Energy-momentum tensor in theories with scalar fields and two coupling constants. II. Yukawa theory

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(Received 11 October 1988)

We examine the question of renormalization of the energy-momentum tensor in Yukawa theory following our earlier work on scalar QED and non-Abelian gauge theories with scalars. As in those cases, we consider two kinds of forms for the improvement term: (1) one in which the improvement coefficient is a finite function of bare quantities of the theory (so that the energy-momentum tensor can be derived from an action that is a finite function of bare quantities); (ii) one in which the improvement coefficient is a finite quantity. As in earlier cases discussed we show that neither form leads to a finite energy-momentum tensor to $O(g^2\lambda^n)$.

I. INTRODUCTION

Energy-momentum tensors and their finiteness have received a good deal of attention on account of their relevance in physics.[1–11] The finiteness of energy-momentum tensors in theories with scalar fields is a nontrivial question on account of a need for an improvement term and has been studied in great detail by various authors.[2–4,6–9] Until recently the question had been studied in detail in the context of $\lambda\phi^4$ theory (with one scalar field). Collins[6,7] has shown that an improvement term of the form

$$H_0(\epsilon)(\partial_\mu\partial_\nu - g_{\mu\nu}\partial^2)\phi^2 \tag{1.1}$$

[where $H_0(\epsilon)$ is a unique power series in *non-negative* powers of $\epsilon=4-n$] leads to a finite energy-momentum tensor to all orders.

In Refs. 10 and 11 (henceforth referred to as I and II) we discussed two special cases of theories containing scalar fields and having more than one coupling constant: viz., scalar QED and non-Abelian theories with scalar fields. As explained in detail in II, the crucial question in such theories is whether the improvement term can be chosen in such a way that the root-mean-square mass radius of the scalar particle is a *prediction* of the theory or whether this piece of information is needed as an independent experimental input to fix an independent renormalization constant of the $\frac{1}{2}R\phi^2$ term in the action.[4] This depends on whether a finite energy-momentum tensor can be constructed so that the improvement coefficient is either a finite function of bare quantities, or a finite function of renormalized quantities (and hence a finite number), i.e., whether the "finite improvement program"[4] works in such theories.

To this end, we have considered energy-momentum tensors in theories with scalar fields and having *two* coupling constants (for simplicity). There are four such renormalizable models: (i) scalar QED; (ii) non-Abelian gauge theories with scalars; (iii) Yukawa theory; (iv) a model with two interacting scalar fields. The first two cases were analyzed in I and II. We analyze the third case in this paper. As in I and II we reach a negative conclusion for either kind of improvement coefficients in Yukawa theory also. The case (iv) is analyzed elsewhere with a similar conclusion.

In theories without scalar fields finite energy-momentum tensors which are finite functions of bare quantities exist.[3,4] As shown by Collins[6,7] the forms for the improvement coefficient of both kinds [the same one mentioned in Eq. (1.1)] work in $\lambda\phi^4$ theory. In pure $\lambda\phi^4$ theory, the root-mean-square radius of the scalar particle is (or rather can be) an experimental prediction of the theory and the coefficient of the $R\phi^2$ term in the action need not be independently renormalized.[8] This proves to be an *exception* rather than a rule. In theories with scalar fields and more than one coupling constant the root-mean-square mass radius of the scalar field is needed as an additional experimental input to renormalize the coefficient of the $R\phi^2$(-like) term in all the four cases analyzed.

II. PRELIMINARIES

We shall be dealing with the Yukawa theory of the scalar-fermion interaction. The Lagrangian density is

$$\begin{aligned}\mathcal{L} &= \tfrac{1}{2}(\partial_\mu\phi)^2 - \tfrac{1}{2}M_0^2\phi^2 - \frac{\lambda_0}{4!}\phi^4 + \bar{\psi}(i\partial\!\!\!/ - m_0)\psi \\ &\quad + ig_0\bar{\psi}\gamma_5\psi\phi\ , \\ S &= \int d^nx\,\mathcal{L}\ .\end{aligned} \tag{2.1}$$

We shall work with dimensionally regularized quantities and use the minimal subtraction (MS) scheme.[12,13] The unrenormalized but dimensionally regularized Green's functions, connected Green's functions, and proper vertices are generated, respectively, by $W[J,\eta,\bar{\eta}]$, $Z[J,\eta,\bar{\eta}]$, and $\Gamma[\phi,\psi,\bar{\psi}]$ with

$$W[J,\eta,\bar{\eta}] = \frac{1}{N}\int \mathcal{D}\phi\,\mathcal{D}\psi\,\mathcal{D}\bar{\psi} \times \exp\left[i\int d^n x\left[\frac{\mathcal{L}}{a}+J\phi+\bar{\psi}\eta+\bar{\eta}\psi\right]\right], \quad (2.2)$$

where $W[0]=1$ and a is the loop expansion parameter:

$$Z[J,\eta,\bar{\eta}] = -i\ln W[J,\eta,\bar{\eta}] , \quad (2.3)$$

$$\phi(x)=\frac{\delta Z}{\delta J(x)},\quad \psi(x)=\frac{\delta Z}{\delta\bar{\eta}(x)},\quad \bar{\psi}(x)=-\frac{\delta Z}{\delta\eta(x)} , \quad (2.4)$$

and

$$\Gamma[\phi,\psi,\bar{\psi}]=Z-\int d^n x(J\phi+\bar{\eta}\psi+\bar{\psi}\eta) . \quad (2.5)$$

In the MS scheme the renormalization transformations are[14]

$$\phi=Z^{1/2}\phi^R,\quad \psi=\tilde{Z}^{1/2}\psi^R ,$$
$$\lambda_0=\mu^{\epsilon}[\lambda Z_\lambda+\delta\lambda(g)],\quad \bar{\psi}=\tilde{Z}^{1/2}\bar{\psi}^R , \quad (2.6)$$
$$g_0=\mu^{\epsilon/2}gZ_g,\quad m_0=Z_m m,\quad M_0^2=Z_M M^2+Z_M' m^2 ,$$

where μ is an arbitrary parameter of dimension of mass. $\delta\lambda$ in Eq. (2.6) starts with $O(g^4)$ and Z_M' starts with $O(g^2)$.

The renormalized Green's functions, connected Green's functions and property vertices are generated, respectively, by

$$W^R[J^R,\eta^R,\bar{\eta}^R],\quad Z^R[J^R,\eta^R,\bar{\eta}^R],\quad \text{and}\quad \Gamma^R[\phi^R,\psi^R,\bar{\psi}^R]$$

with

$$W^R[J^R,\eta^R,\bar{\eta}^R]=W[J,\eta,\bar{\eta}]\quad\text{and}\quad J^R=Z^{1/2}J,\ \text{etc.}$$

Equations of motion imply that

$$\left\langle\phi\frac{\delta S}{\delta\phi}\right\rangle^R=\left\langle\phi\frac{\delta S}{\delta\phi}\right\rangle=\text{finite} , \quad (2.7)$$

$$\left\langle\bar{\psi}\frac{\delta S}{\delta\bar{\psi}}\right\rangle^R=\left\langle\bar{\psi}\frac{\delta S}{\delta\bar{\psi}}\right\rangle=\text{finite} , \quad (2.8)$$

$$\left\langle\frac{\delta S}{\delta\psi}\psi\right\rangle^R=\left\langle\frac{\delta S}{\delta\psi}\psi\right\rangle=\text{finite} . \quad (2.9)$$

Unlike the cases considered in I and II $M_0^2\phi^2$ now is not a finite operator as seen from Eq. (2.6), using the mass independence of Z_m, Z_M, Z_M':

$$\langle M_0^2\phi^2\rangle=-2M_0^2\frac{\partial Z^{\text{UR}}}{\partial M_0^2}=-2\left[1+\frac{m^2 Z_M'}{M^2 Z_M}\right]M^2\frac{\partial Z^R}{\partial M^2} . \quad (2.10)$$

Similarly, $m_0\bar{\psi}\psi$ is also not a finite operator:

$$\langle m_0\bar{\psi}\psi\rangle=-m_0\frac{\partial Z^{\text{UR}}}{\partial m_0} = -m\frac{\partial Z^R}{\partial m}+\left[\frac{2m^2}{M^2}\frac{Z_M'}{Z_M}\right]M^2\frac{\partial Z^R}{\partial M^2} . \quad (2.11)$$

But the sum $M_0^2\phi^2+m_0\bar{\psi}\psi$ is finite:

$$\langle M_0^2\phi^2+m_0\bar{\psi}\psi\rangle=-2M^2\frac{\partial Z^R}{\partial M^2}-m\frac{\partial Z^R}{\partial m}=\text{finite} . \quad (2.12)$$

Furthermore $\partial^2\phi^2$ is a multiplicatively renormalizable operator:[15]

$$\{\partial^2\phi^2\}^{\text{UR}}=Z_M^{-1}\{\partial^2\phi^2\}^R . \quad (2.13)$$

We shall use the renormalization group extensively. Below we give definitions and values of various renormalization-group quantities:[16]

$$\begin{aligned}
\beta^\lambda(\lambda,g,\epsilon)&\equiv\mu\frac{\partial\lambda}{\partial\mu}\bigg|_{\lambda_0,g_0,M_0,m_0} \\
&=-\lambda\epsilon+\beta^\lambda(\lambda,g) \\
&=-\lambda\epsilon+\lambda^2\frac{\partial Z_\lambda^{(1)}}{\partial\lambda}+\lambda\frac{\partial(\delta\lambda^{(1)})}{\partial\lambda}+\frac{g}{2}\frac{\partial(\delta\lambda^{(1)})}{\partial g}-\delta\lambda^{(1)}+\tfrac{1}{2}\lambda g\frac{\partial Z_\lambda^{(1)}}{\partial g} \\
&=-\lambda\epsilon+\beta_2\lambda^2+\cdots , \\
\beta^g(\lambda,g,\epsilon)&\equiv\mu\frac{\partial g}{\partial\mu}\bigg|_{\lambda_0,g_0,M_0,m_0}=-\frac{\epsilon g}{2}+\beta^g(\lambda,g)=-\frac{\epsilon g}{2}+\lambda g\frac{\partial Z_g^{(1)}}{\partial\lambda}+\frac{g^2}{2}\frac{\partial Z_g^{(1)}}{\partial g} , \\
\gamma_M(\lambda,g,\epsilon)&\equiv+\tfrac{1}{2}\mu\frac{\partial}{\partial\mu}\ln M^2=\gamma_M(\lambda,g) , \\
\gamma_m(\lambda,g,\epsilon)&\equiv\tfrac{1}{2}\mu\frac{\partial}{\partial\mu}\ln m^2=\gamma_m(\lambda,g) , \\
\tilde{\gamma}_M(\lambda,g,\epsilon)&=-\tfrac{1}{2}\mu\frac{\partial}{\partial\mu}\ln Z_M=\tilde{\gamma}_M(\lambda,g)=\frac{\lambda}{2}\frac{\partial}{\partial\lambda}Z_M^{(1)}+\frac{g}{4}\frac{\partial}{\partial g}Z_M^{(1)}=\gamma_M^{(1)}\lambda+\cdots , \\
\gamma(\lambda,g,\epsilon)&\equiv\tfrac{1}{2}\mu\frac{\partial}{\partial\mu}\ln Z=\gamma(\lambda,g) , \\
\tilde{\gamma}(\lambda,g,\epsilon)&=\tfrac{1}{2}\mu\frac{\partial}{\partial\mu}\ln\tilde{Z}=\tilde{\gamma}(\lambda,g) .
\end{aligned} \quad (2.14)$$

We shall work with the following set of operators:

$$O_1=-\frac{\lambda_0\phi^4}{4!}+\frac{i}{2}g_0\bar{\psi}\gamma_5\psi\phi,\quad O_2=M_0^2\phi^2\,,$$
$$O_3=m_0\bar{\psi}\psi,\quad O_4=\phi\frac{\delta S}{\delta\phi},\quad O_5=\bar{\psi}\frac{\delta S}{\delta\bar{\psi}}\,, \tag{2.15}$$
$$O_6=\frac{\delta S}{\delta\psi}\psi,\quad O_7=\tfrac{1}{2}ig_0\bar{\psi}\gamma_5\psi\phi,\quad O_8=\partial^2\phi^2\,.$$

We note that, at zero momentum,

$$\int d^nx\, O_1=\lambda_0\frac{\delta S}{\delta\lambda_0}+\tfrac{1}{2}g_0\frac{\delta S}{\delta g_0}\,,$$
$$\int d^nx\, O_2=-2M_0^2\frac{\delta S}{\delta M_0^2}\,,$$
$$\int d^nx\, O_3=-m_0\frac{\delta S}{\delta m_0}\,,$$
$$\int d^nx\, O_7=\tfrac{1}{2}g_0\frac{\delta S}{\delta g_0}\,.$$

We shall define another set of renormalized operators:

$$\left\langle\int d^nx\, X_1^R\right\rangle^R=\lambda\frac{\partial Z^R}{\partial\lambda}+\tfrac{1}{2}g\frac{\partial Z^R}{\partial g}\,,$$
$$\left\langle\int d^nx\, X_2^R\right\rangle^R=-M^2\frac{\partial Z^R}{\partial M^2}\,,$$
$$\left\langle\int d^nx\, X_3^R\right\rangle^R=-m\frac{\partial Z^R}{\partial m}\,,$$
$$\left\langle\int d^nx\, X_4^R\right\rangle^R=-\int J^R(x)\frac{\delta Z^R}{\delta J^R(x)}=\left\langle\int d^nx\, O_4^R\right\rangle\,, \tag{2.16}$$
$$\left\langle\int d^nx\, X_5^R\right\rangle^R=-\int \bar{\eta}^R(x)\frac{\delta Z^R}{\delta\bar{\eta}^R(x)}=\left\langle\int d^nx\, O_5^R\right\rangle$$
$$\left\langle\int d^nx\, X_6^R\right\rangle^R=-\int \frac{\delta Z^R}{\delta\eta^R(x)}\eta^R(x)=\left\langle\int d^nx\, O_6^R\right\rangle\,,$$
$$\left\langle\int d^nx\, X_7^R\right\rangle^R=\tfrac{1}{2}g\frac{\partial Z^R}{\partial g},\quad X_8=O_8\,.$$

As discussed in I and is also evident from Eqs. (2.6), (2.10), and (2.11), $X_i^R\neq O_i^R$ for $i=1$, 2, and 3, but one can still define a renormalization matrix Z_{ij} by

$$\langle O_i\rangle^{\rm UR}=Z_{ij}\langle X_j\rangle^R\,. \tag{2.17}$$

(Note that $X_j=O_j$ for $j=4,5,\ldots,8$.) Then

$$Z_{ij}^{-1}\mu\frac{\partial}{\partial\mu}Z_{jk}=\gamma_{ik}=\text{finite at }\epsilon=0\,. \tag{2.18}$$

Information on the structure of renormalization matrix is obtained from Eqs. (2.7)–(2.11) and Eq. (2.13):

$$Z=\begin{bmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{14} & Z_{15} & Z_{16} & Z_{17} & Z_{18}\\ 0 & Z_{22} & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & Z_{32} & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\ Z_{71} & Z_{72} & Z_{73} & Z_{74} & Z_{75} & Z_{76} & Z_{77} & Z_{78}\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & Z_{88}\end{bmatrix}. \tag{2.19}$$

It should be noted that Z_{22} and Z_{32} so defined depend on m/M.

We expand Z_M^{-1} in powers of g:

$$Z_M^{-1}=Z_{M(0)}^{-1}+g^2Z_{M(2)}^{-1}+g^4Z_{M(4)}^{-1}+\cdots\,.$$

We now state two results needed in Secs. V and VI. As shown by Collins,[7] if $H(\lambda,M,\epsilon)$ is a finite function of λ,M at $\epsilon=0$ and

$$H(\lambda,M,\epsilon)Z_{M(0)}^{-1}(\lambda,\epsilon)=\text{finite at }\epsilon=0$$

keeping λ and M finite and fixed, then

$$H(\lambda,M,\epsilon)\equiv0\,. \tag{2.20}$$

Second, as shown in Ref. 9, if

$$F\left[\lambda_0\mu^{-\epsilon},\frac{M_0^2}{\mu^2},\epsilon\right]$$

is a finite function, at $\epsilon=0$, of $\lambda_0\mu^{-\epsilon}$ and M_0^2/μ^2 and if

$$F\left[\lambda_0\mu^{-\epsilon},\frac{M_0^2}{\mu^2},\epsilon\right]Z_{M(0)}^{-1}(\lambda,\epsilon)=\text{finite at }\epsilon=0$$

keeping λ and M fixed and finite, then

$$F\left[\lambda_0\mu^{-\epsilon},\frac{M_0^2}{\mu^2},\epsilon\right]\equiv0\,. \tag{2.21}$$

III. IMPROVED TRACE

The energy-momentum tensor of Yukawa theory as obtained from the action (2.1) is

$$\theta_{\mu\nu}=-\frac{2}{\sqrt{-g(y)}}\frac{\delta S}{\delta g^{\mu\nu}}\bigg|_{g^{\mu\nu}(y)=\eta^{\mu\nu}}$$
$$=-g_{\mu\nu}\mathcal{L}+\partial_\mu\phi\partial_\nu\phi+i\bar{\psi}\gamma_\mu\partial_\nu\psi\,. \tag{3.1}$$

One may carry out the analysis of Ref. 3 here also to show that this energy-momentum tensor has finite matrix elements at $q=0$ and to first order in q, the external momentum. But in $O(q^2)$ a further improvement term is needed. The most general improvement one can add to $\theta_{\mu\nu}$ is parametrized as

$$\theta_{\mu\nu}^{\rm imp}=\theta_{\mu\nu}+\left[\frac{n-2}{4(1-n)}+\frac{\tilde{G}}{(1-n)}\right](\partial_\mu\partial_\nu-g_{\mu\nu}\partial^2)\phi^2\,. \tag{3.2}$$

As before, $\widetilde{G}$ is a free parameter for which we shall be trying specific forms in Secs. V and VI, just as in I and II. The trace of $\theta_{\mu\nu}^{\rm imp}$ is

$$\theta_{\mu}^{\rm imp\,\mu}=(n-4)\left[-\frac{\lambda\phi^4}{4!}+\tfrac{1}{2}ig\bar{\psi}\gamma_5\psi\phi\right]+(M^2\phi^2+m\bar{\psi}\psi)-\left[\frac{n-2}{2}\right]\phi\frac{\delta S}{\delta\phi}-(n-1)\bar{\psi}\frac{\delta S}{\delta\bar{\psi}}+\widetilde{G}\partial^2\phi^2 . \quad (3.3)$$

Using Eqs. (2.7), (2.8), and (2.13) one obtains

$$\langle\theta_{\mu}^{\rm imp\,\mu}\rangle=\text{finite}+(n-4)\langle O_1\rangle^{\rm UR}+\widetilde{G}Z_M^{-1}\langle\partial^2\phi^2\rangle^R . \quad (3.4)$$

In the following sections we shall investigate if it is possible to choose $\widetilde{G}$ in either of the two ways mentioned in the Introduction so that $\langle\theta_{\mu}^{\rm imp\,\mu}\rangle$ is finite.

IV. EXPRESSIONS FOR Z_{ij}

In this section we shall use the techniques of Ref. 10 to obtain expressions for Z_{1j} $(j=1,2,\ldots,7)$, which are needed in further investigations of Secs. V and VI. These expressions are obtained by considering renormalization of O_1 at zero momentum.

It is straightforward to show that

$$\int d^nx\, O_1=-S+\tfrac{1}{2}\int\phi\frac{\delta S}{\delta\phi}d^nx+\int\bar{\psi}\frac{\delta S}{\delta\bar{\psi}}d^nx . \quad (4.1)$$

The last two terms on the right-hand side are finite operators. $\langle S\rangle^{\rm UR}$ can be obtained by following the procedure of Refs. 10 and 17. The final result is

$$\begin{aligned}\left\langle\int d^nx\,O_1\right\rangle^{\rm UR}&=-\langle S\rangle^{\rm UR}+\frac{1}{2}\left\langle\int d^nx\,\phi\frac{\delta S}{\delta\phi}\right\rangle^R+\left\langle\int d^nx\,\bar{\psi}\frac{\delta S}{\delta\bar{\psi}}\right\rangle^R\\&=\left[1-\frac{\beta^\lambda(\lambda,g)}{\epsilon}\right]\left\langle\int d^nx\,X_1\right\rangle^R+\frac{\gamma_M(\lambda,g)}{\epsilon}\left\langle\int d^nx\,X_2\right\rangle^R+\frac{\gamma_M(\lambda,g)}{\epsilon}\left\langle\int d^nx\,X_3\right\rangle^R+\frac{\gamma(\lambda,g)}{\epsilon}\left\langle\int d^nx\,X_4\right\rangle^R\\&\quad+\frac{\widetilde{\gamma}(\lambda,g)}{\epsilon}\left\langle\int d^nx\,(X_5+X_6)\right\rangle^R+\left[\frac{\beta^\lambda(\lambda,g)}{\lambda\epsilon}-2\frac{\beta^g(\lambda,g)}{g\epsilon}\right]\left\langle\int d^nx\,X_7\right\rangle^R . \end{aligned}\quad (4.2)$$

Hence,

$$\begin{aligned}Z_{11}&=1-\frac{\beta^\lambda}{\lambda\epsilon},\quad Z_{14}=\frac{\gamma}{\epsilon} ,\\Z_{12}&=\frac{\gamma_M}{\epsilon},\quad Z_{15}+Z_{16}=2\frac{\widetilde{\gamma}}{\epsilon} ,\\Z_{13}&=\frac{\gamma_m}{\epsilon},\quad Z_{17}=\frac{\beta^\lambda}{\lambda\epsilon}-\frac{2\beta^g}{g\epsilon} .\end{aligned}\quad (4.3)$$

From the fact that the right-hand side of Eq. (4.2) has only simple poles in ϵ, it follows that $\langle\theta_{\mu}^{\mu}\rangle$ is finite at zero momentum. Finiteness of $\langle\theta_{\mu}^{\mu}\rangle$ at zero momentum and to first order in q can be established along the lines of Ref. 3 using the conservation equation.

It should be noted that the above procedure does not yield Z_{15} and Z_{16} separately because $\int d^nx\,O_5=\int d^nx\,O_6$. From Eq. (4.3) it only follows that $Z_{15}+Z_{16}$ has simple poles. However, the theory has a charge-conjugation invariance and the operator O_1 is charge-conjugation invariant. Hence the operator $O_5-O_6=i\partial^\mu(\bar{\psi}\gamma_\mu\psi)$ which is odd under charge conjugation cannot appear as a counterterm for O_1. Hence only the combination O_5+O_6 can appear in the expression for $\langle O_1\rangle^{\rm UR}$. This requires that $Z_{15}=Z_{16}=\widetilde{\gamma}/\epsilon$.

V. IMPROVEMENT-TERM DEPENDENCE OF THE FORM $\widetilde{G}(\epsilon,\lambda_0\mu^{-\epsilon},g_0^2\mu^{-\epsilon})$

As shown in the previous section, the energy-momentum tensor

$$\theta_{\mu\nu}^{\rm imp'}=\theta_{\mu\nu}+\frac{n-2}{4(1-n)}(\partial_\mu\partial_{\bar{\nu}}-g_{\mu\nu}\partial^2)\phi^2$$

is finite at zero momentum. Explicit calculation shows that it is finite only up to $O(\lambda^3)$ at $g=0$, up to $O(g^4)$ at $\lambda=0$ and also in $O(\lambda g^2)$, but a finite improvement is necessarily needed in $O(\lambda^4)$, $O(\lambda g^4)$, and $O(\lambda^2g^2)$ [see, for example, the Appendix].

In this section, we shall consider a further improvement, where the improvement coefficient $\widetilde{G}$ is a *finite* function of *bare* coupling constants:

$$\theta_{\mu\nu}^{\rm imp}=\theta_{\mu\nu}+\left[\frac{n-2}{4(1-n)}+\frac{\widetilde{G}(\epsilon,g_0^2\mu^{-\epsilon},\lambda_0\mu^{-\epsilon})}{1-n}\right]\times(\partial_\mu\partial_\nu-g_{\mu\nu}\partial^2)\phi^2 , \quad (5.1)$$

where

$$\widetilde{G}(\epsilon,g_0^2\mu^{-\epsilon},\lambda_0\mu^{-\epsilon})=\sum_{r=0}^{\infty}\widetilde{G}_{2r}(\epsilon,\lambda_0\mu^{-\epsilon})(g_0^2\mu^{-\epsilon})^r , \quad (5.2)$$

where $\widetilde{G}_{2r}(\epsilon,\lambda_0\mu^{-\epsilon})$ are finite functions of λ_0.

From Eq. (3.4) and the fact that Z_{1j} $(j=1,2,\ldots,7)$ involve simple poles, it follows that

$$\langle\theta_{\mu}^{\rm imp\,\mu}\rangle=\text{finite}+[-\epsilon Z_{18}+\widetilde{G}(\epsilon,g_0^2\mu^{-\epsilon},\lambda_0\mu^{-\epsilon})Z_M^{-1}]\times\langle\partial^2\phi^2\rangle^R . \quad (5.3)$$

For future convenience, we shall reexpress this as follows:

$$\tilde{G}(\epsilon,g_0^2\mu^{-\epsilon},\lambda_0\mu^{-\epsilon})=-\epsilon G(\epsilon,g_0^2\mu^{-\epsilon},\lambda_0\mu^{-\epsilon}) ,$$
$$\tilde{G}_{2r}(\epsilon,\lambda_0\mu^{-\epsilon})=-\epsilon G_{2r}(\epsilon,\lambda_0\mu^{-\epsilon}) , \quad (5.4)$$

where now $G_{2r}(\epsilon,\lambda_0\mu^{-\epsilon})$ may also contain $1/\epsilon$ terms when expanded in powers of ϵ and $(\lambda_0\mu^{-\epsilon})$. We thus have

$$\langle \theta_\mu^{\text{imp}\ \mu} \rangle = \text{finite} - \epsilon X \langle \partial^2\phi^2 \rangle^R , \quad (5.5)$$

where

$$X = Z_{18} + G(\epsilon,g_0^2\mu^{-\epsilon},\lambda_0\mu^{-\epsilon})Z_M^{-1} . \quad (5.6)$$

Thus to obtain a finite energy-momentum tensor, one must find a $G(\epsilon,g_0^2\mu^{-\epsilon},\lambda_0\mu^{-\epsilon})$ such that X given above does not contain worse than simple poles. We shall show, in what follows, that it is not possible to do so consistently except at $g=0$.

For this purpose, we shall use the renormalization-group (RG) equation satisfied by Z_{18}, which can be derived straightforwardly from Eqs. (2.18) and (2.19) following the procedure of Appendix C of I. It is

$$(-\lambda\epsilon+\beta^\lambda)\frac{\partial Z_{18}}{\partial\lambda}+\left[-\frac{g\epsilon}{2}+\beta^g\right]\frac{\partial Z_{18}}{\partial g}-2\tilde{\gamma}_M Z_{18} = Z_{11}\gamma_{18}+Z_{17}\gamma_{78} . \quad (5.7)$$

Substituting from Eq. (5.6) for Z_{18} and using

$$\mu\frac{\partial}{\partial\mu}[G(\epsilon,g_0^2\mu^{-\epsilon},\lambda_0\mu^{-\epsilon})Z_M^{-1}]=2\tilde{\gamma}_M G(\epsilon,g_0^2\mu^{-\epsilon},\lambda_0\mu^{-\epsilon})Z_M^{-1}+Z_M^{-1}\mu\frac{\partial}{\partial\mu}G(\epsilon,g_0^2\mu^{-\epsilon},\lambda_0\mu^{-\epsilon})$$
$$=2\tilde{\gamma}_M G(\epsilon,g_0^2\mu^{-\epsilon},\lambda_0\mu^{-\epsilon})Z_M^{-1}+Z_M^{-1}\mu\frac{\partial}{\partial\mu}\sum_{n=0}^{\infty}G_{2n}(\epsilon,\lambda_0\mu^{-\epsilon})(g_0^2\mu^{-\epsilon})^n \quad (5.8)$$

one obtains an equation satisfied by X:

$$(-\lambda\epsilon+\beta^\lambda)\frac{\partial X}{\partial\lambda}+\left[-\frac{g\epsilon}{2}+\beta^g\right]\frac{\partial X}{\partial g}-2\tilde{\gamma}_M X - Z_{11}\gamma_{18}-Z_{17}\gamma_{78}=-\epsilon\sum_{n=0}^{\infty}(g_0^2\mu^{-\epsilon})^n\left[nG_{2n}+\frac{\partial G_{2n}}{\partial(\lambda_0\mu^{-\epsilon})}\lambda_0\mu^{-\epsilon}\right]Z_M^{-1} . \quad (5.9)$$

Now, suppose it were possible to choose G_n's such that X has no worse than simple poles (which would imply the existence of a finite energy-momentum tensor). Then as Z_{11} and Z_{17} have only simple poles, the left-hand side of Eq. (5.9) has at worst simple poles and hence so does the right-hand side. Hence

$$\epsilon^2\left[\sum_{n=0}^{\infty}(g_0^2\mu^{-\epsilon})^n\left[nG_{2n}+\frac{\partial G_{2n}}{\partial(\lambda_0\mu^{-\epsilon})}\lambda_0\mu^{-\epsilon}\right]\right]Z_M^{-1} = \text{finite} . \quad (5.10)$$

Then, following the same reasoning as in I, the above equation, after using Eq. (2.21), implies that

$$G_2(\epsilon,\lambda_0\mu^{-\epsilon})=0 . \quad (5.11)$$

This implies that the improvement term in Eq. (5.1) is consistent with the finiteness of $\langle \theta_\mu^{\text{imp}\ \mu} \rangle$ in $O(g^2\lambda^n)$ only if

$$G(\epsilon,g_0^2\mu^{-\epsilon},\lambda_0\mu^{-\epsilon})=G_0(\epsilon)+O(g^4) , \quad (5.12)$$

i.e., only if the improvement term obtained to $O(g^0)$ is sufficient even to $O(g^2)$. But this contradicts the result in the Appendix that an additional improvement term is necessarily neeed to make $\theta_{\mu\nu}$ finite in $O(\lambda^2 g^2)$. Hence, we conclude that it is *not* possible to find an improved energy-momentum tensor of the form given in Eq. (5.1) which may be finite even to $O(g^2\lambda^n)$.

VI. IMPROVEMENT-TERM DEPENDENCE OF THE FORM $\tilde{G}(\epsilon,g^2,\lambda)$

In this section we shall consider an improved energy-momentum tensor of the form

$$\theta_{\mu\nu}^{\text{imp}}=\theta_{\mu\nu}+\left[\frac{n-2}{4(1-n)}+\frac{\tilde{G}(\epsilon,g^2,\lambda)}{1-n}\right](\partial_\mu\partial_\nu-\partial^2 g_{\mu\nu})\phi^2 , \quad (6.1)$$

where $\tilde{G}(\epsilon,g^2,\lambda)$ is finite for finite g^2 and λ at $\epsilon=0$. Retracing the steps of the previous section, we obtain

$$\langle \theta_\mu^{\text{imp}\ \mu} \rangle = \text{finite} - \epsilon X \langle \partial^2\phi^2 \rangle^R , \quad (6.2)$$

where

$$X = Z_{18} + G(\epsilon,g^2,\lambda)Z_M^{-1}$$

and

$$G(\epsilon,g^2,\lambda)=-\frac{1}{\epsilon}\tilde{G}(\epsilon,g^2,\lambda)\equiv\sum_{n=0}^{\infty}g^{2n}G_{2n}(\epsilon,\lambda) . \quad (6.3)$$

As in Sec. V the RG equation satisfied by X can be obtained and rewritten as [see II Eq. (6.4)]

$$(-\lambda\epsilon+\beta^\lambda)\frac{\partial X}{\partial\lambda}+\left[-\frac{\epsilon g}{2}+\beta^g\right]-2\tilde{\gamma}_M X - Z_{11}\gamma_{18}-Z_{17}\gamma_{78}$$
$$=\left[(-\lambda\epsilon+\beta^\lambda)\frac{\partial G}{\partial\lambda}+\left[-\frac{g\epsilon}{2}+\beta^g\right]\frac{\partial G}{\partial g}\right]Z_M^{-1} . \quad (6.4)$$

As before, the existence of $\langle \theta_\mu^{\text{imp}\,\mu} \rangle$ implies that

$$\epsilon \left[(-\lambda\epsilon+\beta^\lambda)\frac{\partial G}{\partial\lambda} + \left[-\frac{g\epsilon}{2}+\beta^g \right] \frac{\partial G}{\partial g} \right] Z_M^{-1} = \text{finite} \ . \tag{6.5}$$

Again, following the same procedure as in I and II and using the value of β^g from Eq. (2.14), Eq. (2.20) together with the above equation implies that

$$G(\epsilon, g^2, \lambda) = G_0(\epsilon) + O(g^4) \ , \tag{6.6}$$

and hence as argued at the end of Sec. V, it is not possible to find an improved energy-momentum tensor of the form given in Eq. (6.1), which may be finite to $O(g^2\lambda^n)$.

APPENDIX

It was shown in both Secs. V and VI that, in order that X has no worse than simple poles to $O(g^2)$, it was *necessary* (but not sufficient) that $\tilde{G}$ was a function of ϵ only. Now we show explicitly that even in this case X does have double poles to $O(\lambda^2 g^2)$. The proof proceeds exactly analogous to Appendix A of I.

The double poles in $X = Z_{18} - G_0(\epsilon) Z_M^{-1}$ arise entirely from those in Z_{18} in $O(\lambda^2 g^2)$, since $G_0(\epsilon)$ [which is the same as $g_0(\epsilon)$ of I and II] begins as $O(\epsilon^2)$ (Ref. 7). Direct calculation shows that Z_{18} has no worse than simple poles in $O(\lambda g^2)$ and the simple pole term is nonvanishing. The double-pole term in Z_{18} in $O(\lambda^2 g^2)$ is obtained using the RG equation satisfied by Z_{18}:

$$(-\lambda\epsilon+\beta^\lambda)\frac{\partial Z_{18}}{\partial\lambda} + \left[-\frac{g\epsilon}{2}+\beta^g \right] \frac{\partial Z_{18}}{\partial g} - 2\tilde{\gamma}_M Z_{18} = \left[1 - \frac{\beta^\lambda}{\lambda\epsilon} \right] \left[-\lambda \frac{\partial Z_{18}^{(1)}}{\partial\lambda} - \frac{g}{2}\frac{\partial Z_{18}^{(1)}}{\partial g} \right] + \left[\frac{\beta^\lambda}{\lambda\epsilon} - 2\frac{\beta^g}{g\epsilon} \right] \left[-\lambda \frac{Z_{78}^{(1)}}{\partial\lambda} - \frac{g}{2}\frac{\partial Z_{78}^{(1)}}{\partial g} \right] . \tag{A1}$$

Following the same procedure as in Appendix A of I and using (a) Z_{18} and Z_{78} vanish to $O(g^2)$, (b) Z_{78} vanishes at $g=0$, (c) at $g=0$, Z_{18} begins as λ^3, (d) $Z_{18} - Z_{78}$ has no poles in $O(\lambda g^2)$, since

$$O_1 - O_7 = -\frac{\lambda_0 \phi^4}{4!}$$

(see II), and (e) β^g has no term of $O(\lambda g)$ as verified by direct calculation, one obtains

$$Z_{18(2,1)}^{(2)} = -\tfrac{1}{3}(\beta_2 - 2\gamma_M'^{(1)}) Z_{18(1,1)}^{(1)} \neq 0$$

as neither factor vanishes. (Note that $2\gamma_M'^{(1)}$ is the same as $2\gamma_m^{(1)}$ of I and II.) Here $Z_{18(m,n)}^{(r)}$ is the coefficient of $\lambda^m g^{2n}/\epsilon^r$ in Z_{18}.

Hence, Z_{18} and therefore X does have double poles in $O(\lambda^2 g^2)$. Therefore, the improvement coefficient $G_0(\epsilon)$ obtained from $O(g^0)$ calculation does not suffice in $O(\lambda^2 g^2)$ to make $\theta_\mu^{\text{imp}\,\mu}$ finite to this order.

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[14] It has already been shown in I that λ is not multiplicatively renormalizable. The same argument holds here. In Yukawa theory, the scalar mass is also not multiplicatively renormalizable due to the presence of diagrams for the scalar propagator [such as the single fermion loop diagram in $O(g^2)$] which give contributions proportional to m^2.

[15] ϕ^2 is a multiplicatively renormalizable operator since there is no dimension-2 operator it can mix with. Its renormalization constant Z_M^{-1} can be obtained by dividing Eq. (2.10) by M_0^2 and putting $m=0$ as this renormalization constant is mass independent in the MS scheme.

[16] Here γ_M and $\tilde{\gamma}_M$ are not the same in contrast with scalar QED and non-Abelian gauge theory (NAGT). The reason lies in the fact that M_0^2 is not multiplicatively renormalizable. One may verify that

$$\tilde{\gamma}_M = \gamma_M + \frac{m^2}{M^2}\frac{Z_M'}{Z_M}\left[\gamma_m + \tfrac{1}{2}\mu\frac{\partial}{\partial\mu}\ln Z_M' \right] .$$

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