Proof of three-flavor scattering formulas

Madhusree Mukerjee and Michael P. Mattis
Enrico Fermi Institute and Department of Physics, University of Chicago, Chicago, Illinois 60637
(Received 7 October 1988)

We detail the derivation of the two recently published formulas governing three-flavor meson-baryon scattering in the large-N_c limit.

In this Brief Report, we present the promised derivation of the two recently published formulas [Eqs. (7) and (8) of Ref. 1, reproduced as Eqs. (12) and (15) below] that govern quasielastic meson-baryon scattering in the large-N_c and unbroken $SU(3)_{flavor}$ limits. Our tool is the three-flavor Skyrme model, and we shall assume basic familiarity with this approach.[2] The principal expressions for the s-channel and t-channel partial-wave amplitudes are valid for mesons of arbitrary spin, and in this regard, they generalize the results of Refs. 3 and 4, which only apply to the pseudoscalar octet. The case of arbitrary spin was previously considered in the context of the better-known two-flavor Skyrme model in Ref. 5.

We are interested in the process

$$\phi+B\to\psi+B' . \tag{1}$$

Here, ϕ and ψ stand for mesons of arbitrary spin S_ϕ and S_ψ and $SU(3)_{flavor}$ quantum numbers $\{R_\phi,I_\phi,I_{\phi z},Y_\phi\}$ and $\{R_\psi,I_\psi,I_{\psi z},Y_\psi\}$, with $R_{\phi,\psi}$ the flavor representations; B and B' denote either $\frac{1}{2}^+$ octet or $\frac{3}{2}^+$ decuplet baryons, with spin and flavor quantum numbers S_B, R_B, etc., defined in analogy to the mesons.

In the three-flavor Skyrmion approach, the $\frac{1}{2}^+$ octet and $\frac{3}{2}^+$ decuplet baryons are viewed as rotating solitons. To construct them, one must first solve for the *nonrotating* solitons. This involves postulating an effective meson Lagrangian, and then finding the minimum-energy classical solutions with winding number (baryon number) unity. Because of energy considerations, any such configuration will have a large component in the lowest-lying meson multiplet, the pseudoscalar octet. If we think of the pseudoscalar mesons as residing in the exponent of the usual nonlinear chiral field U, then the soliton U_0 can always be chosen to lie in the isospin subgroup:

$$U_0=\exp\begin{bmatrix} iF(r)\hat{\mathbf{r}}\cdot\boldsymbol{\sigma} & 0\\ 0 & 0\end{bmatrix} . \tag{2}$$

The soliton profile $F(r)$ is determined by energy minimization, subject to the boundary conditions $[F(0)=\pi, F(\infty)=0]$ appropriate to baryon number one.

Following Ref. 5, we begin by examining the simplified process

$$\phi+U_0\to\psi+U_0 \tag{3}$$

in which the meson scatters, not from a physical baryon, but rather from a nonrotating soliton. What are the selection rules governing (3)? These will certainly not be the familiar ones of the strong interactions, since U_0 not only breaks translational invariance, but intertwines angular momentum and isospin indices. [We will find below that $SU(3)_{flavor}$ and angular momentum conservation are regained after the baryon wave functions are folded in.] First of all, since U_0 commutes with hypercharge, the scattering must vanish unless $Y_\phi=Y_\psi$. Furthermore, thanks to the dot-product structure in the exponent of U_0, the collision will preserve the vector sum $\mathbf{K}$ of the mesons' isospin and total angular momentum:

$$\mathbf{K}=\mathbf{I}_\phi+\mathbf{L}+\mathbf{S}_\phi=\mathbf{I}_\psi+\mathbf{L}'+\mathbf{S}_\psi , \tag{4}$$

with $\mathbf{L}$ and $\mathbf{L}'$ the initial and final orbital angular momenta. Defining the hybrid quantum numbers $\tilde{\mathbf{K}}=\mathbf{I}_\phi+\mathbf{L}$ and $\tilde{\mathbf{K}}'=\mathbf{I}_\psi+\mathbf{L}'$, we therefore find for the $\mathbf{T}$ matrix

$$\begin{aligned}\mathbf{T}(\phi+U_0\to\psi+U_0)&=\delta_{Y_\phi Y_\psi}(\langle I_\psi I_{\psi z}|\otimes\langle L'L'_z|\otimes\langle S_\psi S_{\psi z}|)\mathbf{T}(|I_\phi I_{\phi z}\rangle\otimes|LL_z\rangle\otimes|S_\phi S_{\phi z}\rangle)\\ &=\delta_{Y_\phi Y_\psi}\sum_{KK_z\tilde{K}\tilde{K}_z\tilde{K}'\tilde{K}'_z}\langle L'I_\psi L'_zI_{\psi z}|\tilde{K}'\tilde{K}'_z\rangle\langle\tilde{K}'S_\psi\tilde{K}'_zS_{\psi z}|KK_z\rangle\\ &\qquad\times\langle\tilde{K}\tilde{K}_z|LI_\phi L_zI_{\phi z}\rangle\langle KK_z|\tilde{K}S_\phi\tilde{K}_zS_{\phi z}\rangle\mathcal{T}^{\{I_\phi I_\psi Y_\phi\}}_{K\tilde{K}\tilde{K}'LL'} ,\end{aligned} \tag{5}$$

where the $\mathcal{T}$'s are reduced matrix elements characterizing the simple process (3) (they are functions of energy).

In addition to hypercharge and "K-spin" conservation, the set of possible reduced matrix elements will be constrained by the discrete symmetries of the soliton. Thus, the fact that $U_0(\hat{\mathbf{r}})=U_0^\dagger(-\hat{\mathbf{r}})$ enforces the parity constraint

$$P_\phi \times (-1)^L = P_\psi \times (-1)^{L'}$$

on the process (3), with $P_{\phi,\psi}$ the intrinsic meson parity. Likewise, the identity $U_0(\hat{\mathbf{r}}) = \sigma_y U_0^*(\hat{\mathbf{r}})\sigma_y$ serves to rule out (*inter alia*) any $\pi\eta$ coupling through such conceivable terms as $\nabla\eta\cdot\nabla\times\pi$. Note that this last result, together with hypercharge conservation, gives rise to an "accidental" isospin-conserving factor $\delta_{I_\phi I_\psi}$ in Eq. (5) in the important special case that ϕ and ψ are elements of the pseudoscalar octet; this factor is essential to verifying that Eq. (12) below properly reduces to the corresponding formula given in Refs. 3 and 4 when meson spin is set to zero.

Although we have arrived at Eq. (5) by assuming that the soliton configuration resides exclusively in the pseudoscalar octet, we believe that it applies to the higher-mass multiplets as well. For, in our experience, regardless of one's choice of effective Lagrangian, it appears that the soliton components in the various meson multipliets can always be chosen to commute with Y and $\mathbf{K}$. In the absence of a general proof, however, we shall take this as a fundamental assumption.

A modest generalizaton of (3) is to consider scattering, not from the canonically embedded soliton U_0, but rather from the rotated configuration

$$U_A = A U_0 A^{-1}, \quad A \in \mathrm{SU}(3) . \tag{6}$$

It is easy to convince oneself that

$$\mathbf{T}(\phi+U_A\to\psi+U_A) = \sum [D^{(R_\psi)}(A)]_{(I_\psi I_{\psi z} Y_\psi),(I'_\psi, I'_{\psi z} Y'_\psi)} \mathbf{T}(\phi'+U_0\to\psi'+U_0)[D^{(R_\phi)\dagger}(A)]_{(I'_\phi I'_{\phi z} Y'_\phi),I_\phi I_{\phi z} Y_\phi)} , \tag{7}$$

the sum extending over all $(I'_\phi, I'_{\phi z}, Y'_\phi)\in R_\phi$ and $(I'_\psi, I'_{\psi z}, Y'_\psi)\in R_\psi$.

Armed with Eqs. (5) and (7), we can finally consider the physically relevant case (1) of meson scattering from a bona fide $\frac{1}{2}^+$ octet or $\frac{3}{2}^+$ decuplet baryon. The required expression is the quantum superposition

$$\mathbf{T}(\phi+B\to\psi+B') = \int_{\mathrm{SU}(3)} dA\, \chi^\dagger_{B'}(A)\mathbf{T}(\phi+U_A\to\psi+U_A)\chi_B(A) . \tag{8}$$

Here, χ_B and $\chi_{B'}$ are the $\mathrm{SU}(3)_{\text{flavor}}$ wave functions of the initial and final baryons; they are themselves expressible as D functions:

$$\chi_B(A) = (-1)^{S_B - S_{Bz}}\sqrt{\dim R_B}[D^{(R_B)\dagger}(A)]_{(S_{B,}-S_{Bz},1),(I_B,I_{Bz},Y_B)} . \tag{9}$$

The indicated integral over SU(3) can easily be carried out with the help of the standard identities[6]

$$[D^{(R)}(A)]_{(I_1 I_{1z} Y_1),(I_2 I_{2z} Y_2)}[D^{(R')}(A)]_{(I_3 I_{3z} Y_3),(I_4 I_{4z} Y_4)} = \sum_{R''\gamma I_5 I_{5z} Y_5 I_6 I_{6z} Y_6} [D^{(R'')}(A)]_{(I_5 I_{5z} Y_5),(I_6 I_{6z} Y_6)} \langle (R I_1 I_{1z} Y_1);(R' I_3 I_{3z} Y_3)|R''\gamma I_5 I_{5z} Y_5\rangle \times \langle (R I_2 I_{2z} Y_2);(R' I_4 I_{4z} Y_4)|R''\gamma I_6 I_{6z} Y_6\rangle \tag{10}$$

and

$$\int_{\mathrm{SU}(3)} dA\,[D^{(R)\dagger}(A)]_{(I_1 I_{1z} Y_1),(I_2 I_{2z} Y_2)}[D^{(R')}(A)]_{(I_3 I_{3z} Y_3),(I_4 I_{4z} Y_4)} = (\dim R)^{-1}\delta_{RR'}\delta_{I_1 I_4}\delta_{I_{1z} I_{4z}}\delta_{Y_1 Y_4}\delta_{I_2 I_3}\delta_{I_{2z} I_{3z}}\delta_{Y_2 Y_3} . \tag{11}$$

The brackets in (10) are SU(3) Clebsch-Gordan coefficients.

In order to obtain the most compact expression for the scattering (still paralleling Ref. 5), one projects the initial (final) meson-baryon system onto a state of definite total s-channel spin and angular momentum $|J_s J_{sz} S\rangle$ ($|J'_s J'_{sz} S'\rangle$), and total $\mathrm{SU}(3)_{\text{flavor}}$ $|R_s\gamma_s I_s I_{sz} Y_s\rangle$ ($|R'_s\gamma'_s I'_s I'_{sz} Y'_s\rangle$). With the shorthand notation $[K]\equiv 2K+1$, etc., we find, after some algebra, that the sum on magnetic quantum numbers can be accomplished in closed form, yielding for the $\mathbf{T}$-matrix element:

$$T(LL'SS'J_sJ'_sJ_{sz}J'_{sz}R_sR'_s\gamma_s\gamma'_sI_sI'_sI_{sz}I'_{sz}YY') = \delta_{J_sJ'_s}\delta_{J_{sz}J'_{sz}}\delta_{R_sR'_s}\delta_{I_sI'_s}\delta_{I_{sz}I'_{sz}}\delta_{Y_sY'_s}\sum_{II'I''Y}(-1)^{I+I'+Y}\left(\begin{array}{cc|c} R_B & R_\phi & R_s\gamma_s \\ S_B 1 & IY & I'', Y+1\end{array}\right)\left(\begin{array}{c|cc} R_s\gamma'_s & R'_B & R_\psi \\ I'', Y+1 & S'_B 1 & I'Y\end{array}\right)$$
$$\times\sum_{K\tilde{K}\tilde{K}'}\frac{[I''][K]\{(\dim R_B)(\dim R'_B)[S][S'][\tilde{K}][\tilde{K}']\}^{1/2}}{\dim R_s}\times\begin{Bmatrix} L & I & \tilde{K} \\ S & S_B & S_\phi \\ J_s & I'' & K\end{Bmatrix}\begin{Bmatrix} L' & I' & \tilde{K}' \\ S' & S'_B & S_\psi \\ J_s & I'' & K\end{Bmatrix}\mathcal{T}^{\{II'Y\}}_{K\tilde{K}\tilde{K}'LL'} . \tag{12}$$

The sums run over all values consistent with the isocalar factors and $9j$ symbols. This is Eq. (7) of Ref. 1.

The Kronecker δ's which have emerged in (12) express conservation of angular momentum and $SU(3)_{\rm flavor}$ (note that the discrete quantum number γ_s is not generally conserved). In obtaining (12), we have factored each SU(3) Clebsch-Gordan coefficient into a product of an SU(2) Clebsch-Gordan coefficient and an SU(3) isoscalar factor,

$$\langle (R_1 I_1 I_{1z} Y_1);(R_2 I_2 I_{2z} Y_2)|R_\gamma I I_z Y\rangle = \langle I_1 I_2 I_{1z} I_{2z}|I I_z\rangle \left[\begin{array}{cc|c} R_1 & R_2 & R\gamma \\ I_1 Y_1 & I_2 Y_2 & IY \end{array}\right], \quad (13)$$

reexpressed all SU(2) Clebsch-Gordan coefficients as $3j$ symbols, and finally, made use of the definition of the $9j$ symbol as a product of $3j$ symbols:

$$\begin{bmatrix} J_{13} & J_{24} & J \\ M_{13} & M_{24} & M \end{bmatrix} \begin{Bmatrix} j_1 & j_2 & J_{12} \\ j_3 & j_4 & J_{34} \\ J_{13} & J_{24} & J \end{Bmatrix} = \sum_{m_1 m_2 m_3 m_4 M_{12} M_{34}} \begin{bmatrix} j_1 & j_2 & J_{12} \\ m_1 & m_2 & M_{12} \end{bmatrix} \begin{bmatrix} j_3 & j_4 & J_{34} \\ m_3 & m_4 & M_{34} \end{bmatrix} \begin{bmatrix} j_1 & j_3 & J_{13} \\ m_1 & m_3 & M_{13} \end{bmatrix}$$
$$\times \begin{bmatrix} j_2 & j_4 & J_{24} \\ m_2 & m_4 & M_{24} \end{bmatrix} \begin{bmatrix} J_{12} & J_{34} & J \\ M_{12} & M_{34} & M \end{bmatrix}. \quad (14)$$

Inspired by Donohue's finding[7] of substantial simplification in the helicity amplitudes when isospin is formulated in the t channel instead of the s channel, we also presented in Ref. 1 the t-channel analog of Eq. (12); namely,

$$\mathbf{T}(LL'J_\phi J_\psi J_t J_t' J_{tz} J_{tz}' R_t R_t' \gamma_t \gamma_t' I_t I_t' I_{tz} I_{tz}' Y_t Y_t') = \delta_{J_t J_t'} \delta_{J_{tz} J_{tz}'} \delta_{R_t R_t'} \delta_{I_t I_t'} \delta_{I_{tz} I_{tz}'} \delta_{Y_t Y_t'} \frac{\{(\dim R_B)(\dim R_B')[J_\phi][J_\psi]\}^{1/2}}{\dim R_t}$$
$$\times \sum_{II'Y'} \left[\begin{array}{cc|c} R_\phi & R_\psi^* & R_t\gamma_t \\ IY & I',-Y & J_t 0 \end{array}\right] \left[\begin{array}{c|cc} R_t\gamma_t' & R_B^* & R_B' \\ J_t 0 & S_B,-1 & S_B' 1 \end{array}\right]$$
$$\times \sum_{K\tilde{K}\tilde{K}'} (-1)^{(Y+1)/2+S_B+J_t+J_\psi+K+\tilde{K}+\tilde{K}'+L+L'} [K]\sqrt{[\tilde{K}][\tilde{K}']}$$
$$\times \begin{Bmatrix} J_\phi & I & K \\ I' & J_\psi & J_t \end{Bmatrix} \begin{Bmatrix} J_\phi & I & K \\ \tilde{K} & S_\phi & L \end{Bmatrix} \begin{Bmatrix} J_\psi & I' & K \\ \tilde{K}' & S_\psi & L' \end{Bmatrix} \mathcal{T}^{\{II'Y\}}_{K\tilde{K}\tilde{K}'LL'}. \quad (15)$$

Here, J_ϕ and J_ψ are defined by $\mathbf{J}_\phi = \mathbf{L} + \mathbf{S}_\phi$ and $\mathbf{J}_\psi = \mathbf{L}' + \mathbf{S}_\psi$.

The derivation of Eq. (15) follows exactly the same course as before through Eq. (11), except that $D^{(R_\psi)}(A)$ in Eq. (7) and $D^{(R_B)\dagger}(A)$ in Eq. (8) are each adjointed with the help of the identity[6]

$$[D^{(R)}(A)]_{(II_zY),(I'I_z'Y')} = (-1)^{I_z+I_z'+Y/2+Y'/2} [D^{(R^*)\dagger}(A)]_{(I',-I_z',-Y'),(I,-I_z,-Y)}. \quad (16)$$

In order to project out total t-channel angular momentum and flavor, we need a prescription for crossing all the bras associated with the final meson ψ into kets, and conversely, all the kets associated with the initial baryon B into bras. Our conventions are

$$|II_z\rangle \leftrightarrow (-1)^{I+I_z}\langle I,-I_z|$$

for SU(2) quantities, and

$$|RII_zY\rangle \leftrightarrow (-1)^{I_z+Y/2}\langle R^*I,-I_z,-Y|$$

for SU(3) quantities. The $6j$ symbols in (15) result from the identity

$$\sum_{M_1 M_2 M_3} (-1)^{J_1+J_2+J_3+M_1+M_2+M_3} \begin{bmatrix} J_1 & J_2 & j_3 \\ M_1 & -M_2 & m_3 \end{bmatrix} \begin{bmatrix} J_2 & J_3 & j_1 \\ M_2 & -M_3 & m_1 \end{bmatrix} \begin{bmatrix} J_3 & J_1 & j_2 \\ M_3 & -M_1 & m_2 \end{bmatrix}$$
$$= \begin{bmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{bmatrix} \begin{Bmatrix} j_1 & j_2 & j_3 \\ J_1 & J_2 & J_3 \end{Bmatrix}. \quad (17)$$

Finally, we should point out that although the two-flavor Skyrmion formalism for partial-wave amplitudes described in Refs. 1 and 5 can be shown to have close parallels to the study of one-boson exchange in the large-N_c limit,[8] we are not aware of any such correspondence in the three-flavor case.

This work was supported in part by NSF Grant No. PHY-85-21588 and DOE Contract No. DE-AC0280ER/10587.

[1]M. P. Mattis and M. Mukerjee, Phys. Rev. Lett. **61**, 1344 (1988).

[2]E. Guadagnini, Nucl. Phys. **B236**, 35 (1984). Note, however, that the baryon wave functions given here are in error; see Refs. 3 and 4.

[3]M. P. Mattis, in *Nuclear Chromodynamics: Quarks and Gluons in Particles and Nuclei,* proceedings of the ITP Workshop, Santa Barbara, 1985, edited by S. Brodsky and E. Moniz (World Scientific, Singapore, 1986).

[4]M. Karliner and M. P. Mattis, Phys. Rev. Lett. **56**, 428 (1986); Phys. Rev. D **34**, 1991 (1986).

[5]M. P. Mattis, Phys. Rev. Lett. **56**, 1103 (1986).

[6]J. J. deSwart, Rev. Mod. Phys. **35**, 4 (1963). The quantum number γ that appears in the SU(3) Clebsch-Gordan coefficients in (10) serves only to distinguish degenerate representations in a tensor product, e.g., the two **8**'s that appear in $\mathbf{8}\times\mathbf{8}$.

[7]J. T. Donohue, Phys. Rev. Lett. **58**, 3 (1987); Phys. Rev. D **37**, 631 (1988).

[8]M. P. Mattis, Phys. Rev. D **39**, 994 (1989).