Witten index of supersymmetric chiral theories

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(Received 29 December 1987; revised manuscript received 29 November 1988)

The Witten index can be defined in many supersymmetric theories by formulating them in the space-time $R\times S^3$. If the index is nonzero for any value of the radius of S^3, it can be shown that the theory does not break supersymmetry in Minkowski space. This approach rules out supersymmetry breaking in a large class of models, chiral and otherwise. The index arguments are consistent with previous instanton calculations which indicate supersymmetry breaking in certain theories.

If supersymmetry plays a role in nature, it must be broken since fermions and bosons with the same mass are not observed. One possibility is that a theory which has global supersymmetry at the tree level might break this symmetry dynamically through nonperturbative effects.[1] A simple way of knowing if this happens was discovered by Witten.[2] For a theory with global supersymmetry, define an index Δ equal to the number of bosonic states N_B minus the number of fermionic states N_F in the subspace of energy eigenstates which have zero spatial momentum. If Δ is nonzero, supersymmetry can be shown to remain unbroken.[2,3]

To count the number of states, one regulates the theory by formulating it in a finite volume. Then the energy spectrum is discrete. In supersymmetric theories, all states have non-negative energy, and for any positive energy, there is an equal number of bosonic and fermionic states. So Δ only gets a contribution from the zero-energy states: $N_B(0)-N_F(0)$.

But the number of such states is not finite if the theory contains zero-energy bosonic and/or fermionic modes. States with an arbitrary number of such modes have zero energy (assuming that the interaction energy between two or more modes is zero). Then $N_B(0)$ [and perhaps $N_F(0)$ as well] is infinite and Δ becomes ill defined.

Witten was able to formulate several theories in a rectangular box with suitable boundary conditions in such a way that there are no zero-energy modes.[2] The index can then be easily counted. However, this method fails in two classes of theories: (i) those with chiral fermions, i.e., scalar multiplets in a complex representation of the gauge group, and (ii) those with massless fermions belonging to a real representation of the gauge group. Defined in a box, these theories necessarily contain zero-energy fermionic modes and Δ cannot be computed. This is unfortunate because some very interesting theories belong to these two classes. Thus supersymmetric SU(5) and E_6 grand unified theories are of type (i), whereas massless supersymmetric QCD is of type (ii).

To complete the program of ruling out supersymmetry breaking through index arguments alone, one must be able to define the index in every theory. In this paper we significantly extend the class of theories in which the index can be calculated. In particular, some versions of the models mentioned above can be dealt with.

The new idea is to define theories in the space-time $R\times S^3$. On S^3, the spectrum is discrete and there are no boundary conditions to worry about. Generally speaking, the curvature eliminates all the zero-energy modes. If the index can be calculated here and is nonzero, then one can let the radius ρ of S^3 go to infinity and conclude that the limiting Minkowskian theory also has a nonzero index and cannot break supersymmetry. Unlike the index (whose precise value, though independent of the volume, does depend on the nature of the finite space and the boundary conditions), the absence of supersymmetry breaking is a physical result and cannot depend on how the Minkowskian theory is approached as a limit.

Supersymmetry in $R\times S^3$ has been discussed in Ref. 4 to which the reader is referred for mathematical details. For $N=1$ supersymmetry, the superalgebra is modified from Minkowski space by a piece proportional to ρ^{-1}:

$$\{Q_\alpha,\bar{Q}_{\dot\alpha}\}=2i\sigma^m_{\alpha\dot\alpha}\partial_m-\frac{2}{\rho}\sigma^0_{\alpha\dot\alpha}\hat{R}\ . \tag{1}$$

Here $Q_\alpha,\bar{Q}_{\dot\alpha}$ $(\alpha,\dot\alpha=1,2)$ are the supersymmetry generators and the operators $\partial_m=(\partial_0,\partial_i)$ generate time translation and rotations of S^3 (which reduce to translations in flat space as $\rho\to\infty$). The last term in (1), which is absent in flat space, contains the well-known $\hat{R}$ operator.[5] This operator assigns U(1) charges to the various fields as follows. In a vector multiplet, the gauge boson is neutral whereas the charge of the gaugino is $+1$. In a scalar multiplet of $\hat{R}$ charge r_S, the scalar field has charge r_S whereas the fermion has charge r_S-1.

An important consequence of (1) is that for any theory in $R\times S^3$, the $\hat{R}$ charge of each scalar multiplet must be specified and the action is supersymmetric if and only if it is also $\hat{R}$ invariant. (Since the kinetic terms are automatically $\hat{R}$ invariant, this means that all the F terms must be invariant as well.) This indirectly provides a rationale for preferring $\hat{R}$-invariant theories in Minkowski space, as is usually done for purely phenomenological reasons.

The $\hat{R}$ charge r_V of a vector (gauge) multiplet is zero. For a scalar (chiral) multiplet, the charge r_S could be any real number in Minkowski space. On S^3, however, a curvature-dependent mass term is required by supersymmetry for the scalar field A (Ref. 4). This term equals $\rho^{-2} r_S(2-r_S) A^* A$, and is negative unless

$$0 \le r_S \le 2 \ . \quad (2)$$

This is necessary for the Hamiltonian to be bounded below.

The $\hat{R}$ operator generates a U(1) symmetry which is anomalous in general. This would be disastrous in $R \times S^3$ since, by (1), supersymmetry itself would then be anomalously broken, and the connection between a nonzero index and the absence of supersymmetry breaking would no longer hold. Demanding that a theory be nonanomalous imposes severe constraints on the $\hat{R}$ charges of the various scalar multiplets as we will see.

To summarize, a consistent supersymmetric theory exists on S^3 only if all the charges r_S can be chosen such that (a) all F terms in the action are $\hat{R}$ invariant, (b) each r_S satisfies $0 \le r_S \le 2$, and (c) the $\hat{R}$ current is anomaly-free at the one-loop level. These three conditions form the central result of this paper.

Next, we define Δ in $R \times S^3$ as the difference $N_B - N_F$ in the subspace of energy eigenstates in which the operator $P = i\partial_3 - \hat{R}/\rho$ is zero. [The Hamiltonian H, a Hermitian supersymmetry generator $Q = (1/\sqrt{2})(Q_1 + \bar{Q}_1)$ and the operator P mutually commute, and $Q^2 = H + P$. Also, the vacuum belongs to the subspace $P = 0$.] With this definition of the index, the connection between a nonzero Δ and the absence of supersymmetry breaking continues to hold.[4]

The value of Δ follows immediately from the spectrum of states. The latter must be calculated with all the complicated interactions taken into account. In the literature, however, the simplifying assumption is always made (but never rigorously justified), that it is sufficient to look at the perturbative spectrum, i.e., nonperturbative effects do not change the value of Δ. So one takes the extreme step of setting all the interaction terms equal to zero (so that the Lagrangian is at most quadratic in the various fields), and then calculating the free spectrum.

On doing this in $R \times S^3$ (Refs. 4 and 6), we find that if each of the charges r_S satisfies $0 < r_S < 2$, then there are no zero-energy fermionic or bosonic modes in the scalar multiplets. The gauginos and gauge bosons do not have any zero-energy modes. Hence there is only one zero-energy state, the vacuum, and this is bosonic. So Δ equals one and supersymmetry is unbroken. Note that the index in $R \times S^3$ is different from that in Minkowski space where it is not always one.[2]

If any one of the r_S equals zero or two, there are zero-energy bosonic and/or fermionic modes and Δ becomes ill defined. There is a reason why Δ changes abruptly if an r_S becomes zero or two. If $0 < r_S < 2$, the mass term mentioned earlier produces a potential which rises to infinity at large values of the field A. For r_S equal to zero or two, the potential is flat. It is known that the index may change suddenly at a particular value of a parameter if the asymptotic form of the potential changes there.[2] Note also that only at $r_S = 2$, an F term for that particular scalar multiplet is allowed in the action by condition (a) above. Since such a term breaks supersymmetry, it is logical that the index should change discontinuously at $r_S = 2$.

The index is well defined only if all the r_S satisfy $0 < r_S < 2$. This also ensures that the tree-level potential has no flat directions for large fields.

We now examine several models in light of requirements (a)–(c). Each model will contain a single non-Abelian gauge multiplet coupled to various scalar multiplets. In each case, we will point out the conditions under which supersymmetry breaking can be ruled out. To simplify matters, we will ignore the possible presence of F terms in the tree-level action, so that condition (a) will be trivially satisfied.

The first example we consider is massless supersymmetric QCD with N colors and M flavors. The left-handed quarks and antiquarks come in multiplets A_i and $\bar{A}_i$ $(i = 1, \ldots, M)$, transforming as the fundamental and antifundamental representations of SU(N). Let the $\hat{R}$ charges of A_i and $\bar{A}_i$ be r_A and $r_{\bar{A}}$. The $\hat{R}$ current is susceptible to a triangle anomaly coming from the gaugino, the quarks, and the antiquarks with $\hat{R}$ charges equal to $+1$, $r_A - 1$, and $r_{\bar{A}} - 1$, respectively. The appropriate group theoretical characters can be found in Ref. 7. The character C_T is defined by $\mathrm{Tr}\lambda_T^a \lambda_T^b = C_T \delta^{ab}$, where the λ_T^a are the gauge group generators in the representation T. The values of C_T in the fundamental (or antifundamental) and adjoint representations of SU(N) are $\frac{1}{2}$ and N, respectively. The $\hat{R}$ current is therefore nonanomalous if $N + (M/2)(r_A - 1) + (M/2)(r_{\bar{A}} - 1) = 0$, or $r_A + r_{\bar{A}} = 2(1 - N/M)$. This is consistent with condition (b) if and only if $M > N$. All the r_A and $r_{\bar{A}}$ can be chosen to lie in the range $0 < r_A, r_{\bar{A}} < 2$ if

$$M > N \ . \quad (3)$$

Since the index is then equal to one, supersymmetry is unbroken if the number of flavors exceeds the number of colors.

If $M = N$, all the r_A and $r_{\bar{A}}$ can be chosen to be equal to zero so that supersymmetry can be consistently defined on S^3. However, Δ is not well defined and one cannot say anything about supersymmetry breaking. Finally, if $M < N$, supersymmetry is anomalously broken on S^3 [if one imposes condition (b)], and taking the $\rho \to \infty$ limit does not permit us to conclude anything about supersymmetry breaking in Minkowski space.

Next, we consider an SU(N) gauge theory ($N \ge 4$) with M generations of scalar multiplets, each generation consisting of one antisymmetric tensor representation $T = N(N-1)/2$ and $N - 4$ flavors in the antifundamental representation $\bar{F} = \bar{N}$. For the T representation, C_T equals $\frac{1}{2}(N-2)$. (If $N = 5$ and $M = 3$, this would be the simplest supersymmetric grand unified model except that the Higgs sector has been omitted.) The $\hat{R}$ current is not anomalous if $N + (M/2)(N-2)(r_T - 1) + (M/2)(N-4)(r_{\bar{F}} - 1) = 0$. Similar arguments as above show that supersymmetry is unbroken if

$$M > \frac{N}{N-3} \ , \quad (4)$$

whereas we cannot say anything definite if $M \leq N/(N-3)$.

Two other examples one might consider are gauge groups SO(10) and E_6 with M generations of scalar multiplets in the representations **16** and **27**, respectively (corresponding to the leptoquark sector of possible grand unification schemes). In both cases, the $\hat{R}$-current anomaly vanishes if $r_M = 1 - 4/M$, and supersymmetry is unbroken if $M > 4$.

Some of these results [for example, (3) and (4)] were previously known from instanton arguments.[8–10] It is clear that the index argument is comparatively simpler.

In all the examples above, supersymmetry is unbroken if the number of generations (or flavors) M is greater than a certain number M_c. If M is too large however, asymptotic freedom is lost and the theory usually becomes physically uninteresting. The contributions of gauge bosons, chiral fermions and complex scalars to the β function for the gauge coupling was computed in Ref. 11. It turns out that asymptotic freedom holds if $M < 3M_c$. So there is a large range of values of M for which the theory is asymptotically free and does not break supersymmetry.

An extensive list of theories known to break supersymmetry dynamically is given in Refs. 12 and 13 and references therein. On studying these theories, we find that none of them satisfies all the three conditions (a)–(c). Thus the index arguments are not in contradiction with any of the calculations which positively indicate supersymmetry breaking.

We end by citing some other theories in which index arguments rule out supersymmetry breaking. These examples are not meant to be exhaustive but merely illustrative of the efficiency of index arguments in comparison with the arguments in Refs. 12 and 13.

(i) Any non-Abelian gauge theory with more than one flavor of scalar multiplets in the adjoint representation.

(ii) SU(N) gauge theory ($N \geq 3$) with any number of generations of scalar multiplets, each generation consisting of one symmetric tensor representation $S = N(N+1)/2$ and $N+4$ antifundamental representations.

(iii) O(N) gauge theory ($N \geq 3$) with M flavors of scalar multiplets in the vector representation, if $M > N-2$.

In (i) above, if we have three flavors and set the r_S of each of the three scalar multiplets equal to $\frac{2}{3}$, we can add an $\hat{R}$-invariant F term which is trilinear in these multiplets and has the same form as in the $N=4$ extended supersymmetric Yang-Mills theory. So this theory does not break supersymmetry, in agreement with an index calculation in flat space.[14]

In general, the index arguments can rule out supersymmetry breaking only if the following inequality holds. Let C_A be the character in the adjoint representation of a gauge group, and C_T be the characters in the various scalar multiplet representations. The $\hat{R}$ anomaly vanishes if $C_A + \sum_T C_T(r_T - 1) = 0$. The various r_T can lie in the range $0 < r_T < 2$ if and only if

$$\sum_T C_T > C_A \ . \tag{5}$$

However, this inequality is not always sufficient to rule out supersymmetry breaking because one may be unable to choose the r_T so as to satisfy condition (a) if the action contains F terms. We have generally ignored the possibility of such terms in this paper.

Interestingly, the instanton arguments[12,13] seem to show that the condition in (5) is actually sufficient to rule out supersymmetry breaking. This could be a sign of a deeper connection between the instanton and index arguments.

To conclude, supersymmetric models can be consistently defined on $R \times S^3$ if three conditions hold. In addition, if the $\hat{R}$ charge of every scalar multiplet lies between zero and two, then the Witten index has the value one and supersymmetry breaking can be ruled out. However, index arguments cannot say anything about supersymmetry breaking if these conditions do not hold.

The relevance of dynamical supersymmetry breaking for realistic model building has been much discussed in the literature. See, for example, Ref. 12 and references therein.

I am grateful for the encouragement given by the Theoretical Physics group in the University of Edinburgh where this work was begun. Financial support was provided there by the United Kingdom Science and Engineering Research Council through Grant No. NG 17094. I also thank Romesh Kaul for stimulating discussions.

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