

if the states and convention disagree. This, when combined with $\mathrm{sgn}(\langle\, n-1\,|\,n-1\,\rangle)=\mathrm{sgn}(\langle\, n\,|\,f^{\dagger}f\,|\,n\,\rangle)$ [which follows from Eqs. (A4f) and (A4g)], implies

$$\mathrm{sgn}(\langle\, n-1\,|\,n-1\,\rangle)=+\mathrm{sgn}(\langle\, n\,|\,n\,\rangle) \quad \text{(A8a)}$$

if states and convention agree,

$$\mathrm{sgn}(\langle\, n-1\,|\,n-1\,\rangle)=-\mathrm{sgn}(\langle\, n\,|\,n\,\rangle) \quad \text{(A8b)}$$

if states and convention disagree. Thus, if the states and convention agree, then all the states in the Fock space can have positive-definite norm and the formalism takes on the standard form; but if the states and convention disagree, then some states have negative norm, i.e., the Fock space has an *indefinite metric*, and one must adopt a formalism designed to deal with spaces of indefinite metric.[35]

In this paper we adopt for this purpose a variant of the η formalism.[35] (In our variant the operator $\hat{\eta}$ will live in the space with indefinite metric, whereas in Ref. 35 $\hat{\eta}$ lives in a space of positive-definite metric.) Our η formalism is constructed so as to give, when the convention and states disagree, precisely the same values for all matrix elements (and thence for all expectation values and physical predictions) as one would get using reversed conventions and the standard formalism.

More specifically, *the transformation of convention with viewpoint and state-type held fixed* proceeds as follows: Let the "old" convention agree with the states to be studied (both $\mathcal{J}$ or both $\mathcal{H}$), and transform to the "new" convention which disagrees with the states. Then there will be a one-to-one correspondence between states $|\Psi\rangle_{\rm new}$ in the new Fock space (of indefinite metric) and those $|\Psi\rangle_{\rm old}$ in the old Fock space (of positive-definite metric)

$$|\Psi\rangle_{\rm new} \leftrightarrow |\Psi\rangle_{\rm old} . \tag{A9a}$$

In particular, we shall choose the phases of the new Fock basis states such that

$$|n\rangle_{\rm new} \leftrightarrow |n\rangle_{\rm old} . \tag{A9b}$$

Because of the difference in metric, the inner products of corresponding states are not the same in the two spaces, ${}_{\rm new}\langle\Phi|\Psi\rangle_{\rm new} \neq {}_{\rm old}\langle\Phi|\Psi\rangle_{\rm old}$. However, there exists an operator $\hat{\eta}$ such that

$${}_{\rm new}\langle\Phi|\hat{\eta}|\Psi\rangle_{\rm new} \equiv {}_{\rm old}\langle\Phi|\Psi\rangle_{\rm old} . \tag{A9c}$$

Specifically, with the choice of norm ${}_{\rm new}\langle 0|0\rangle_{\rm new}=+1$ [if $\mathrm{sgn}(n)>0$] or ${}_{\rm new}\langle -1|-1\rangle_{\rm new}=+1$ [if $\mathrm{sgn}(n)<0$], Eqs. (A9b), (A8b) and (A7b) imply

$$\begin{aligned} \hat{\eta} &= (-1)^{\hat{n}_{\rm new}} \quad \text{with } \hat{n}_{\rm new} = -\hat{f}^{\dagger}_{\rm new}\hat{f}_{\rm new} \quad \text{if } \mathrm{sgn}(n)>0 , \\ \hat{\eta} &= (-1)^{\hat{n}_{\rm new}+1} \quad \text{with } \hat{n}_{\rm new} = +\hat{f}^{\dagger}_{\rm new}\hat{f}_{\rm new} \quad \text{if } \mathrm{sgn}(n)<0 , \end{aligned} \tag{A9d}$$

so that, in particular,

$${}_{\rm new}\langle n|\hat{\eta}|n'\rangle_{\rm new} = {}_{\rm old}\langle n|n'\rangle_{\rm old} = +\delta_{nn'} . \tag{A9e}$$

Notice that $\hat{\eta}$ is both unitary and Hermitian. The correspondence (A9a) between states dictates a correspondence between operators in the new and old spaces

$$\hat{P}_{\rm new} \leftrightarrow \hat{P}_{\rm old} \quad \text{iff} \quad \hat{P}_{\rm new}|\Psi\rangle_{\rm new} \leftrightarrow \hat{P}_{\rm old}|\Psi\rangle_{\rm old} \quad \text{for all } |\Psi\rangle , \tag{A9f}$$

which in turn implies

$${}_{\rm new}\langle\Phi|\hat{\eta}\hat{P}_{\rm new}|\Psi\rangle_{\rm new} = {}_{\rm old}\langle\Phi|\hat{P}_{\rm old}|\Psi\rangle_{\rm old} . \tag{A9g}$$

This, together with the definition of the adjoint $\hat{P}^{\dagger}$ of an operator $\hat{P}$ ($\langle\Phi|\hat{P}^{\dagger}|\Psi\rangle = \langle\Psi|\hat{P}|\Phi\rangle^{*}$ for all $|\Psi\rangle$, $|\Phi\rangle$) and the unitarity of $\hat{\eta}$, implies that

$$\text{if } \hat{P}_{\rm new} \leftrightarrow \hat{P}_{\rm old} \text{ then } \hat{\eta}\hat{P}^{\dagger}_{\rm new}\hat{\eta} \leftrightarrow \hat{P}^{\dagger}_{\rm old} . \tag{A9h}$$

Since the formalism takes the standard form in the old convention, observables in the old convention are described by self-adjoint operators, $\hat{P}^{\dagger}_{\rm old}=\hat{P}^{\dagger}$. Correspondingly *observables in the new convention are described by operators that satisfy* $\hat{\eta}\hat{P}^{\dagger}_{\rm new}\hat{\eta}=\hat{P}_{\rm new}$. Note that the convention-independent and viewpoint-independent relations (A8b), together with (A9d), imply

$$\hat{\eta}\hat{f}_{\rm new}\hat{\eta} = -\hat{f}_{\rm new}, \quad \hat{\eta}\hat{f}^{\dagger}_{\rm new}\hat{\eta} = -\hat{f}^{\dagger}_{\rm new} . \tag{A9i}$$

These, together with (A9h), imply that we should choose either $\hat{f}_{\rm new} \leftrightarrow \hat{f}_{\rm old}$ and $-\hat{f}^{\dagger}_{\rm new} \leftrightarrow \hat{f}^{\dagger}_{\rm old}$, or $\hat{f}_{\rm new} \leftrightarrow -\hat{f}_{\rm old}$ and $\hat{f}^{\dagger}_{\rm new} \leftrightarrow \hat{f}^{\dagger}_{\rm old}$. Because changes of viewpoint map $\hat{f}^{\dagger}$ into $\hat{f}$ [Eq. (A4b)], our choice should be viewpoint dependent—or, equivalently [cf. Eqs. (A6)], it should depend on $\mathrm{sgn}(n)$. We shall make the explicit choice

$$-\mathrm{sgn}(n)\hat{f}_{\rm new} \leftrightarrow \hat{f}_{\rm old}, \quad +\mathrm{sgn}(n)\hat{f}^{\dagger}_{\rm new} \leftrightarrow \hat{f}^{\dagger}_{\rm old} . \tag{A9j}$$

This correspondence, together with the correspondence (A9b) between bases and the standard relations

$$\begin{aligned} \hat{f}_{\rm old}|n\rangle_{\rm old} &= \sqrt{|n|}\,|n-1\rangle_{\rm old}, \\ \hat{f}^{\dagger}_{\rm old}|n\rangle_{\rm old} &= \sqrt{|n+1|}\,|n+1\rangle_{\rm old} \end{aligned} \tag{A9k}$$

[which follow from the standard ladder-type commutation relations (A4f)], implies that

$$\begin{aligned} \hat{f}_{\rm new}|n\rangle_{\rm new} &= -\mathrm{sgn}(n)\sqrt{|n|}\,|n-1\rangle_{\rm new} , \\ \hat{f}^{\dagger}_{\rm new}|n\rangle_{\rm new} &= +\mathrm{sgn}(n)\sqrt{|n+1|}\,|n+1\rangle_{\rm new} . \end{aligned} \tag{A9l}$$

The correspondence (A9j) also implies that if $\hat{P}_{\rm old}=P(\hat{f}_{\rm old},\hat{f}^{\dagger}_{\rm old})$ is a polynomial or power series in $\hat{f}_{\rm old}$ and $\hat{f}^{\dagger}_{\rm old}$, then the corresponding operator in the new space is

$$\begin{aligned} \hat{P}_{\rm new} &= P(-\mathrm{sgn}(n)\hat{f}_{\rm new}, +\mathrm{sgn}(n)\hat{f}^{\dagger}_{\rm new}) \\ &\leftrightarrow \hat{P}_{\rm old} = P(\hat{f}_{\rm old},\hat{f}^{\dagger}_{\rm old}) . \end{aligned} \tag{A9m}$$

Equations (A7) are a special case of this:

$$\begin{aligned} \hat{n}_{\rm new} &= -\mathrm{sgn}(n)\hat{f}^{\dagger}_{\rm new}\hat{f}_{\rm new} \\ &\leftrightarrow \hat{n}_{\rm old} = +\mathrm{sgn}(n)\hat{f}^{\dagger}_{\rm old}\hat{f}_{\rm old} . \end{aligned} \tag{A9n}$$

Another special case is the field operator. In the old convention, which agrees with the state type, it has the standard form

$$\begin{aligned} &\hat{\Phi}_{\rm old} \sim e^{-i\epsilon t}e^{+im\phi}\hat{f}_{\rm old} \;\&\; e^{+i\epsilon t}e^{-im\phi}\hat{f}^{\dagger}_{\rm old} \\ &\qquad \text{for } \mathcal{J} \text{ states and } \mathcal{J} \text{ convention,} \\ &\hat{\Phi}_{\rm old} \sim e^{-i\bar{\epsilon} t}e^{+im\bar{\phi}}\hat{f}_{\rm old} \;\&\; e^{+i\bar{\epsilon} t}e^{-im\bar{\phi}}\hat{f}^{\dagger}_{\rm old} \\ &\qquad \text{for } \mathcal{H} \text{ states and } \mathcal{H} \text{ convention.} \end{aligned} \tag{A9o}$$

Here "$\sim$" means "is made up of terms proportional to," and "&" means "plus a term proportional to." From Eqs. (A9m) we read off the corresponding new expressions

$$\hat{\Phi}_{\rm new} \sim -{\rm sgn}(n) e^{-i\epsilon t} e^{+im\phi} \hat{f}_{\rm new} \ \& \ {\rm sgn}(n) e^{+i\epsilon t} e^{-im\phi} \hat{f}^{\dagger}_{\rm new} \quad \text{for } \mathcal{J} \text{ states and } \mathcal{H} \text{ convention,}$$
$$\hat{\Phi}_{\rm new} \sim -{\rm sgn}(n) e^{-i\bar{\epsilon} t} e^{+im\bar{\phi}} \hat{f}_{\rm new} \ \& \ {\rm sgn}(n) e^{+i\bar{\epsilon} t} e^{-im\bar{\phi}} \hat{f}^{\dagger}_{\rm new} \quad \text{for } \mathcal{H} \text{ states and } \mathcal{J} \text{ convention.} \tag{A9p}$$

In Appendix C we use this η formalism to derive an expression for the negative-E_∞ state ($\mathcal{H}$-type state) $|H\rangle$ and its density operator $\hat{\rho}_H$ in the *in* modes' $\mathcal{J}$ convention.

In this paper we never meet a situation that requires simultaneous study of $\mathcal{J}$ states and $\mathcal{H}$ states, but one can imagine such situations—e.g., a black hole in the state $(1/\sqrt{2})(|H\rangle + |U\rangle)$. The η formalism is readily adapted to deal with such situations: One works with a Fock space that is the direct sum of the space of $\mathcal{J}$ states and the space of $\mathcal{H}$ states; and in the basis $\{\ldots, |-2\rangle, |-1\rangle, |0\rangle, |1\rangle, \ldots\}$ one defines $\hat{\eta}$ by

$$\hat{\eta}|n\rangle = |n\rangle \ \text{ if } |n\rangle\text{'s type } (\mathcal{J} \text{ or } \mathcal{H}) \text{ agrees with the chosen convention,}$$
$$\hat{\eta}|n\rangle = [\text{ expression (A9d) }]|n\rangle \ \text{ if it disagrees.} \tag{A10}$$

This, in fact, is done in the present paper—not to permit the analysis of superpositions of $\mathcal{J}$ states and $\mathcal{H}$ states, but rather to permit the analysis of separate $\mathcal{J}$ states ($|B\rangle, |U\rangle, |UN\rangle$) and $\mathcal{H}$ states ($|\tilde{B}\rangle, |H\rangle$) using a single choice of convention and viewpoint.

APPENDIX B: GENERAL FORMULAS FOR QUANTIZATION OF A MASSLESS SCALAR FIELD IN THE EXTENDED KERR SPACETIME

Here we collect the main formulas[20] concerning the quantization of a massless scalar field in the extended Kerr spacetime (Fig. 2), with the modifications dictated by our desire to use for each mode a single choice of convention and viewpoint regardless of the type of state being studied; see Appendix A. From the outset we shall insist that for each superradiant (SR) mode our chosen convention and viewpoint agree. Correspondingly, that portion of the Fock space of SR mode i with non-negative numbers of quanta [${\rm sgn}(n_i) > 0$ in the notation of Appendix A] will have state type in agreement with the convention and viewpoint, while that portion with negative numbers [${\rm sgn}(n_i) < 0$] will have state type in disagreement with the convention and viewpoint; cf. Eqs. (A6). This means that the operator $\hat{\eta}_i$ used to deal with states that disagree with mode i's convention [Eq. (A10)] is

$$\hat{\eta}_i = 1 \tag{B1a}$$

when acting on states $|n_i\rangle$ with $n_i \geq 0$,

$$\hat{\eta}_i = (-1)^{\hat{n}_i + 1} \tag{B1b}$$

when acting on states $|n_i\rangle$ with $n_i \leq -1$. Here $\hat{n}_i = \hat{f}^{\dagger}_i \hat{f}_i$ is the number operator for mode i. In our formalism we shall make extensive use of the operator

$$\hat{\eta} \equiv \prod_i \hat{\eta}_i \ , \tag{B1c}$$

where the product is over all SR modes i.

As a mathematical convenience we shall build our formalism in the complete analytically extended Kerr spacetime[36] of Fig. 2 rather than in the true black-hole spacetime. This is justified because outside the surface of the material that collapsed (long ago) to form the black hole, and outside and on the hole's future horizon, the spacetime of the black hole is identical to region I of the extended spacetime.

Let the scalar field Φ satisfy the equation [Eq. (3.1) with vanishing scalar curvature R]

$$\Box\Phi = 0 \ . \tag{B2}$$

Then the bilinear form

$$B(\Phi_1, \Phi_2) = \int_\Sigma (\Phi_1 \Phi_{2,\nu} - \Phi_2 \Phi_{1,\nu}) d\Sigma^\nu \tag{B3}$$

is conserved; i.e., for any two solutions Φ_1 and Φ_2 of Eq. (B2) its value does not depend on the particular choice of the total Cauchy surface Σ (which reaches from spacelike infinity $I^{0'}$ in region I′ of Fig. 2 to I^0 in region I). Here, as in Ref. 20, the sign of $d\Sigma^\nu$ is taken opposite to that used in MTW.[1] The canonical commutation relations for the quantized field $\hat{\Phi}$ can be written in the form

$$[B(\Phi_1, \hat{\Phi}), \ \hat{\eta} B(\Phi_2, \hat{\Phi}) \hat{\eta}] = iB(\Phi_1, \Phi_2) \ , \tag{B4}$$

where Φ_1 and Φ_2 are arbitrary solutions of (B2). The factors of $\hat{\eta}$ are required here in order to mesh with the formalism of Appendix A—more specifically, in order to guarantee that Eqs. (B7) follow from (B4)—(B6). [These factors of $\hat{\eta}$ arise because $\hat{\Phi}$ is not quite the standard field operator. Rather, when as here one has chosen a set of modes whose conventions differ in type from the states being studied, $\hat{\Phi}$ is that operator for which

$$\langle \chi | \hat{\eta} \hat{\Phi} | \Psi \rangle = {}_{\rm standard}\langle \chi | \hat{\Phi}_{\rm standard} | \Psi \rangle_{\rm standard} \ .$$

Here "standard" denotes the form of the field operator and states when one makes the "standard" choice of modes whose conventions agree with state type; cf. Eq. (A9g).] If $\{u_i, \bar{u}_j\}$ is a complete basis in the space of solutions of Eq. (B2) satisfying the normalization conditions

$$B(u_i, u_j) = B(\bar{u}_i, \bar{u}_j) = 0, \quad B(u_i, \bar{u}_j) = i\delta_{ij} \tag{B5}$$

(complete, orthonormal basis), then the field operator $\hat{\Phi}$ can be written as

$$\hat{\Phi} = \sum_i (\hat{f}_i u_i + \hat{\eta} \hat{f}^{\dagger}_i \hat{\eta} \bar{u}_i) \ , \tag{B6}$$

where the sum is over all modes i, superradiant and nonsuperradiant; and the commutation relations (B4)—together with $\hat{\eta}^2 = 1$, $\hat{\eta}\hat{f}_i\hat{\eta} = \pm\hat{f}_i$, and $\hat{\eta}\hat{f}^{\dagger}_i\hat{\eta} = \pm\hat{f}^{\dagger}_i$—imply that

$$[\hat{f}_i,\hat{f}_j]=[\hat{f}^\dagger_i,\hat{f}^\dagger_j]=0,\quad [\hat{f}_i,\hat{f}^\dagger_j]=\delta_{ij}\ . \tag{B7}$$

The placement of the $\hat{\eta}$'s in Eq. (B6) is in accord with Eq. (A9p), and with $\{\hat{\eta}\hat{f}^\dagger_i\hat{\eta}=-\hat{f}^\dagger_i$ when operating on states $|n_i\rangle$ with $n_i\le -1\}$. The sign of the last commutation relation in (B7) is in accord with Eqs. (A3) and our insistence that for all SR modes the chosen convention and viewpoint agree. Equations (A3) also tell us that we must define the number operator $\hat{n}_i$ for mode i by

$$\hat{n}_i=\hat{f}^\dagger_i\hat{f}_i\ . \tag{B8}$$

Let $u=t-r^*$ and $v=t+r^*$ [Eq. (3.3)] be the retarded and advanced time coordinates in region I of the extended Kerr spacetime (Fig. 2). Define in region I new null coordinates

$$U_+=\exp(-\kappa u)\ ,\quad V_+=\exp(\kappa v)\ , \tag{B9}$$

which turn out to be affine parameters along the geodesic generators of the past and future horizons, $\mathcal{H}^-$ and $\mathcal{H}^+$. Then analytically continue the coordinates $(U_+, V_+, \bar{\phi}\equiv\phi-\Omega_H t, \theta)$ to cover the regions I, I′, II and II′ of the analytically extended Kerr spacetime (Fig. 2). If y_i is a solution of the wave equation (B2), then denote by the corresponding capital letter the function $Y_i\equiv(r^2+a^2)^{1/2}y_i$. By setting $\Sigma=\mathcal{I}^-\cup\{$that part of $\mathcal{H}^-$ which touches region I$\}\cup\{$that part of $\mathcal{H}^+$ which touches region I′$\}\cup\mathcal{I}'^-$, the bilinear form $B(y_1,y_2)$ [Eq. (B3)] for any two solutions can then be written as follows:

$$\begin{aligned} B(y_1,y_2)= &\int_0^\infty dV_+d\Omega(Y_1\overleftrightarrow{\partial}_{V_+}Y_2)_{\mathcal{I}^-}\\ &-\int_0^\infty dU_+d\Omega(Y_1\overleftrightarrow{\partial}_{U_+}Y_2)_{\mathcal{H}^-}\\ &+\int_{-\infty}^0 dV_+d\Omega(Y_1\overleftrightarrow{\partial}_{V_+}Y_2)_{\mathcal{H}^+}\\ &-\int_{-\infty}^0 dU_+d\Omega(Y_1\overleftrightarrow{\partial}_{U_+}Y_2)_{\mathcal{I}'^-}\ , \end{aligned} \tag{B10a}$$

where

$$Y_1\overleftrightarrow{\partial}_X Y_2\equiv Y_1\partial_X Y_2-(\partial_X Y_1)Y_2 \tag{B10b}$$

and

$$\begin{aligned} d\Omega&=\sin\theta\, d\theta\, d\phi\quad\text{at } \mathcal{I}^- \text{ and } \mathcal{I}'^-\ ,\\ d\Omega&=\sin\theta\, d\theta\, d\bar{\phi}\quad\text{at } \mathcal{H}^+ \text{ and } \mathcal{H}^-\ . \end{aligned} \tag{B10c}$$

Using this formula one can easily verify that the functions v_I and q_I introduced in Sec. III, which satisfy the boundary conditions (3.2) and (3.4) and vanish in region I′, are normalized as follows:

$$B(v_I,\bar{v}_{I'})=B(q_I,\bar{q}_{I'})=i\delta_{II'}\ , \tag{B11}$$

$$\delta_{II'}\equiv\delta(\epsilon-\epsilon')\delta_{ll'}\delta_{mm'}\ . \tag{B12}$$

For any other pair of v_I, $\bar{v}_I$, q_I, $\bar{q}_I$ the bilinear form B vanishes. For the *in* mode v_I the viewpoint and convention are both chosen to be distant-observer ($\mathcal{I}$); for the *up* mode q_I the viewpoint and convention are chosen to be near-horizon ($\mathcal{H}$).

Because the functions v_I and q_I vanish in the region I′, in order to get a complete basis in the extended spacetime we must introduce additional basis functions. For this purpose we consider the discrete symmetry transformation

$$U_+\to -U_+,\quad V_+\to -V_+,\quad \theta\to\theta,\quad \bar{\phi}\to\bar{\phi} \tag{B13}$$

(reflection through $\mathcal{H}^+\cap\mathcal{H}^-$ of Fig. 2), which transforms region I into I′; and using it we introduce new functions

$$\begin{aligned} w_I(U_+,V_+,\theta,\bar{\phi})&=\bar{v}_I(-U_+,-V_+,\theta,\bar{\phi})\quad\text{for } \epsilon>0\ ;\\ p_I(U_+,V_+,\theta,\bar{\phi})&=\bar{q}_I(-U_+,-V_+,\theta,\bar{\phi})\quad\text{for } \tilde{\epsilon}>0\ . \end{aligned} \tag{B14}$$

These functions vanish in region I and obey the normalization conditions

$$B(w_I,\bar{w}_{I'})=B(p_I,\bar{p}_{I'})=i\delta_{II'} \tag{B15}$$

with other "products" equal to zero. One can verify that w_I is an *out* mode (all waves go out to $\mathcal{I}'^+$; none go down the future horizon $\mathcal{H}^-$), while p_I is a *down* mode (all waves go down the future horizon $\mathcal{H}^-$; none go out to $\mathcal{I}'^+$).[28]

The functions v_I, q_I, w_I, p_I form a complete, orthogonal basis in regions I, I′, II, II′ of the extended Kerr spacetime. Expanded in terms of this basis the field operators take the form

$$\begin{aligned} \hat{\Phi}(x)= &\sum_{\epsilon>0}[v_I(x)\hat{a}_I+\bar{v}_I(x)\hat{\eta}\hat{a}^\dagger_I\hat{\eta}+w_I(x)\hat{a}'_I\\ &\qquad+\bar{w}_I(x)\hat{\eta}\hat{a}'^\dagger_I\hat{\eta}]\\ &+\sum_{\tilde{\epsilon}>0}[q_I(x)\hat{b}_I+\bar{q}_I(x)\hat{\eta}\hat{b}^\dagger_I\hat{\eta}+p_I(x)\hat{b}'_I\\ &\qquad+\bar{p}_I(x)\hat{\eta}\hat{b}'^\dagger_I\hat{\eta}]\ , \end{aligned} \tag{B16}$$

which is a specific version of Eqs. (B6). Correspondingly, the creation operators $\{\hat{f}^\dagger_i\}=\{\hat{a}^\dagger_I, \hat{b}^\dagger_I, \hat{a}'^\dagger_I, \hat{b}'^\dagger_I\}$ and annihilation operators $\{\hat{f}_i\}=\{\hat{a}_I, \hat{b}_I, \hat{a}'_I, \hat{b}'_I\}$ satisfy the standard commutation relations (B7). In region I where the solutions $w_I(x)$ and $p_I(x)$ vanish, Eq. (B16) coincides with Eq. (3.7). The vacuum state $|B\rangle$ defined by the conditions

$$\hat{a}_I|B\rangle=\hat{b}_I|B\rangle=\hat{a}'_I|B\rangle=\hat{b}'_I|B\rangle=0 \tag{B17}$$

is known as the Boulware vacuum.[11]

In order to introduce the Hartle-Hawking $|H\rangle$ and Unruh $|U\rangle$ vacuum states we need a different complete, orthonormal basis defined by

$$\nu_I\equiv c_I v_I+s_I\bar{w}_I,\quad \nu'_I\equiv c_I w_I+s_I\bar{v}_I\quad\text{for } \epsilon>0\ , \tag{B18}$$

$$\lambda_I\equiv c_I q_I+s_I\bar{p}_I,\quad \lambda'_I\equiv c_I p_I+s_I\bar{q}_I\quad\text{for } \tilde{\epsilon}>0\ , \tag{B19}$$

where

$$\begin{aligned} s_I&\equiv[\exp(2\pi|\tilde{\epsilon}|/\kappa)-1]^{-1/2}\ ,\\ c_I&\equiv[1-\exp(-2\pi|\tilde{\epsilon}|/\kappa)]^{-1/2}\ ,\\ c_I^2-s_I^2&=1\ ,\quad s_I/c_I=\exp(-\pi|\tilde{\epsilon}|/\kappa)\ . \end{aligned} \tag{B20}$$

From the normalization conditions (B11) and (B15) for the $\{u_I, q_I, w_I, p_I\}$ basis one can verify that

$$B(\nu_I, \bar{\nu}_{I'}) = B(\nu'_I, \bar{\nu}'_{I'}) = B(\lambda_I, \bar{\lambda}_{I'}) = B(\lambda'_I, \bar{\lambda}'_{I'}) = i\delta_{II'} \quad \text{(B21)}$$

(other products vanish), so this new basis is indeed orthonormal.

The functions $\{\nu_I, \nu'_I\}$ vanish at $\mathscr{H}^-$ (Fig. 2), and the functions $\{\lambda_I, \lambda'_I\}$ are of positive frequency at $\mathscr{H}^-$ with respect to any future-increasing affine parameter U, e.g., $U = -U_+$. The functions $\{\lambda_I, \lambda'_I\}$, and $\{\nu_I, \nu'_I$ (for non-SR modes, $\tilde{\epsilon} > 0$, $\epsilon > 0$); $\bar{\nu}_I$, $\bar{\nu}'_I$ (for SR modes $\tilde{\epsilon} < 0$, $\epsilon > 0$)$\}$ are of positive frequency at $\mathscr{H}^+$ with respect to any future-increasing affine parameter V, e.g., $V = V_+$. To prove these positive-frequency properties of the functions $\nu_I, \nu'_I, \lambda_I, \lambda'_I$ one can exploit the fact that for any real number α, the function

$$F(X) \equiv e^{-i\alpha \ln X}\theta(X) + e^{-\alpha\pi}e^{-i\alpha\ln(-X)}\theta(-X) \quad \text{(B22)}$$

for real X coincides with the boundary value of a function $X^{-i\alpha}$ which, for any real α, is analytic in the lower half complex-X plane; and hence this function $F(X)$, when Fourier analyzed in X, contains only positive-frequency components.

If we use $v_I, w_I, \lambda_I, \lambda'_I$ as our basis, then we shall write $\hat{\Phi}$ in the form

$$\hat{\Phi}(x) = \sum_{\epsilon>0} [v_I(x)\hat{a}_I + \bar{v}_I(x)\hat{\eta}\hat{a}_I^\dagger\hat{\eta} + w_I(x)\hat{a}'_I + \overline{w_I}(x)\hat{\eta}\hat{a}_I'^\dagger\hat{\eta}] + \sum_{\tilde{\epsilon}>0} [\lambda_I(x)\hat{\alpha}_I + \bar{\lambda}_I(x)\hat{\eta}\hat{\alpha}_I^\dagger\hat{\eta} + \lambda'_I(x)\hat{\alpha}'_I + \overline{\lambda'_I}(x)\hat{\eta}\hat{\alpha}_I'^\dagger\hat{\eta}] ; \quad \text{(B23)}$$

if, instead, we use $\nu_I, \nu'_I, \lambda_I, \lambda'_I$, then $\hat{\Phi}$ has the form

$$\hat{\Phi}(x) = \sum_{\epsilon>0} [\nu_I(x)\hat{\beta}_I + \bar{\nu}_I(x)\hat{\eta}\hat{\beta}_I^\dagger\hat{\eta} + \nu'_I(x)\hat{\beta}'_I + \overline{\nu'_I}(x)\hat{\eta}\hat{\beta}_I'^\dagger\hat{\eta}] + \sum_{\tilde{\epsilon}>0} [\lambda_I(x)\hat{\alpha}_I + \bar{\lambda}_I(x)\hat{\eta}\hat{\alpha}_I^\dagger\hat{\eta} + \lambda'_I(x)\hat{\alpha}'_I + \overline{\lambda'_I}(x)\hat{\eta}\hat{\alpha}_I'^\dagger\hat{\eta}] . \quad \text{(B24)}$$

By comparing the expansions (B16), (B23) and (B24) for $\hat{\Phi}$ and using the transformation of bases (B18), (B19) we see that

$$\hat{a}_I = c_I\hat{b}_I - s_I\hat{\eta}\hat{b}_I'^\dagger\hat{\eta}, \quad \hat{a}'_I = c_I\hat{b}'_I - s_I\hat{\eta}\hat{b}_I^\dagger\hat{\eta} , \quad \text{(B25a)}$$

$$\hat{\beta}_I = c_I\hat{a}_I - s_I\hat{\eta}\hat{a}_I'^\dagger\hat{\eta}, \quad \hat{\beta}'_I = c_I\hat{a}'_I - s_I\hat{\eta}\hat{a}_I^\dagger\hat{\eta} . \quad \text{(B25b)}$$

These transformations guarantee that the sets of creation and annihilation operators $\{\hat{a}_I^\dagger, \hat{a}_I'^\dagger, \hat{\alpha}_I^\dagger, \hat{\alpha}_I'^\dagger, \hat{a}_I, \hat{a}'_I, \hat{\alpha}_I, \hat{\alpha}'_I\}$ and $\{\hat{\beta}_I^\dagger, \hat{\beta}_I'^\dagger, \hat{\alpha}_I^\dagger, \hat{\alpha}_I'^\dagger, \hat{\beta}_I, \hat{\beta}'_I, \hat{\alpha}_I, \hat{\alpha}'_I\}$, like $\{\hat{a}_I^\dagger, \hat{a}_I'^\dagger, \hat{b}_I^\dagger, \hat{b}_I'^\dagger, \hat{a}_I, \hat{a}'_I, \hat{b}_I, \hat{b}'_I\}$, satisfy the standard commutation relations (B7).

Because the functions $\{v_I, w_I, \lambda_I, \lambda'_I\}$ have similar asymptotic properties to the Feynman propagator G_U for the Unruh vacuum [only one sign of frequency on $\mathscr{H}^-$ and only one sign on $\mathscr{I}^-$; cf. paragraphs following Eqs. (3.14) and (B21)], it is reasonable to expect that this is the basis appropriate to the Unruh vacuum state $|U\rangle$, i.e., that $|U\rangle$ will be given by

$$\hat{a}_I|U\rangle = \hat{a}'_I|U\rangle = \hat{\alpha}_I|U\rangle = \hat{\alpha}'_I|U\rangle = 0 . \quad \text{(B26)}$$

Analogously, because the functions $\{\nu_I, \nu'_I$ (for $\epsilon > 0$, $\tilde{\epsilon} > 0$); $\bar{\nu}_I$, $\bar{\nu}'_I$ (for $\epsilon > 0$, $\tilde{\epsilon} < 0$); λ_I, λ'_I (for $\tilde{\epsilon} > 0$)$\}$ have similar asymptotic properties to the Feynman propagator G_H for the Hartle-Hawking vacuum (only one sign of frequency on $\mathscr{H}^-$ and only one sign on $\mathscr{H}^+$), it is reasonable to expect that this is the basis appropriate to the Hartle-Hawking vacuum state $|H\rangle$, i.e., that $|H\rangle$ will be given by

$$\begin{aligned} &\hat{\beta}_I|H\rangle = \hat{\beta}'_I|H\rangle = 0 \quad \text{for } \epsilon > 0,\ \tilde{\epsilon} > 0, \\ &\hat{\beta}_I^\dagger|H\rangle = \hat{\beta}_I'^\dagger|H\rangle = 0 \quad \text{for } \epsilon > 0,\ \tilde{\epsilon} < 0 , \\ &\hat{\alpha}_I|H\rangle = \hat{\alpha}'_I|H\rangle = 0 \quad \text{for } \tilde{\epsilon} > 0 . \end{aligned} \quad \text{(B27)}$$

A straightforward computation reveals that these equations for $|U\rangle$ and $|H\rangle$ are, indeed, correct: The first step in the computation is to derive the following expressions for the "positive-frequency functions" $S_B(x,x')$, $S_U(x,x')$, and $S_H(x,x')$ by (i) inserting expansions (B16), (B23), and (B24) into Eqs. (3.18b); (ii) making use of Eqs. (B17), (B26), and (B27) for $|B\rangle$, $|U\rangle$, and $|H\rangle$; (iii) using the fact (to be proved in Appendix C) that $|B\rangle$ and $|U\rangle$ contain non-negative numbers of quanta in all SR modes and thus $\hat{\eta}$ behaves as unity when acting on them, while $|H\rangle$ contains non-negative numbers of SR $\hat{\alpha}_I$ and $\hat{\alpha}'_I$ quanta but negative numbers of SR $\hat{\beta}_I$ and $\hat{\beta}'_I$ quanta so for SR modes $\hat{\eta}\hat{\alpha}_I^\dagger\hat{\eta}|H\rangle = +\hat{\alpha}_I^\dagger|H\rangle$, $\hat{\eta}\hat{\alpha}_I'^\dagger\hat{\eta}|H\rangle = +\hat{\alpha}_I'^\dagger|H\rangle$, $\hat{\eta}\hat{\beta}_I^\dagger\hat{\eta}|H\rangle = -\hat{\beta}_I^\dagger|H\rangle$, $\hat{\eta}\hat{\beta}_I'^\dagger\hat{\eta}|H\rangle = -\hat{\beta}_I'^\dagger|H\rangle$; and (iv) restricting attention to points x and x' that lie in region I (which corresponds to the exterior of a black hole). The resulting expressions are

$$S_B(x,x') = \sum_{\epsilon>0} v_I(x)\bar{v}_I(x') + \sum_{\tilde{\epsilon}>0} q_I(x)\bar{q}_I(x') , \quad \text{(B28)}$$

$$S_U(x,x') = \sum_{\epsilon>0} v_I(x)\bar{v}_I(x') + \sum_{\tilde{\epsilon}>0} [\lambda_I(x)\bar{\lambda}_I(x') + \lambda'_I(x)\bar{\lambda}'_I(x')] , \quad \text{(B29)}$$

$$S_H(x,x') = \sum_{\epsilon>0,\,\tilde{\epsilon}>0} [\nu_I(x)\bar{\nu}_I(x') + \nu'_I(x)\bar{\nu}'_I(x')] - \sum_{\epsilon>0,\,\tilde{\epsilon}<0} [\bar{\nu}_I(x)\nu_I(x') + \bar{\nu}'_I(x)\nu'_I(x')] + \sum_{\tilde{\epsilon}>0} [\lambda_I(x)\bar{\lambda}_I(x') + \lambda'_I(x)\bar{\lambda}'_I(x')] . \quad \text{(B30)}$$

The next step is to verify that the propagators $G_U(x,x')$ and $G_H(x,x')$ constructed from these S_U and S_H via Eq. (3.18a) satisfy the defining properties for the Unruh and Hartle-Hawking vacuum states [paragraph following Eq. (3.14)]. Thus, $|U\rangle$ and $|H\rangle$ are, indeed, given by Eqs. (B26) and (B27). It is also straightforward to read off the boundary conditions for the Feynman propagator $G_B(x,x')$ [Eqs. (3.18a) and (B28)] associated with the Boulware vacuum $|B\rangle$ [Eq. (B17)]: (i) for x on the past horizon $\mathscr{H}^-$ and x' outside the horizon, when Fourier analyzed on $\mathscr{H}^-$ *not* in terms of a future-increasing affine parameter U but rather in terms of the nonaffine advanced Killing time parameter u, G_H contains only negative-frequency components; and (ii) for x at $\mathcal{J}^-$ and x' outside the horizon, when Fourier analyzed at $\mathcal{J}^-$ in terms of the affine, advanced, Killing time parameter v, G_B contains only negative-frequency components.

APPENDIX C: DENSITY MATRICES AND ZAMO MEASUREMENTS FOR VARIOUS STATES OUTSIDE A BLACK HOLE

In this appendix we shall prove that the states $|U\rangle$, $|UN\rangle$, and $|H\rangle$, which were defined mathematically in Sec. III [paragraph following Eq. (3.14)] and were shown in Appendix B to satisfy Eqs. (B17), (B26), and (3.15), can be characterized by ZAMO measurements in the manner of Sec. II.

Our proof will rely on an intimate connection between ZAMO measurements outside a real black hole and the *in* modes v_I and *up* modes q_I of the extended Kerr spacetime:[2] In the extended spacetime v_I and q_I are confined to region I, which coincides with the spacetime of the black hole; and there they form a complete set. Moreover, along any ZAMO world line (which has angular velocity ω), these modes are sinusoidal functions of proper time τ, (v_I and q_I) $\propto \exp[-i\alpha^{-1}(\epsilon - m\omega)\tau]$, and they are also eigenfunctions of the ZAMO-measured angular momentum (with eigenvalue m) and ZAMO-measured red-shifted energy (with eigenvalue ϵ). Correspondingly,[2] the quanta measured by ZAMO's using physical particle detectors (which see v_I and q_I oscillate sinusoidally) are those associated with the modes v_I, q_I, i.e., those created and annihilated by $\hat{a}_I^\dagger$, $\hat{b}_I^\dagger$, $\hat{a}_I$, and $\hat{b}_I$.

The states $|U\rangle$, $|UN\rangle$, and $|H\rangle$, as defined in the extended Kerr spacetime, are pure; but as viewed by ZAMO's outside a black hole they are mixed. This shows up mathematically as follows: Let $\hat{F}$ be an observable confined to the exterior of the black hole. Then in the language of the extended Kerr spacetime, $\hat{F}$ is expressible in terms of the creation and annihilation operators $\hat{a}_I^\dagger$, $\hat{a}_I$ of the region-I *in* modes and those $\hat{b}_I^\dagger$, $\hat{b}_I$ of the region-I *up* modes, $\hat{F}=F(\hat{a}_I, \hat{a}_I^\dagger, \hat{b}_I, \hat{b}_I^\dagger)$. Denote by $\{|\mathcal{N}\rangle\}$ an orthonormal basis for all states of all the region-I modes, and by $\{|\mathcal{N}'\rangle\}$ an orthonormal basis for all states of all the region-I$'$ modes; and split the operator $\hat{\eta}$ up into a part that acts in the space of $\{|\mathcal{N}\rangle\}$ and a part that acts in $\{|\mathcal{N}'\rangle\}$, $\hat{\eta}\to\hat{\eta}\hat{\eta}'$. Then orthonormality says that $\langle\mathcal{N}|\hat{\eta}|\mathcal{M}\rangle=\delta_{\mathcal{N}\mathcal{M}}$, $\langle\mathcal{N}'|\hat{\eta}'|\mathcal{M}'\rangle=\delta_{\mathcal{N}'\mathcal{M}'}$ [cf. Eq. (A9e)], and it implies that

$$\hat{1}=\sum_{\mathcal{N},\mathcal{N}'}|\mathcal{N}\rangle|\mathcal{N}'\rangle\langle\mathcal{N}'|\langle\mathcal{N}|\hat{\eta}'\hat{\eta}$$

is the identity operator in the space of all states of all modes. It is straightforward using this identity operator and these bases to show that, because $\hat{F}$ acts only in the space of $\{|\mathcal{N}\rangle\}$, the expectation value

$$\langle\hat{F}\rangle_\bullet \equiv \langle\bullet|\eta\hat{F}|\bullet\rangle \tag{C1}$$

can be rewritten as

$$\langle\hat{F}\rangle_\bullet=\mathrm{tr}(\hat{F}\hat{\rho}_\bullet)=\mathrm{tr}(\hat{\rho}_\bullet\hat{\eta}\hat{F}\hat{\eta}) , \tag{C2a}$$

where

$$\hat{\rho}_\bullet=\sum_{\mathcal{N}'}\langle\mathcal{N}'|\hat{\eta}'|\bullet\rangle\langle\bullet|\hat{\eta}'|\mathcal{N}'\rangle \tag{C2b}$$

is the density operator $\hat{\rho}_\bullet$ for the state $|\bullet\rangle$ and where the trace has the form

$$\mathrm{tr}(\hat{F}\hat{\rho}_\bullet)\equiv\sum_{\mathcal{N}}\langle\mathcal{N}|\hat{\eta}\hat{F}\hat{\rho}_\bullet\hat{\eta}|\mathcal{N}\rangle . \tag{C2c}$$

Note the following features of Eqs. (C2): (i) The density operator $\hat{\rho}_\bullet$, like $\hat{F}$, lives in the space of $\{|\mathcal{N}\rangle\}$, i.e., in the space of all states of the region-I modes, and thus describes the state $|\bullet\rangle$ from the viewpoint of observers who are confined to the exterior of the black hole. (ii) The trace (C2c) is taken over the space of black-hole-exterior states $\{|\mathcal{N}\rangle\}$. (iii) The one factor of $\hat{\eta}$ in the expectation value (C1) for a pure state becomes two factors of $\hat{\eta}$ in the trace (C2c) for a mixed state. (iv) By contrast with the standard, $\hat{\eta}$-free, formalism, the trace is sensitive to the order of $\hat{F}$ and $\hat{\rho}_\bullet$: $\mathrm{tr}(\hat{F}\hat{\rho}_\bullet)\neq\mathrm{tr}(\hat{\rho}_\bullet\hat{F})$ [Eq. (C2a)].

Our objective is to prove that the density matrices $\hat{\rho}_U$, $\hat{\rho}_{UN}$, and $\hat{\rho}_H$, as derived by Eq. (C2b) from the pure states $|U\rangle$, $|UN\rangle$, and $|H\rangle$ of Sec. III and Appendix B, have the forms dictated by the physical descriptions of Sec. II: $\hat{\rho}_U$ is that density matrix for which the ZAMO-measured *up* modes are perfectly thermalized and the *in* modes are perfectly empty; $\hat{\rho}_{UN}$ is that for which the *up* are thermalized and the *in* contain $N=n_{I_1}^{\mathrm{in}}, n_{I_2}^{\mathrm{in}}, \ldots$ quanta; and $\hat{\rho}_H$ is that in which all modes are thermalized (that of a rotating, thermal atmosphere).

We begin our proof by showing that the Unruh vacuum can be written as

$$|U\rangle=\prod_{\tilde{\epsilon}>0}\hat{R}^{(b)}_{(U)I}|B\rangle , \tag{C3}$$

where

$$\hat{R}^{(b)}_{(U)I}=C^{(b)}_{(U)I}\exp(e^{-\pi\tilde{\epsilon}/\kappa}\hat{b}_I^\dagger\hat{b}_I'^\dagger) , \tag{C4}$$

$C^{(b)}_{(U)I}$ are normalization constants, and the Boulware vacuum state $|B\rangle$ is defined by Eq. (B17). Because [as Eqs. (C3) and (C4) show] $|U\rangle$ is a $\mathcal{J}$ state (positive E_∞) in the superradiant *in* modes, and an $\mathcal{H}$ state (negative E_∞) in the superradiant *up* modes, it has the same state type as conventions, and the factors of $\hat{\eta}$ all behave as unity (are irrelevant) in calculations with $|U\rangle$.

To verify expressions (C3) and (C4) for $|U\rangle$ we must show that this $|U\rangle$ is annihilated by $\hat{a}_I$, $\hat{a}_I'$, $\hat{\alpha}_I$, $\hat{\alpha}_I'$ [Eqs. (B26)]. Since $|B\rangle$ is annihilated by $\hat{a}_I$, $\hat{a}_I'$ [Eqs. (B17)], this $|U\rangle$ is as well. Annihilation by $\hat{\alpha}_I$, $\hat{\alpha}_I'$ can be shown using the expressions (B25a) for $\hat{\alpha}_I$, $\hat{\alpha}_I'$, and using

$$\hat{B}e^{\hat{C}\hat{B}^\dagger}=e^{\hat{C}\hat{B}^\dagger}(\hat{B}+\sigma\hat{C}) , \tag{C5a}$$

which is valid for any operator $\hat{C}$ commuting with $\hat{B}$ and $\hat{B}^\dagger$ provided the operators $\hat{B}$ and $\hat{B}^\dagger$ obey the commutation relations

$$[\hat{B},\hat{B}^\dagger]=\sigma , \quad [\hat{B},\hat{B}]=[\hat{B}^\dagger,\hat{B}^\dagger]=0 , \tag{C5b}$$

where σ is a c-number constant.

Using Eqs. (C3) and (3.15) we can also write

$$|UN\rangle=\prod_i[(n_{I_i}^{\text{in}}!)^{-1/2}(\hat{a}_{I_i}^{\dagger})^{n_{I_i}^{in}}]\prod_{\tilde{\epsilon}>0}\hat{R}_{(U)I}^{(b)}|B\rangle\ , \quad \text{(C6)}$$

where again the state type and conventions agree so $\hat{\eta}$ is irrelevant.

The density operators $\hat{\rho}_U$ and $\hat{\rho}_{UN}$ associated with the states $|U\rangle$ and $|UN\rangle$ can be derived straightforwardly from Eqs. (C2b), (C3), and (C6), using for $\{|\mathcal{N}'\rangle\}$ the Fock basis generated by applying $\{\hat{a}_I'^{\dagger}, \hat{b}_I'^{\dagger}\}$ to the Boulware vacuum of region I' and for $\{|\mathcal{N}\rangle\}$ the Fock basis generated by applying $\{\hat{a}_I^{\dagger}, \hat{b}_I^{\dagger}\}$ to the Boulware vacuum of region I. The results, which can also be derived efficiently using functional methods[37] (for details see Ref. 20), are

$$\hat{\rho}_U=\prod_{\epsilon>0}\hat{\rho}_U^{I\,in}\prod_{\tilde{\epsilon}>0}\hat{\rho}_U^{I\,up}\ , \quad \text{(C7)}$$

$$\hat{\rho}_{UN}=\prod_{\epsilon>0}\hat{\rho}_{UN}^{I\,in}\prod_{\tilde{\epsilon}>0}\hat{\rho}_{UN}^{I\,up}\ , \quad \text{(C8)}$$

where the density operators for the I *in* and I *up* modes, written first in abstract terms and then in terms of probabilities $p_{(\bullet)n}^{I\,in}, p_{(\bullet)n}^{I\,up}$ to contain n quanta, are

$$\hat{\rho}_U^{I\,in}=:\exp(-\hat{a}_I^{\dagger}\hat{a}_I):\ , \quad \text{(C9a)}$$

$$p_{(U)n}^{I\,in}=\delta_{n,0}\ ; \quad \text{(C9b)}$$

$$\hat{\rho}_U^{I\,up}=\hat{\rho}_{UN}^{I\,up}=(1-e^{-\tilde{\epsilon}/T_H})\exp(-\hat{n}_I^{up}\tilde{\epsilon}/T_H)\ , \quad \text{(C10a)}$$

$$p_{(U)n}^{I\,up}=p_{(UN)n}^{I\,up}=p_n^{th}=(1-e^{-\tilde{\epsilon}/T_H})e^{-n\tilde{\epsilon}/T_H}\ ; \quad \text{(C10b)}$$

$$\hat{\rho}_{UN}^{I\,in}=\frac{1}{(n_I^{in})!}:(\hat{a}_I^{\dagger})^{n_I^{in}}\exp(-\hat{a}_I^{\dagger}\hat{a}_I)(\hat{a}_I)^{n_I^{in}}:\ , \quad \text{(C11a)}$$

$$p_{(UN)n}^{I\,in}=\delta_{n,\,n_I^{in}}\ . \quad \text{(C11b)}$$

In Eqs. (C9a) and (C11a) : : denotes normal ordering with respect to the $\hat{a}_I$, $\hat{a}_I^{\dagger}$ operators, and in Eq. (C10a) $\hat{n}_I^{up}$ is the number operator for the I *up* mode, $\hat{n}_I^{up}=\hat{b}_I^{\dagger}\hat{b}_I$.

The density operator $\hat{\rho}_{(U)}^{I\,up}=\hat{\rho}_{UN}^{I\,up}$ for the I *up* mode is perfectly thermal, with the Hawking red-shifted temperature $T_H=\kappa/2\pi$; and because this mode is described in the near-horizon viewpoint, its thermality is that of emission from a blackbody horizon that rotates with angular velocity Ω_H relative to distant observers; cf. Sec. IV. The density operator $\hat{\rho}_U^{I\,in}$ is that of a perfectly empty I *in* mode; and $\hat{\rho}_{(UN)}^{I\,in}$ is that of an I *in* mode containing exactly n_I^{in} quanta. Thus, $\hat{\rho}_U$ and $\hat{\rho}_{UN}$ have precisely the forms corresponding to the physically realistic black holes discussed in Sec. II: a hole evaporating into vacuum (the state $|U\rangle$), and an evaporating, accreting hole (the state $|UN\rangle$).

The Hartle-Hawking state $|H\rangle$ would be extremely troublesome computationally if we had not introduced the η formalism. If one were to try to make calculations for this state in the same way as was done for the states $|U\rangle$ and $|UN\rangle$ without the η formalism, one would get [in view of Eq. (B27)]

$$|H\rangle=\prod_{\tilde{\epsilon}>0}\hat{R}_{(H)I}^{(b)}\prod_{\epsilon>0,\,\tilde{\epsilon}>0}\hat{R}_{(H)I}^{(a)}\prod_{\epsilon>0,\tilde{\epsilon}<0}\hat{R}_{(H)I}^{(a)}|B\rangle\ , \quad \text{(C12)}$$

where

$$\hat{R}_{(H)I}^{(b)}=\hat{R}_{(U)I}^{(b)}=C_{(U)I}^{(b)}\exp(e^{-\pi\tilde{\epsilon}/\kappa}\hat{b}_I^{\dagger}\hat{b}_I'^{\dagger})\ , \quad \text{(C13)}$$

$$\hat{R}_{(H)I}^{(a)}=C_{(H)I}^{a}\exp(e^{-\pi\tilde{\epsilon}/\kappa}\hat{a}_I^{\dagger}\hat{a}_I'^{\dagger})\ . \quad \text{(C14)}$$

But for the superradiant *in* modes ($\epsilon>0$, $\tilde{\epsilon}<0$) the state $\hat{R}_{(H)I}^{(a)}|B\rangle$ is not normalizable, so this is not a physically acceptable expression for $|H\rangle$. This difficulty—which was noted in different language a decade ago by Bekenstein and Meisels[38]—arises from the fact that in the state $|H\rangle$ a superradiant *in* mode, being perfectly thermalized, has positive total red-shifted energy $\tilde{E}_\infty$ as measured by near-horizon ZAMO's, and this corresponds to negative total red-shifted energy E_∞ as measured by distant observers. Since, by applying $\hat{a}_I$, $\hat{a}_I^{\dagger}$, $\hat{a}_I'$, $\hat{a}_I'^{\dagger}$ to $|B\rangle$ one can construct only states with positive E_∞, not negative E_∞, it is impossible to express $|H\rangle$ by the action of these operators on $|B\rangle$.

From the material presented in Appendix A it should be clear that, if we had adopted the *near-horizon conventions and viewpoint* [$\tilde{\epsilon}>0$ in Eq. (B16)] for all *in* and *out* modes v_I, w_I just as we did for all *up* and *down* modes q_I, p_I, then we would have encountered no difficulties in computing $|H\rangle$ by $\hat{\eta}$-free techniques. Indeed, long ago Israel[3] evaluated $|H\rangle$ and $\hat{\rho}_H$ in just this manner with no difficulty [though, because his work was carried out simultaneously with and independently of that of Hartle and Hawking, he used the notation $|0(\kappa)\rangle$ instead of $|H\rangle$ and called this the "Kruskal vacuum state" instead of the "Hartle-Hawking vacuum state"].

Unfortunately, although the near-horizon conventions and viewpoint would permit the evaluation of $|H\rangle$ and $\hat{\rho}_H$ by standard techniques, they would produce difficulties with $|U\rangle$, $|UN\rangle$, $\hat{\rho}_U$, and $\hat{\rho}_{UN}$; cf. Appendix A. This has motivated us to stick with our distant-observer conventions and viewpoint for the *in* and *out* modes—at the price of adopting the η formalism for dealing with $|H\rangle$ and $\hat{\rho}_H$.

Returning to the η formalism, we shall construct $|H\rangle$ by adding and removing quanta from the vacuum state $|\tilde{B}\rangle$ which is defined by

$$\hat{b}_I|\tilde{B}\rangle=\hat{b}_I'|\tilde{B}\rangle=0 \quad \text{for } \tilde{\epsilon}>0 \quad \text{(C15a)}$$

(no quanta in *up* and *down* modes),

$$\hat{a}_I|\tilde{B}\rangle=\hat{a}_I'|\tilde{B}\rangle=0 \quad \text{for } \tilde{\epsilon}>0,\ \epsilon>0 \quad \text{(C15b)}$$

(no quanta in nonsuperradiant *in* and *out* modes),

$$a_I^{\dagger}|\tilde{B}\rangle=\hat{a}_I'^{\dagger}|\tilde{B}\rangle=0 \quad \text{for } \tilde{\epsilon}<0,\ \epsilon>0 \quad \text{(C15c)}$$

(-1 quanta in superradiant *in* and *out* modes). In this state, the state $|H\rangle$, and any other state obtained by removing quanta from or adding quanta to $|\tilde{B}\rangle$, $\hat{\eta}$ acts nontrivially (differs from unity) only on operators associated with the superradiant *in* and *out* modes: $\hat{\eta}\hat{a}_I^{\dagger}\hat{\eta}=-\hat{a}_I^{\dagger}$, $\hat{\eta}\hat{a}_I'^{\dagger}\hat{\eta}=-\hat{a}_I'^{\dagger}$ for I with $\epsilon>0$, $\tilde{\epsilon}<0$ [Eqs. (A9i)].

By the arguments given in Appendix B, the state $|H\rangle$ is the unique state satisfying Eq. (B27). It is straightforward to verify, by a calculation completely analogous to that for $|U\rangle$ [paragraph following Eq. (C4)], that this $|H\rangle$ is given by

$$|H\rangle = \prod_{\tilde\epsilon>0} \hat R^{(b)}_{(H)I} \prod_{\epsilon>0,\,\tilde\epsilon>0} \hat R^{(a)}_{(H)I} \prod_{\epsilon>0,\,\tilde\epsilon<0} \hat R^{(a)SR}_{(H)I} |\tilde B\rangle \ , \tag{C16}$$

where $\hat R^{(b)}_{(H)}$ and $\hat R^{(a)}_{(H)}$ are defined by Eqs. (C13) and (C14), and

$$\hat R^{(a)SR}_{(H)I} = C^{(a)SR}_{(H)I}\exp(e^{-\pi|\tilde\epsilon|/\kappa}\hat a_I \hat a'_I) \ . \tag{C17}$$

This state is readily normalized, by contrast with (C12), since for the superradiant *in* and *out* modes it is $e^{-\pi|\tilde\epsilon|/\kappa}<1$ that appears in the exponential [Eq. (C17)] rather than $e^{-\pi\tilde\epsilon/\kappa}>1$ [Eq. (C14)]. Notice that $\hat R^{(b)}_{(H)I}$ creates quanta with $\tilde\epsilon>0$ in the *up* and *down* modes, $\hat R^{(a)}_{(H)I}$ creates quanta with $\tilde\epsilon>0$ in the nonsuperradiant *in* and *out* modes, and $R^{(a)SR}_{(H)I}$ removes quanta with $\tilde\epsilon<0$ in the superradiant *in* and *out* modes; thus, all pieces of $|H\rangle$ correspond to creation of positive near-horizon energy $\tilde E_\infty$.

The density operator $\hat\rho_H$ associated with the state $|H\rangle$ can be derived by functional methods[37] or, more directly, from Eqs. (C2b) and (C16) using for $\{|\mathcal{N}'\rangle\}$ the Fock basis generated by applying $\{\hat a'^{\dagger}_I$ (for $\epsilon>0$, $\tilde\epsilon>0$), $\hat a'_I$ (for $\epsilon>0$, $\tilde\epsilon<0$), $\hat b'^{\dagger}_I\}$ to the state $|\tilde B\rangle$ of region I′ [Eqs. (C15) with only primed operators kept], and using for $\{|\mathcal{N}\rangle\}$ the Fock basis generated by applying $\{\hat a^{\dagger}_I$ (for $\epsilon>0$, $\tilde\epsilon>0$), $\hat a_I$ (for $\epsilon>0$, $\tilde\epsilon<0$), $\hat b^{\dagger}_I\}$ to the state $|\tilde B\rangle$ of region I. The result is

$$\hat\rho_H = \prod_{\tilde\epsilon>0} \hat\rho^{I\,up}_H \prod_{\tilde\epsilon>0,\,\epsilon>0} \hat\rho^{I\,in\,\mathrm{NSR}}_H \prod_{\tilde\epsilon<0,\,\epsilon>0} \hat\rho^{I\,in\,\mathrm{SR}}_H \ , \tag{C18}$$

where the density operators for the *I up* modes, the nonsuperradiant *I in* NSR modes, and the superradiant *I in* SR modes—and the corresponding probabilities for them to contain n quanta—are

$$\hat\rho^{I\,up}_H = \hat\rho^{I\,up}_U = (1-e^{-\tilde\epsilon/T_H})\exp(-\hat n^{up}_I\tilde\epsilon/T_H) \ , \tag{C19a}$$

$$p^{I\,up}_{(H)n} = p^{\mathrm{th}}_n = (1-e^{-\tilde\epsilon/T_H})e^{-n\tilde\epsilon/T_H} \quad \text{for } n\ge 0 \ ; \tag{C19b}$$

$$\hat\rho^{I\,in\,\mathrm{NSR}}_H = (1-e^{-\tilde\epsilon/T_H})\exp(-\hat n^{in}_I\tilde\epsilon/T_H) \ , \tag{C20a}$$

$$p^{I\,in\,\mathrm{NSR}}_{(H)n} = p^{\mathrm{th}}_n = (1-e^{-\tilde\epsilon/T_H})e^{-n\tilde\epsilon/T_H} \quad \text{for } n\ge 0,\ \tilde\epsilon>0 \ ; \tag{C20b}$$

$$\hat\rho^{I\,in\,\mathrm{SR}}_H = -(1-e^{-\tilde\epsilon/T_H})\exp(-\hat n^{in}_I\tilde\epsilon/T_H)\hat\eta^{in}_I\hat{\mathcal{P}}_- \ , \tag{C21a}$$

$$p^{I\,in\,\mathrm{SR}}_n = p^{\mathrm{th}}_n = -(1-e^{-\tilde\epsilon/T_H})e^{-n\tilde\epsilon/T_H} \quad \text{for } n\le -1,\ \tilde\epsilon<0 \ . \tag{C21b}$$

In Eq. (C21a) $\hat\eta^{in}_I$ is the $\hat\eta$ operator for the *I in* SR mode, $\hat\eta^{in}_I=(-1)^{\hat n^{in}_I+1}$, and $\hat{\mathcal{P}}_-$ projects onto states containing *negative* numbers of quanta.

Each of the Hartle-Hawking density operators (C19)—(C21) is *perfectly thermal*: Each describes a mixture in which the probability for its mode to be in the energy eigenstate with near-horizon red-shifted energy $\tilde E_I=\tilde\epsilon(n_I+\frac{1}{2})$ is

$$P(\tilde E_I)=\mathrm{const}\times\exp(-\tilde E_I/T_H) \ , \qquad T_H=\kappa/2\pi \ . \tag{C22}$$

(The *I in* SR thermal density operator only looks unfamiliar because it is that of a near-horizon thermal state written using the distant-observer viewpoint.) These density operators verify (in accord with Israel's[3] conclusions) that near-horizon ZAMO's see the field $\hat\Phi$ to be perfectly thermally excited when it is in the mathematically defined Hartle-Hawking vacuum state $|H\rangle$—a conclusion that we used as our physical definition of $|H\rangle$ in Sec. II. For further discussion see Sec. IV B.

We note, for completeness, that the thermal density submatrix (C21a) predicts a mean number of ($\epsilon>0$, $\tilde\epsilon<0$) quanta, in the superradiant *I in* mode, given by

$$\langle n^{in}_I\rangle = \mathrm{tr}(\hat n^{in}_I\hat\rho^{I\,in\,\mathrm{SR}}_H) = \sum_{n=-1}^{-\infty} n p^{I\,in\,\mathrm{SR}}_n = \frac{1}{e^{\tilde\epsilon/T_H}-1} = n^{th}_I \ , \tag{C23}$$

which (although it is negative) is the standard thermal occupation number for a rotating black-hole atmosphere as described in the distant-observer viewpoint; cf. Eq. (2.51).

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