Forward-backward multiplicity distributions in a pure-birth process

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Forward-backward multiplicity correlations are considered in a pure-birth process. We obtain analytic formulas for conditional moments of the backward multiplicity and the conditional backward multiplicity distribution at a given forward multiplicity. The conditional mean multiplicities and the conditional dispersions observed in pp collisions at $\sqrt{s}=24$, 31, 45, and 63 GeV and in $\bar{p}p$ collisions at $\sqrt{s}=546$ GeV are analyzed. Good agreement between our approach and the data allows us to recognize some details specifying the stochastic approach of hadron-hadron collisions.

I. INTRODUCTION

Probability distributions obtained in a pure-birth process are sometimes applied to the analysis of observed multiplicity distributions in hadron-hadron (h-h) collisions.[1-3] In this paper we consider forward-backward multiplicity correlation in a framework of the pure-birth process.

The branching equation for a pure-birth (PB) process is given as

$$\frac{\partial}{\partial t}P(n,t)=\lambda(n-1)P(n-1,t)-\lambda nP(n,t)\ . \tag{1}$$

It is often investigated under the initial conditions

$$P(n,t=0)=\delta_{n,m} \tag{2a}$$

and

$$P(n,t=0)=\langle m\rangle^n e^{-\langle m\rangle}/n!\ . \tag{2b}$$

Analysis of multiplicity moments has shown[1] that the probability distribution $P(n,t)$ obtained from Eq. (1) with the initial condition (2b) is more favorable than that with the condition (2a).

A probability generating function is defined as

$$Q(z;t)=\sum_{n=0}^{\infty}P(n;t)z^n\ . \tag{3}$$

That corresponding to Eq. (2b) is given explicitly as

$$Q(z;t)=\exp\left[\frac{\langle n\rangle(z-1)}{1-\langle n\rangle(z-1)/\xi}\right], \tag{4}$$

where

$$\langle n\rangle=\langle m\rangle(e^{\lambda t}-1),\quad \xi=(1/\langle m\rangle-1/\langle n\rangle)^{-1}\ . \tag{5}$$

Hereafter, we abbreviate the variable t in the probability distribution and the generating function.

The probability distribution is given by means of Eq. (4) as

$$P(0)=Q(z=0)=\exp\left[-\frac{\langle n\rangle}{1+\langle n\rangle/\xi}\right], \tag{6a}$$

$$\begin{aligned}P(n)&=\frac{1}{n!}\frac{\partial n}{\partial z^n}Q(z)\bigg|_{z=0}\\&=\frac{\xi}{n}\frac{(\langle n\rangle/\xi)^n}{(1+\langle n\rangle/\xi)^{n+1}}\exp\left[-\frac{\langle n\rangle}{1+\langle n\rangle/\xi}\right]\\&\quad\times L_{n-1}^{(1)}\left[-\frac{\xi}{1+\langle n\rangle/\xi}\right]\quad(n=1,2,\dots)\ ,\end{aligned} \tag{6b}$$

where $L_n^{(1)}(x)$ is the associated Laguerre polynomial. The jth moment of multiplicity is written as

$$\begin{aligned}\langle n(n-1)\cdots(n-j+1)\rangle&=\frac{\partial^j}{\partial z^j}Q(z)\bigg|_{z=1}\\&=\Gamma(j)\langle n\rangle(\langle n\rangle/\xi)^{j-1}\\&\quad\times L_{j-1}^{(1)}(-\xi)\ ,\end{aligned} \tag{7}$$

where $\langle n\rangle$ is the mean multiplicity. From Eq. (7), we obtain

$$C_2=\langle n^2\rangle/\langle n\rangle^2=1+2/\xi+1/\langle n\rangle\ . \tag{8}$$

Forward-backward (FB) multiplicity correlations observed in h-h collisions have been analyzed by several authors.[4-6] In this paper we would examine whether observed FB multiplicity correlations are described in the framework of the pure-birth process under the condition (2b).

In Sec. II some formulas of the FB multiplicity correlation are obtained from the generating function. Asymptotic behavior of the conditional mean backward multiplicity is given in Sec. III. Analyses of the conditional means forward multiplicity and the conditional dispersion observed in pp collisions at $\sqrt{s}=24$, 31, 45, and 63 GeV, and that of the conditional mean multiplicity in $\bar{p}p$ collisions at $\sqrt{s}=546$ GeV, are made in Sec. IV. Concluding remarks are presented in the final section.

II. FORMULAS FOR THE *FB* MULTIPLICITY CORRELATION

The following probability distributions are defined to consider the *FB* multiplicity correlation.[6,7]

(D1) A two-component probability distribution $p(n_1,n_2)$ that n_1 charged particles are emitted in one (L) region and n_2 charged particles are in the other (R) region. It is normalized as

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} p(n_1,n_2)=1 .$$

(D2) $f_n(n_1)$ is the probability that n_1 particles are emitted in the L region under the condition that n ($n=n_1+n_2$) particles are produced in total. It satisfies $\sum_{n_1=0}^{n} f_n(n_1)=1$.

(D3) The multiplicity distribution in the R region is denoted as $p_2(n_2)$.

(D4) $p(n_1|n_2)$ is the conditional probability that n_1 particles are emitted in the L region under the condition that n_2 particles are emitted in the R region. The normalization condition is written as

$$\sum_{n_1=0}^{\infty} p(n_1|n_2)=1 \quad (n_2=0,1,2,\ldots) .$$

Then we can write the relations

$$p(n_1,n_2)=P(n)f_n(n_1) , \tag{9a}$$

$$p(n_1,n_2)=p_2(n_2)p(n_1|n_2) . \tag{9b}$$

For simplicity, we assume the binomial form for $f_n(n_1)$,

$$f_n(n_1)={}_nC_{n_1}\alpha^{n_1}\beta^{n-n_1} \quad (\alpha+\beta=1) , \tag{10}$$

where α (β) is the probability that a particle is emitted in the L (R) region.

The generating function $G(u,v)$ of the two-component probability $p(n_1,n_2)$ is defined as

$$G(u,v)=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} p(n_1,n_2)u^{n_1}v^{n_2} = \sum_{n=0}^{\infty} P(n)(\alpha u+\beta v)^n = Q(\alpha u+\beta v) . \tag{11}$$

Then the two-component probability $p(n_1,n_2)$ and the multiplicity distribution $p_2(m)$ in the R region are expressed by the use of $Q(x)$:

$$p(j,m)=\frac{1}{j!m!}\alpha^j\beta^m\frac{\partial^j\partial^m}{\partial u^j\partial v^m}G(u,v)\Big|_{u=0,v=0} =\frac{1}{j!m!}\frac{\partial^{j+m}}{\partial x^{j+m}}Q(x)\Big|_{x=0} \tag{12}$$

and

$$p_2(m)=\frac{1}{m!}\frac{\partial^m}{\partial v^m}G(u,v)\Big|_{u=1,v=0} =\frac{1}{m!}\beta^m\frac{\partial^m}{\partial x^m}Q(x)\Big|_{x=\alpha} . \tag{13}$$

The jth conditional moment $\langle n_1(n_1-1)\cdots(n_1-j+1)\rangle_m$ of the L multiplicity at the given R multiplicity m is defined by the formula

$$\langle n_1(n_1-1)\cdots(n_1-j+1)\rangle_m p_2(m)=\sum_{n_1=0}^{\infty} n_1(n_1-1)\cdots(n_1-j+1)p(n_1|m)p_2(m) =\sum_{n_1=0}^{\infty} n_1(n_1-1)\cdots(n_1-j+1)p(n_1,m) . \tag{14}$$

Then, from Eqs. (11), (13), and (14), we obtain

$$\langle n_1(n_1-1)\cdots(n_1-j+1)\rangle_m p_2(m)=\frac{1}{m!}\frac{\partial^j\partial^m}{\partial u^j\partial v^m}G(u,v)\Big|_{u=1,v=0} =\frac{1}{m!}\alpha^j\beta^m\frac{\partial^{j+m}}{\partial x^{j+m}}Q(x)\Big|_{x=\alpha}=\frac{(m+j)!}{m!}(\alpha/\beta)^j p_2(m+j) . \tag{15}$$

Therefore the conditional moment is written by means of $p_2(m)$:

$$\langle n_1(n_1-1)\cdots(n_1-j+1)\rangle_m=\frac{(m+j)!}{m!}(\alpha/\beta)^j p_2(m+j)/p_2(m) . \tag{16}$$

Equation (16) generally holds as long as Eq. (10) is assumed. The conditional dispersion $D_L(m)$ when m particles are observed in the R region is defined as

$$D_L(m)=(\langle n_1^2\rangle_m-\langle n_1\rangle_m^2)^{1/2} . \tag{17}$$

The multiplicity distribution in the R region is given from Eqs. (4) and (13) as

$$p_2(0)=\exp\left[-\frac{\beta\langle n\rangle}{1+\beta\langle n\rangle/\xi}\right] , \tag{18a}$$

$$p_2(m)=\frac{\xi}{m}\frac{(\beta\langle n\rangle/\xi)^m}{(1+\beta\langle n\rangle/\xi)^{m+1}}\exp\left[-\frac{\beta\langle n\rangle}{1+\beta\langle n\rangle/\xi}\right]L_{m-1}^{(1)}\left[-\frac{\xi}{1+\beta\langle n\rangle/\xi}\right] \quad (m=1,2,\dots) . \tag{18b}$$

Equation (18) is obtained from Eq. (6) by the substitution of $\beta\langle n\rangle$ for $\langle n\rangle$. From Eqs. (16) and (18) the conditional mean L multiplicity is expressed as

$$\langle n_1\rangle_m=m\frac{\alpha\langle n\rangle/\xi}{1+\beta\langle n\rangle/\xi}L_m^{(1)}(-x)/L_{m-1}^{(1)}(-x) , \tag{19}$$

where $x=\xi/(1+\beta\langle n\rangle/\xi)$.

III. ASYMPTOTIC BEHAVIOR OF THE CONDITIONAL MEAN L MULTIPLICITY

We examine the asymptotic behavior of $\langle n_1\rangle_m$ in two cases.

(i) When $|x|\ll 1$, namely, $\xi\ll 1$ or $\beta\langle n\rangle\gg 1$, the associated Laguerre polynomial is approximated by

$$L_n^{(1)}(-x)\simeq(n+1)\left[1+\frac{n}{2}x+O(x^2)\right] .$$

Then we get

$$\langle n_1\rangle_m=(m+1)\frac{\alpha\langle n\rangle}{1+\beta\langle n\rangle/\xi}\left[1+\frac{x}{2}+O(x^2)\right] . \tag{20}$$

(ii) When $|x|\gg 1$ ($\xi\gg 1$) with finite $\beta\langle n\rangle$, using the approximation

$$L_n^{(1)}(-x)\simeq\frac{x^n}{n!}\left[1+\frac{n(n+1)}{x}+O(1/x^2)\right] ,$$

we obtain

$$\langle n_1\rangle_m=\frac{\alpha\langle n\rangle/\xi}{1+\beta\langle n\rangle/\xi}[2m+1+O(1/x^2)] . \tag{21}$$

In both asymptotic limits, $\langle n_1\rangle_m$ is expressed as a linear function of m.

TABLE I. Comparison of the observed multiplicity moments in pp collisions at $\sqrt{s}=24$, 31, 45, and 63 GeV with our calculations involving assumptions (i), (ii), (i′), and (ii′), respectively. $\langle n\rangle$ and C_2 are inputs in our calculations.

	$\langle n\rangle/C_2$	$\langle n_{ch}\rangle$	C_2^{ch}	C_3^{ch}	C_4^{ch}	C_5^{ch}
			$\sqrt{s}=23.6$ GeV			
Expt		8.12±0.08	1.249±0.009	1.840±0.033	3.08±0.09	
(i)	8.12/1.249	8.12	1.249	1.830	3.047	5.645
(ii)	4.06/1.249	8.12	1.249	1.809	2.945	5.281
(i′)	6.12/1.438	8.11	1.248	1.856	3.184	6.141
(ii′)	3.06/1.438	8.12	1.248	1.844	3.119	5.903
			$\sqrt{s}=30.8$ GeV			
Expt		9.54±0.12	1.256±0.012	1.859±0.041	3.12±0.11	
(i)	9.54/1.256	9.54	1.255	1.857	3.130	5.877
(ii)	4.77/1.256	9.54	1.256	1.843	3.058	5.621
(i′)	7.54/1.438	8.11	1.255	1.880	3.246	6.298
(ii′)	3.77/1.438	8.12	1.255	1.870	3.195	6.111
			$\sqrt{s}=45.2$ GeV			
Expt		11.01±0.17	1.287±0.015	1.988±0.054	3.53±0.16	
(i)	11.01/1.287	11.00	1.286	1.975	3.483	6.885
(ii)	5.51/1.287	11.00	1.287	1.965	3.436	6.713
(i′)	9.01/1.429	11.00	1.286	1.998	3.606	7.343
(ii′)	4.51/1.429	11.00	1.286	1.993	3.577	7.236
			$\sqrt{s}=62.8$ GeV			
Expt		12.70±0.12	1.297±0.010	2.017±0.034	3.60±0.10	
(i)	12.70/1.297	12.69	1.296	2.014	3.604	7.242
(ii)	6.35/1.297	12.69	1.296	2.006	3.567	7.101
(i′)	10.70/1.418	12.69	1.295	2.033	3.708	7.637
(ii′)	5.35/1.418	12.69	1.295	2.027	3.680	7.529

IV. ANALYSIS OF EXPERIMENTAL DATA

In this section the observed multiplicity correlations are analyzed by the use of the formulas mentioned in Sec. III. In our formulas, three parameters, $\langle n \rangle$, ξ, and α (or $\beta = 1-\alpha$) are contained. We take $\alpha = 0.5$ because of the *FB* symmetry in rapidity plot. ξ is expressed as

$$\xi = 2(C_2 - 1 - 1/\langle n \rangle)^{-1} , \tag{22}$$

where $C_2 = \langle n^2 \rangle / \langle n \rangle^2$. The parameters $\langle n \rangle$ and ξ can be determined by the use of the experimental data $\langle n_{\rm ch} \rangle$ and $\langle n_{\rm ch}^2 \rangle$.

In order to fix $\langle n \rangle$ and ξ, we consider first the following two cases

(i) Particles are distributed in the L region or R region under the binomial law. In this case, formulas derived in Sec. III are used. Then, we get

$$\langle n_{\rm ch}^j \rangle = \sum_{j=0}^{\infty} n^j P(n) = \langle n^j \rangle \quad (j=1,2) . \tag{23}$$

(ii) Particles are distributed in the L region or R region in pairs under the binomial distribution. In other words, the formulas in Sec. III are used by the substitution of $\langle n \rangle /2$ for $\langle n \rangle$, and m in these formulas is interpreted as the number of pairs:[4,5] namely, $2m$ particles are emitted in the R region. Then, we get

$$\langle n_{\rm ch}^j \rangle = \sum_{j=0}^{\infty} (2n)^j P(n) = 2^j \langle n^j \rangle \quad (j=1,2) . \tag{24}$$

However, in *pp* collisions, the leading-particle effect plays an important role. In our analysis, the parameter characterizing that effect has value 2 (in accordance, e.g., with Ref. 8). It happens very likely that one charged particle is in the L region, and another charged particle is in the R region, due to the leading-particle effect. In order to take this effect into account, we consider the following two modified cases.

(i′) One of charged particles emitted in the final states is always in the L region, and another charged particle is in the R region. Other particles are distributed in the L region or R region under the binomial law. Then, we get

$$\langle n_{\rm ch}^j \rangle = \sum_{j=0}^{\infty} (n+2)^j P(n) = \langle (n+2)^j \rangle \quad (j=1,2) . \tag{25}$$

In this case, $P(n)$ denotes the probability that $n+2$ charged particles are emitted in the final states. The mean multiplicity under the condition that $m+1$ charged particles are in the R region is represented as $\langle n_1 \rangle_m + 1$ $(m=0,1,2,\ldots)$.

(ii′) One charged particle is always in the L region, and another charged particle is in the R region. Other particles are distributed in the L region in pairs under the binomial distribution. Then, we obtain

$$\langle n_{\rm ch}^j \rangle = \sum_{j=0}^{\infty} (2n+2)^j P(n) = 2^j \langle (n+1)^j \rangle \quad (j=1,2) . \tag{26}$$

In this case, $P(n)$ represents the probability that $2(n+1)$ charged particles are emitted in the final states. The mean multiplicity under the condition that $2m+1$ charged particles are in the R region is expressed as $\langle n_1 \rangle_m + 1$ $(m=0,1,2,\ldots)$.

Hereafter, we use the more common, conventional notation $\langle n_B \rangle = \langle n_B \rangle_F$ for the conditional mean multiplicity $\langle n_1 \rangle_m$ defined in Eq. (19), and $D_B = (D_B)_F$ for the conditional dispersion $D_L(m)$ defined in Eq. (17). In other words, $\langle n_B \rangle$ represents the conditional mean multiplicity in the backward region when n_F particles are found in the forward region, and D_B denotes the dispersion of the backward multiplicity at fixed n_F.

Multiplicity moments observed in *pp* collisions at $\sqrt{s} = 24$, 31, 45, and 63 GeV at the CERN ISR (Ref. 9) are compared with our calculation in Table I. As it is seen, our results with assumptions (i)–(ii′) are in very good agreement with the experimental data.

Conditional mean multiplicities $\langle n_B \rangle$ and conditional dispersions D_B observed in *pp* collisions at $\sqrt{s} = 24$, 31, 45, and 63 GeV at the ISR (Ref. 9) are compared with our calculations in Figs. 1, 2, 3, and 4, respectively. The

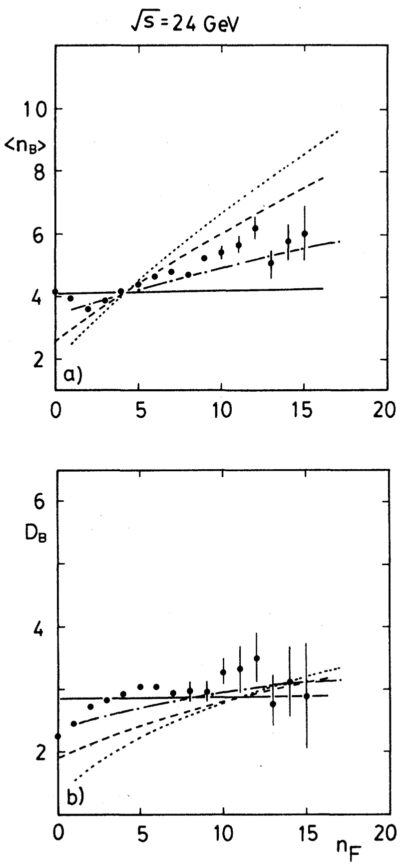

FIG. 1. Analysis of data in *pp* collisions at $\sqrt{s} = 24$ GeV. (a) Conditional mean multiplicity $\langle n_B \rangle$ and (b) conditional dispersion D_B are compared with our calculations: dashed line is obtained with assumption (i), solid line with (ii), dotted line with (i′), and dash-dotted line with (ii′). n_F represents the forward multiplicity.

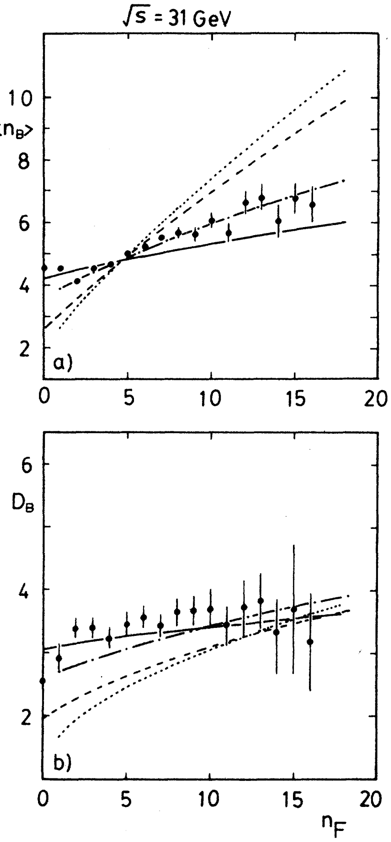

FIG. 2. Analysis of data in pp collisions at $\sqrt{s}=31$ GeV: otherwise the same as in Fig. 1.

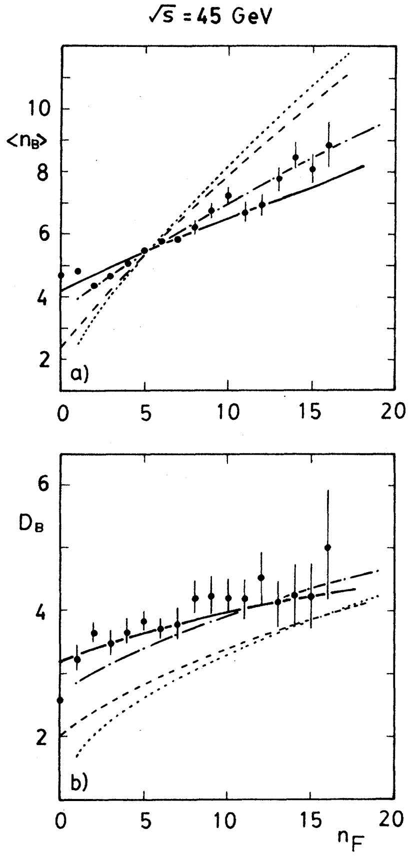

FIG. 3. Analysis of data in pp collisions at $\sqrt{s}=45$ GeV: otherwise the same as in Fig. 1.

dashed lines in these figures are obtained from assumption (i). These lines cannot explain the data. Solid lines are obtained from (ii), and are well fitted to the data except for the data at $\sqrt{s}=24$ GeV. Calculated results of $\langle n_B \rangle$ and D_B at $\sqrt{s}=24$ GeV are almost independent of the multiplicity n_F in the forward region, contrary to the experimental results.

In order to examine the leading-particle effect in pp collisions, we calculate the conditional mean multiplicities and the conditional dispersions, with assumptions (i′) and (ii′). The dotted lines in Figs. 1–4 are obtained from assumption (i′). These lines cannot reproduce the data. Dash-dotted lines in these figures are obtained from (ii′), and are well fitted to the data at the ISR energy region. The calculated results with assumption (ii′) are better fitted to the data at $\sqrt{s}=24$ GeV than those with (ii). Calculated results with (ii′) can explain the data at other ISR energies comparably as well as those with (ii).

In Fig. 5 the conditional mean multiplicity observed in $\bar{p}p$ collisions at $\sqrt{s}=546$ GeV at the CERN S$\bar{p}p$S collider[10] is compared with our calculations involving (i) and (ii). At this energy our calculation with assumption (ii) is more preferable than that with assumption (i).

V. CONCLUDING REMARKS

Forward-backward multiplicity correlations are formulated in the framework of the pure-birth process. It is found that the jth conditional moment of the L multiplicity when m particles are emitted in the R region is in general expressed by the ratio of the multiplicity distributions in the R region [compare Eq. (16)].

Multiplicity correlations observed in pp collisions at ISR energy region, and in $\bar{p}p$ collisions at $\sqrt{s}=546$ GeV at the S$\bar{p}p$S collider are well explained by our calculation involving the formulas of Sec. III with assumption (ii) that particles produced in h-h collisions are distributed in the L or R region in pairs.

As we take the leading-particle effect into account, calculated results of $\langle n_B \rangle$ and D_B with assumption (ii′) can reproduce the experimental results at lower ISR energy region much better than those with assumption (ii).

Chou and Yang analyzed a fluctuation of the FB multiplicity distribution $f_n(n_1)$ at fixed n in $\bar{p}p$ collisions at $\sqrt{s}=546$ GeV (Ref. 5). They suggested that particles are distributed in the forward or backward region in pairs with a binomial distribution. However, they did not analyze the conditional mean multiplicity.

Carruthers and Shih analyzed the observed conditional mean multiplicity at $\sqrt{s}=546$ GeV (Ref. 6). They use a negative-binomial distribution for the multiplicity distribution of full rapidty range. In a negative-binomial distribution, two parameters $\langle n \rangle$ and k are contained. They take $k=3$, which means three clusters are produced, and from each cluster charged particles are emit-

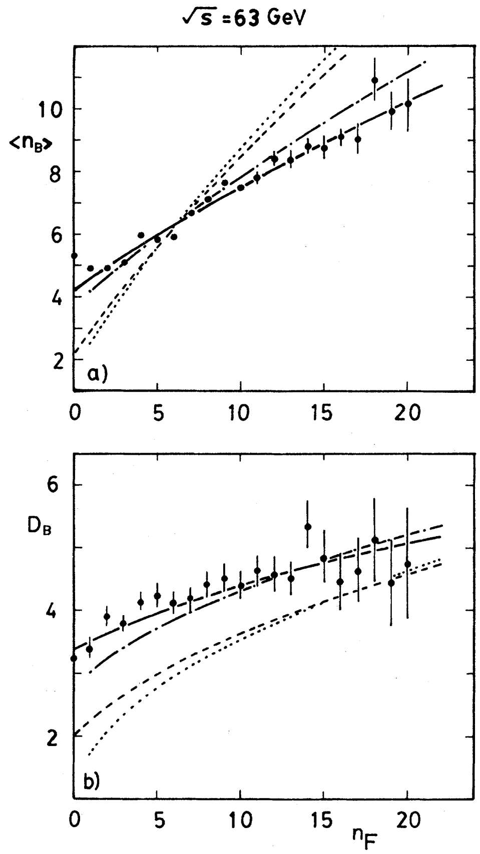

FIG. 4. Analysis of data in pp collisions at $\sqrt{s}=63$ GeV: otherwise the same as in Fig. 1.

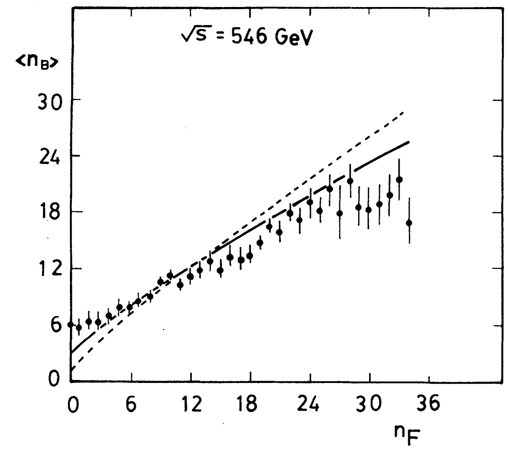

FIG. 5. Observed conditional mean multiplicity in the pseudorapidity interval $|\eta|\leq 4$ in $\bar{p}p$ collisions at $\sqrt{s}=546$ GeV is compared with our calculations: dashed line is obtained with assumption (i) and solid line with (ii).

ted under a geometric distribution. They assume that particles are distributed from each cluster in the forward or backward region in pairs under a binomial distribution. They use three different values of parameter α (β) corresponding to three clusters. We obtain almost the same result as they did, but in their analysis two more parameters are used compared to ours.

In summary, we cannot distinguish which assumptions among the four are the most preferable for the description of multiplicity moments in h-h collisions. However, it is found from our analysis of forward-backward multiplicity correlations that assumption (ii) or (ii′) is better than (i) or (i′); this leads to the conclusion that particles are distributed in pairs according to the binomial distribution, and the leading-particle effect is appreciable in the lower ISR energy region.

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