Energy-momentum tensor in theories with scalar fields and two coupling constants. I. Non-Abelian case

Satish D. Joglekar and Anuradha Misra
Department of Physics, Indian Institute of Technology, Kanpur, 208016, India
(Received 11 October 1988)

In this paper, we generalize our earlier discussion of renormalization of the energy-momentum tensor in scalar QED to that in non-Abelian gauge theories involving scalar fields. We show the need for adding an improvement term to the conventional energy-momentum tensor. We consider two possible forms for the improvement term: (i) one in which the improvement coefficient is a finite function of bare parameters of the theory (so that the energy-momentum tensor can be derived from an action that is a finite function of bare quantities); (ii) one in which the improvement coefficient is a finite quantity, i.e., a finite function of renormalized parameters. We establish a negative result; viz., neither form leads to a finite energy-momentum tensor to $O(e^2\lambda^n)$.

I. INTRODUCTION

Energy-momentum tensors in quantum field theories are of great importance and the finiteness of energy-momentum tensors have received a good deal of attention.[1-10] To understand the significance of a finite energy-momentum tensor, consider the coupling of matter fields to gravity in the weak-field approximation (i.e., keeping terms linear in $h_{\mu\nu}$). It is of the form $\theta_{\mu\nu}h^{\mu\nu}$. A process $A\rightarrow B$ in which gravitational waves are emitted will have a matrix element $\langle B|\theta_{\mu\nu}|A\rangle h^{\mu\nu}$ which, being observable, should be finite. Hence there should exist an energy-momentum tensor $\theta_{\mu\nu}$ of matter fields which has finite matrix elements to all orders.

In theories with scalar fields, it is known that the energy-momentum tensor obtained from

$$\theta_{\mu\nu}=-\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}\bigg|_{g^{\mu\nu}=\eta^{\mu\nu}}$$

(where S is the straightforward generalization of the flat-space action coupled to gravity) does not lead to a finite energy-momentum tensor. An improvement term has to be added. In $\lambda\phi^4$ theory for concreteness the improvement term is proportional to $(\partial_\mu\partial_\nu-\partial^2 g_{\mu\nu})\phi^2$. This term can be derived from a term proportional to $R\phi^2$ added to S. Now the crucial question is whether this term added to S is multiplied by a coupling constant κ_0, which is *independently* renormalized, or the coefficient of this $R\phi^2$ term is a known function of known bare parameters of the theory and needs no independent renormalization. The two cases are qualitatively different in that in the first case an additional piece of experimental information is needed to fix the theory, this information being the "root-mean-square mass radius"[4] of the scalar particle.

The second possibility, in which the coefficient of the $R\phi^2$ term is not independently renormalized, is realized in the case when one can find the energy-momentum tensor $\theta'_{\mu\nu}$, which is a finite function of bare quantities. Consider

$$\theta'_{\mu\nu}=\theta_{\mu\nu}+\frac{\tilde{g}(\epsilon,\lambda_0\mu^{-\epsilon},\dots)}{1-n}(\partial_\mu\partial_\nu-\partial^2 g_{\mu\nu})\phi^2\ , \quad (1.1)$$

where $\tilde{g}$ is a finite function at $n=4$ of bare quantities such as $\lambda_0\mu^{-\epsilon}$. Such an energy-momentum tensor can be derived from an action

$$S=S_0+\frac{\tilde{g}(\epsilon,\lambda_0\mu^{-\epsilon},\dots)}{2(1-n)}R\phi^2\ . \quad (1.2)$$

This action is a finite function of bare quantities as an action should be. Moreover, it leads to finite matrix elements (to first order in $h_{\mu\nu}$) for processes $A\rightarrow B$ in which gravitational radiation is emitted. Thus there is no need to invoke an independent renormalization of the $R\phi^2$ term. This possibility of constructing an improved finite $\theta'_{\mu\nu}$ which is a finite function of bare quantities has been called the "finite improvement program" in Ref. 4.

In theories in which the finite improvement program works, no independent renormalization of the $R\phi^2$ term is needed only in the case in which gravity couples to matter through an action S of Eq. (1.2). It can still couple to matter through a term of the form $h^{\mu\nu}\theta''_{\mu\nu}$ where $\theta''_{\mu\nu}$ is not obtained by finite improvement. This requires addition of a term of the form $\frac{1}{2}\kappa_0 R\phi^2$, where κ_0 is independently renormalized. Which of the two possibilities is realized in reality is a matter for experiments to decide. But in such theories there is at least the possibility that gravity can couple to matter via S of Eq. (1.2). In this case there is an experimental prediction for a mean-square mass radius of particle(s) not possible in other theories in which κ_0 is independently renormalized.

The finite improvement program has also been alternatively interpreted by Collins[6,7] to mean the construction of a finite $\theta'_{\mu\nu}$ in which $\tilde{g}$ is now a finite quantity (i.e., a finite function of renormalized parameters at $n=4$). In this case also, no new experimental input is needed.

The earlier works on the construction of finite energy-momentum tensors in the context of gauge theories without scalars by Freedman, Muzinich, and Weinberg,[3] $\lambda\phi^4$ theory by Freedman and Weinberg,[3,4] by Collins[6,7] and by Brown[8] have been summarized in the Introduction to Ref. 10. It has been shown that in all the above cases there do exist energy-momentum tensors which are finite functions of bare quantities. In $\lambda\phi^4$ theory the improvement coefficient is a function $\tilde{g}(\epsilon)$ of ϵ only and is finite at $\epsilon=0$ (Ref. 7) and hence also fits the description of "finite improvement program" as interpreted by Collins.

The question arises whether the finite improvement program holds in theories which contain scalar fields (among other fields) and have more than one coupling constant. In this series of papers, we shall investigate this question in the context of theories having two coupling constants such as scalar QED, non-Abelian theories with scalars, Yukawa theory, and $\lambda\phi^4$ theory with two scalar fields. In Ref. 10, we investigated this question for scalar QED and reached a negative conclusion for either kinds of finite improvement programs. In this paper we shall investigate this question in the context of non-Abelian gauge theories with scalars and reach the same negative conclusion. This negative conclusion seems to be valid in remaining two theories which will be reported separately.

The treatment of the problem in the context of non-Abelian theories is very similar to that in the context of scalar QED of Ref. 10, which we shall refer to as I. For this reason we shall be very brief.

II. PRELIMINARIES

For simplicity we shall deal with a real scalar multiplet in the vector representation of O(3) coupled to O(3) gauge fields. Generalization to the O(N) group is straightforward. The Lagrangian density is

$$\mathcal{L}=\mathcal{L}_0+\mathcal{L}_g+\mathcal{L}_{\text{gh}},\quad S=\int \mathcal{L}\, d^n x\ , \tag{2.1}$$

where

$$\mathcal{L}_0=-\tfrac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}+\tfrac{1}{2}(D_\mu\phi)^T(D^\mu\phi)-\tfrac{1}{2}m_0^2\phi^T\phi-\frac{\lambda_0}{4!}(\phi^T\phi)^2\ ,$$

$$\mathcal{L}_g=-\tfrac{1}{2}\sum\xi_0(\partial\cdot A^a)^2\ , \tag{2.2}$$

$$\mathcal{L}_{\text{gh}}=\partial^\mu\bar{C}^a D^{ab}_\mu\ ,C_b\ ,$$

and

$$D_\mu\phi=(\partial_\mu-ie_0T^aA^a_\mu)\phi\ ,$$

$$F^a_{\mu\nu}=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+e_0f^{abc}A^b_\mu A^c_\nu\ ,$$

T^a being the adjoint representation of O(3) satisfying

$$[T^a,T^b]=if^{abc}T^c\ .$$

We shall work with dimensionally regularized quantities and use the minimal subtraction (MS) scheme.[11,12] The unrenormalized but dimensionally regularized Green's functions, connected Green's functions and proper vertices are generated, respectively, by $W[J,J_\mu,\eta,\bar{\eta}]$, $Z[J,J_\mu,\eta,\bar{\eta}]$, and $\Gamma[\Phi,\mathcal{A}_\mu,C,\bar{C}]$ with

$$W[J,J_\mu,\eta,\bar{\eta}]=\frac{1}{N}\int D\phi\, DA_\mu DC\, D\bar{C}\times\exp\left[i\int d^n x(\mathcal{L}/a+J\phi+J^a_\mu A^\mu_a+\bar{C}^a\eta^a+\bar{\eta}^aC^a)\right] \tag{2.3}$$

where $W[0]=1$ and a is the loop expansion parameter:

$$Z[J,J_\mu,\eta,\bar{\eta}]=-i\ln W[J,J_\mu,\eta,\bar{\eta}]\ , \tag{2.4}$$

$$\Phi_i(x)=\frac{\delta Z}{\delta J_i(x)},\quad C^a(x)=\frac{\delta Z}{\delta\bar{\eta}^a(x)}\ ,$$
$$\mathcal{A}^a_\mu(x)=\frac{\delta Z}{\delta J^{a\mu}(x)},\quad \bar{C}^a(x)=-\frac{\delta Z}{\delta\eta^a(x)}\ , \tag{2.5}$$

$$\Gamma[\Phi,A_\mu,C,\bar{C}]=Z-\int d^n x(J^T\Phi+J_\mu\mathcal{A}^\mu+\bar{\eta}C+\bar{C}\eta)\ . \tag{2.6}$$

In the MS scheme the renormalization transformations are

$$\begin{aligned}&\phi=Z^{1/2}\phi^R,\quad m_0^2=m^2Z_m\ ,\\&\lambda_0=\mu^\epsilon[\lambda Z_\lambda+\delta\lambda(e)],\quad e_0^2=\mu^\epsilon e^2Z_e^2\ ,\\&A_\mu=Z_3^{1/2}A^R_\mu,\quad \xi_0=Z_\xi\xi=Z_3^{-1}\xi\ ,\\&C=\tilde{Z}^{1/2}C^R,\quad \bar{C}=\tilde{Z}^{1/2}\bar{C}^{R^{1/2}}\ ,\end{aligned} \tag{2.7}$$

where μ is an arbitrary parameter of dimension of mass. $\delta\lambda$ in Eq. (2.7) starts as $O(e^4)$.

The renormalized Green's functions, connected Green's functions, and proper vertices are generated, respectively, by $W^R[J^R,J^R_\mu,\eta^R,\bar{\eta}^R]$, $Z^R[J^R,J^R_\mu,\eta^R,\bar{\eta}^R]$, and $\Gamma^R[\Phi^R,\mathcal{A}^R_\mu,C^R,\bar{C}^R]$ with $W^R[J^R,J^R_\mu,\eta^R,\bar{\eta}^R]=W[J,J_\mu,\eta,\bar{\eta}]$, and $J^R=Z^{1/2}J$, etc.

Equations of motion imply that

$$\left\langle\phi_i\frac{\delta S}{\delta\phi_i}\right\rangle^R=\left\langle\phi_i\frac{\delta S}{\delta\phi_i}\right\rangle=\text{finite}\ , \tag{2.8}$$

$$\left\langle A^a_\mu\frac{\delta S}{\delta A^a_\mu}\right\rangle^R=\left\langle A^a_\mu\frac{\delta S}{\delta A^a_\mu}\right\rangle=\text{finite}\ , \tag{2.9}$$

$$\left\langle\bar{C}_a\frac{\delta S}{\delta\bar{C}_a}\right\rangle^R=\left\langle\bar{C}_a\frac{\delta S}{\delta\bar{C}_a}\right\rangle=\langle\bar{C}_aM^{ab}C_b\rangle=\text{finite}\ , \tag{2.10}$$

$$\left\langle\frac{\delta S}{\delta C_a}C_a\right\rangle^R=\left\langle\frac{\delta S}{\delta C_a}C_a\right\rangle=\text{finite}\ . \tag{2.11}$$

Also one has[8]

$$\langle m_0^2\phi^T\phi\rangle^R=\langle m_0^2\phi^T\phi\rangle=\text{finite}\ . \tag{2.12}$$

Furthermore $\partial^2(\phi^T\phi)$ is a multiplicatively renormalizable operator:[8]

$$[\partial^2(\phi^T\phi)]^{\text{UR}}=Z_m^{-1}[\partial^2(\phi^T\phi)]^R\ . \tag{2.13}$$

We shall use the renormalization group extensively. Below we give the definitions and values of various renormalization-group quantities (here, $Z^{(1)}$ shall mean the coefficient of $1/\epsilon$ terms in Z, etc.):

$$\begin{aligned}
\beta^\lambda(\lambda,e,\epsilon) &\equiv \mu\frac{\partial\lambda}{\partial\mu}\bigg|_{\lambda_0,e_0,m_0,\xi_0} = -\lambda\epsilon+\beta^\lambda(\lambda,e)\\
&= -\lambda\epsilon+\lambda^2\frac{\partial Z_\lambda^{(1)}}{\partial\lambda}+\frac{e\lambda}{2}\frac{\partial Z_\lambda^{(1)}}{\partial e}+\lambda\frac{\partial\delta\lambda^{(1)}}{\partial\lambda}+\frac{e}{2}\frac{\partial}{\partial e}\delta\lambda^{(1)}-\delta\lambda^{(1)}\\
&= -\lambda\epsilon+\beta_2\lambda^2+\cdots\ ,\\
\beta^e(\lambda,e,\epsilon) &= \mu\frac{\partial e}{\partial\mu}=-\frac{e\epsilon}{2}+\beta^e(\lambda,e)=-\frac{e\epsilon}{2}+\frac{e^2}{2}\frac{\partial}{\partial e}Z_e^{(1)}+e\lambda\frac{\partial Z_e^{(1)}}{\partial\lambda}=-\frac{e\epsilon}{2}+O(e^3)+O(e\lambda^2)\ ,\\
\gamma_m(\lambda,e,\epsilon) &\equiv -\tfrac{1}{2}\mu\frac{\partial}{\partial\mu}\ln Z_m=\gamma_m(\lambda,e)Z_m^{(1)}+\frac{e}{4}\frac{\partial}{\partial e}Z_m^{(1)}=\gamma_m^{(1)}\lambda+\cdots\ ,\\
\gamma(\lambda,e,\xi,\epsilon) &\equiv \tfrac{1}{2}\mu\frac{\partial}{\partial\mu}\ln Z=\gamma(\lambda,e,\xi)=\frac{\lambda}{2}\frac{\partial}{\partial\lambda}Z^{(1)}-\frac{\partial}{\partial e}Z^{(1)}\ ,\\
\gamma_3(\lambda,e,\xi,\epsilon) &\equiv \frac{1}{2}\frac{\partial}{\partial\mu}\ln Z_3=\gamma_3(\lambda,e,\xi)=\frac{\lambda}{2}\frac{\partial}{\partial\lambda}Z_3^{(1)}-\frac{e}{4}\frac{\partial}{\partial e}Z_3^{(1)}\ ,\\
\tilde\gamma(\lambda,e,\xi,\epsilon) &\equiv \tfrac{1}{2}\mu\frac{\partial}{\partial\mu}\ln\tilde Z=\tilde\gamma(\lambda,e,\epsilon)=\frac{\lambda}{2}\frac{\partial}{\partial\lambda}\tilde Z^{(1)}-\frac{e}{4}\frac{\partial}{\partial e}\tilde Z^{(1)}\ .
\end{aligned} \tag{2.14}$$

The following set of operators is closed under renormalization:

$$O_1=-\frac{\lambda_0}{4!}(\phi^T\phi)+\tfrac14 F^a_{\mu\nu}F^{a\mu\nu}+\tfrac12\xi_0\sum_a(\partial\cdot A^a)^2\ ,$$

$$O_2=m_0^2\phi^T\phi,\quad O_3=\phi_i\frac{\delta S}{\delta\phi_i}\ ,$$

$$O_4=A^a_\mu\frac{\delta\tilde S}{\delta A^a_\mu}+\partial_\mu\bar C^a D^{ab}_\mu C_b$$

$$\text{where } \tilde S=S+\tfrac12\xi_0\sum_a(\partial\cdot A^a)^2 d^4x\ , \tag{2.15}$$

$$O_5=\frac{\delta S}{\delta C^a}C^a\ ,$$

$$O_6=\tfrac14 F^a_{\mu\nu}F^{a\mu\nu}+\tfrac12\xi_0\sum_a(\partial\cdot A^a)^2\ ,$$

$$O_7=\tfrac12\xi_0\sum_a(\partial\cdot A^a)^2,\quad O_8=\partial^2(\phi^T\phi)\ .$$

We note that, at zero momentum,

$$\begin{aligned}
\int d^nx\, O_1 &= \lambda_0\frac{\partial S}{\partial\lambda_0}+\tfrac12 e_0\frac{\partial S}{\partial e_0}-\tfrac12\int A^a_\mu\frac{\partial S}{\partial A^a_\mu}d^nx\ ,\\
\int d^nx\, O_2 &= -2m_0^2\frac{\partial S}{\partial m_0^2}\ ,\\
\int d^nx\, O_6 &= \tfrac12 e_0\frac{\partial S}{\partial e_0}-\tfrac12\int d^nx\, A^a_\mu\frac{\delta S}{\delta A^a_\mu}\ ,\\
\int d^nx\, O_7 &= -\xi_0\frac{\partial S}{\partial\xi_0}
\end{aligned} \tag{2.15'}$$

and hence define[13]

$$\begin{aligned}
\int d^nx\, O_1^R &= \lambda\frac{\partial Z^R}{\partial\lambda}+\tfrac12 e\frac{\partial Z^R}{\partial e}+\tfrac12\int J^{aR}_\mu\frac{\partial Z^R}{\partial J^{aR}_\mu}d^nx\ ,\\
\int d^nx\, O_2^R &= -2m^2\frac{\partial Z^R}{\partial m^2}\ ,\\
\int d^nx\, O_3^R &= -\int d^nx\, J_i^R\frac{\delta Z^R}{\delta J_i^R}=\int d^nx\, O_3\ ,\\
\int d^nx\, O_4^R &= -\int d^nx\, J^{aR}_\mu\frac{\delta Z^R}{\delta J^{aR}_\mu}+2\left\langle\int d^nx\, O_7^R\right\rangle\\
&\quad+\left\langle\int d^nx\, O_5^R\right\rangle\ , \qquad (2.16)\\
\int d^nx\, O_5^R &= \int d^nx\,\bar\eta_a(x)\frac{\delta Z^R}{\delta\bar\eta_a(x)}=\int d^nx\, O_5\ ,\\
\int d^nx\, O_6^R &= \tfrac12 e\frac{\partial Z^R}{\partial e}+\tfrac12\int d^nx\, J^{aR}_\mu\frac{\delta Z^R}{\delta J^{aR}_\mu}\ ,\\
\int d^nx\, O_7^R &= -\xi\frac{\partial Z^R}{\partial\xi}\ .
\end{aligned} \tag{2.16}$$

Here renormalization of $\int d^nx\, O_4$ has been defined in terms of that of $\int d^nx\, O_5$ and $\int d^nx\, O_7$ using the identity

$$\left\langle\int d^nx\, O_4\right\rangle=-\int J^a_\mu\frac{\delta Z^R}{\delta J^a_\mu}d^nx+2\left\langle\int d^nx\, O_7\right\rangle+\left\langle\int d^nx\, O_5\right\rangle\ .$$

We shall define the renormalization matrix by

$$\langle O_i\rangle^{\text{UR}}=Z_{ij}\langle O_j\rangle^R\ . \tag{2.17}$$

Then,

$$Z_{ij}^{-1}\mu\frac{\partial}{\partial\mu}Z_{jk}=\gamma_{ik}=\text{finite at }\epsilon=0\ . \tag{2.18}$$

Information on the structure of the renormalization matrix is obtained from Eqs. (2.8)–(2.13). Mixing of O_7 is discussed in Appendix B and there it is shown that it can mix only with O_3, O_4, O_5. Mixing of O_4 is also discussed there and it is shown that it can also only mix with O_3, O_4, and O_5. This leads to the following structure for the renormalization matrix:

$$Z=\begin{bmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{14} & Z_{15} & Z_{16} & Z_{17} & Z_{18} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & Z_{43} & Z_{44} & Z_{45} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ Z_{61} & Z_{62} & Z_{63} & Z_{64} & Z_{65} & Z_{66} & Z_{67} & Z_{68} \\ 0 & 0 & Z_{73} & Z_{74} & Z_{75} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & Z_m^{-1} \end{bmatrix} . \tag{2.19}$$

We expand Z_m^{-1} in powers of e:

$$Z_m^{-1}=Z_{m(0)}^{-1}+e^2Z_{m(2)}^{-1}+e^4Z_{m(4)}^{-1}+\cdots .$$

We now state two results needed in Secs. V and VI. As shown by Collins,[7] if $H(\lambda,m,\epsilon)$ is a finite function of λ,m at $\epsilon=0$ and

$$H(\lambda,m,\epsilon)Z_{m(0)}^{-1}(\lambda,\epsilon)=\text{finite at } \epsilon=0$$

keeping λ and m finite and fixed, then

$$H(\lambda,m,\epsilon)=0 . \tag{2.20}$$

Second, as shown in Ref. 9, if

$$F\left[\lambda_0\mu^{-\epsilon},\frac{m_0^2}{\mu^2},\epsilon\right]$$

is a finite function (at $\epsilon=0$) of $\lambda_0\mu^{-\epsilon}$ and m_0^2/μ^2, and if

$$F\left[\lambda_0\mu^{-\epsilon},\frac{m_0^2}{\mu^2},\epsilon\right]Z_{m(0)}^{-1}(\lambda,\epsilon)=\text{finite at } \epsilon=0$$

keeping λ and m fixed and finite, then

$$F\left[\lambda_0\mu^{-\epsilon},\frac{m_0^2}{\mu^2},\epsilon\right]=0 . \tag{2.21}$$

III. IMPROVED TRACE

We work with the energy-momentum tensor[3]

$$\theta_{\mu\nu}=-g_{\mu\nu}L_{\text{eff}}-F^a_{\mu\alpha}F^{a\alpha}_{\nu}+\tfrac{1}{2}[(D_\mu\phi)^T(D_\nu\phi)+(\nu\leftrightarrow\mu)]-\partial_\mu\bar{C}_aD^{ab}_\nu C_b-\partial_\nu\bar{C}_aD^{ab}_\mu C_b-g_{\mu\nu}\xi_0(\partial\cdot A)^2+\xi_0\partial_\mu(\partial\cdot A^a)A^a_\nu$$
$$+\xi_0\partial_\nu(\partial\cdot A^a)A^a_\mu-g_{\mu\nu}\xi_0\partial^\rho(\partial\cdot A^a)A^a_\rho . \tag{3.1}$$

This energy-momentum tensor has finite matrix elements at $q=0$ and to first order in the external momentum q (Ref. 3) but not to second order in q. The most general improvement one can add to $\theta_{\mu\nu}$ is parametrized as

$$\theta^{\text{imp}}_{\mu\nu}=\theta_{\mu\nu}+\left[\frac{n-2}{4(1-n)}+\frac{\tilde{g}}{1-n}\right](\partial_\mu\partial_\nu-g_{\mu\nu}\partial^2)(\phi^T\phi)\equiv\tilde{\theta}_{\mu\nu}+\frac{\tilde{g}}{1-n}(\partial_\mu\partial_\nu-g_{\mu\nu}\partial^2)(\phi^T\phi) , \tag{3.2}$$

where $\tilde{\theta}_{\mu\nu}$ is the energy-momentum tensor obtained from the conformally invariant action.[7] Here $\tilde{g}$ is a free parameter for which we shall be trying specific forms in Secs. V and VI, respectively, as mentioned in the Introduction. The trace of $\theta^{\text{imp}}_{\mu\nu}$ is easily verified to be

$$\theta^{\text{imp}\,\mu}_{\mu}=(n-4)\left[-\frac{\lambda_0}{4!}(\phi^T\phi)^2+\tfrac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}+\tfrac{1}{2}\xi_0\sum_a(\partial\cdot A^a)^2\right]-(n-2)\bar{C}\frac{\delta S}{\delta\bar{C}}-\left[\frac{n-2}{2}\right]\phi_i\frac{\delta S}{\delta\phi_i}$$
$$-(n-2)\partial^\mu[\xi_0(\partial\cdot A^a)A^a_\mu-\bar{C}_aD^{ab}_\mu C_b]+m_0^2\phi^T\phi+\tilde{g}\partial^2(\phi^T\phi) . \tag{3.3}$$

It is shown in Appendix B that $\partial^\mu[\xi_0(\partial\cdot A^a)A^a_\mu-\bar{C}_aD^{ab}_\mu C_b]$ is a finite operator. Now using Eqs. (2.8), (2.10), (2.12), and (2.13) one obtains

$$\langle\theta^{\text{imp}\,\mu}_{\mu}\rangle=\text{finite}+(n-4)\langle O_1\rangle^{\text{UR}}+\tilde{g}Z_m^{-1}\langle\partial^2(\phi^T\phi)\rangle^R . \tag{3.4}$$

In the following sections we shall investigate if it is possible to choose $\tilde{g}$ in either of the two ways mentioned in the Introduction so that $\langle\theta^{\text{imp}\,\mu}_{\mu}\rangle$ is finite.

IV. EXPRESSIONS FOR Z_{1j}

In this section we shall use the techniques of Ref. 10 to obtain expressions for Z_{1j} ($j=1,2,\ldots,7$) which are needed in further investigations of Secs. V and VI. These expressions are obtained by considering renormalization of O_1 at zero momentum.

It is straightforward to show that

$$\int d^nx\, O_1=\int d^nx\left[-\frac{\lambda_0}{4!}(\phi^T\phi)^2+\tfrac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}+\tfrac{1}{2}\xi_0\sum_a(\partial\cdot A^a)^2\right]$$
$$=-S+\tfrac{1}{2}\int\phi_i\frac{\delta S}{\delta\phi_i}d^nx+\int\bar{C}_aM^{ab}C_bd^nx\ . \qquad (4.1)$$

The last two terms on the right-hand side are finite operators. $\langle S\rangle^{\mathrm{UR}}$ can be obtained following the procedure of Refs. 10 and 14. We skip the derivation and give the final results:

$$\left\langle\int d^nx\, O_1\right\rangle^{\mathrm{UR}}=-\langle S\rangle^{\mathrm{UR}}+\frac{1}{2}\left\langle\int\phi_i\frac{\delta S}{\delta\phi_i}d^nx\right\rangle^R+\left\langle\int\bar{C}_aM^{ab}C_bd^nx\right\rangle^R$$
$$=\left[1-\frac{\beta^\lambda(\lambda,e)}{\lambda\epsilon}\right]\left\langle\int d^nx\, O_1\right\rangle^R-\frac{\gamma_m(\lambda,e)}{\epsilon}\left\langle\int d^nx\, O_2\right\rangle^R+\frac{\gamma(\lambda,e)}{\epsilon}\left\langle\int d^nx\, O_3\right\rangle^R$$
$$+\left[\frac{\gamma_3-\beta^e}{\epsilon}\right]\left\langle\int d^nx\, O_4\right\rangle^R+\left[\frac{\beta^e}{e\epsilon}-\frac{\gamma_3}{\epsilon}-\frac{2\tilde{\gamma}}{\epsilon}\right]\left\langle\int d^nx\, O_5\right\rangle$$
$$+\left[\frac{\beta^\lambda}{\lambda\epsilon}-2\frac{\beta^e}{e\epsilon}\right]\left\langle\int d^nx\, O_6\right\rangle+\frac{\gamma_3}{\epsilon}\left\langle\int d^nx\, O_7\right\rangle\ . \qquad (4.2)$$

Hence

$$Z_{11}=1-\frac{\beta^\lambda}{\lambda\epsilon},\quad Z_{14}=\frac{\gamma_3-\beta^e}{\epsilon}\ ,$$
$$Z_{12}=+\frac{\gamma_m}{\epsilon},\quad Z_{15}=\frac{\beta^e}{e\epsilon}-\frac{\gamma_3}{\epsilon}-\frac{2\tilde{\gamma}}{\epsilon}\ , \qquad (4.3)$$
$$Z_{13}=\frac{\gamma}{\epsilon},\quad Z_{16}=\frac{\beta^\lambda}{\lambda\epsilon}-2\frac{\beta^e}{e\epsilon},\quad Z_{17}=\frac{\gamma_3}{\epsilon}\ .$$

As Z_{1j} $(j=1,2,\ldots,7)$ have only simple poles in ϵ, from Eq. (3.4) it follows that $\langle\theta^{\mathrm{imp}\,\mu}_\mu\rangle$ is finite at zero momentum, a result known also from Ref. 3.

V. IMPROVEMENT-TERM DEPENDENCE OF THE FORM $\tilde{g}(\epsilon,e_0^2\mu^{-\epsilon},\lambda_0\mu^{-\epsilon})$

As shown in the previous section, the energy-momentum tensor

$$\theta^{\mathrm{imp}'}_{\mu\nu}=\theta_{\mu\nu}+\frac{n-2}{4(1-n)}(\partial_\mu\partial_\nu-g_{\mu\nu}\partial^2)\phi^T\phi$$

is finite at zero momentum. Explicit calculation shows that it is finite only up to $O(\lambda^3)$ at $e=0$, up to $O(e^4)$ at $\lambda=0$ and also in $O(\lambda e^2)$, but a further improvement is *necessarily* needed in $O(\lambda^4)$, $O(\lambda e^4)$, and $O(\lambda^2e^2)$. (See, for example, Appendix A.)

In this section, we shall consider a further improvement, where the improvement coefficient $\tilde{g}$ is a *finite* function of *bare* coupling constants:

$$\theta^{\mathrm{imp}}_{\mu\nu}=\theta_{\mu\nu}+\left[\frac{n-2}{4(1-n)}+\frac{\tilde{g}(\epsilon,e_0^2\mu^{-\epsilon},\lambda_0\mu^{-\epsilon})}{1-n}\right]\times(\partial_\mu\partial_\nu-\partial^2g_{\mu\nu})(\phi^T\phi)\ , \qquad (5.1)$$

where

$$\tilde{g}(\epsilon,e_0^2\mu^{-\epsilon},\lambda_0\mu^{-\epsilon})=\sum_{r=0}^{\infty}g_{2r}(\epsilon,\lambda_0\mu^{-\epsilon})(e_0^2\mu^{-\epsilon})^r\ , \qquad (5.2)$$

where $\tilde{g}_r(\epsilon,\lambda_0\mu_0^\epsilon)$ are finite functions of λ_0.

From Eq. (3.4) and the fact that Z_{1j} $(j=1,2,\ldots,7)$ involve only simple poles, it follows that

$$\langle\theta^{\mathrm{imp}\,\mu}_\mu\rangle=\text{finite}+[-\epsilon Z_{18}+\tilde{g}(\epsilon,e_0^2\mu^{-\epsilon},\lambda_0\mu^{-\epsilon})Z_m^{-1}]\times\langle\partial^2(\phi^T\phi)\rangle^R\ . \qquad (5.3)$$

For future convenience, we shall reexpress

$$\tilde{g}(\epsilon,e_0^2\mu^{-\epsilon},\lambda_0\mu^{-\epsilon})=-\epsilon g(\epsilon,e_0^2\mu^{-\epsilon},\lambda_0\mu^{-\epsilon})\ ,$$
$$\tilde{g}_n(\epsilon,\lambda_0\mu^{-\epsilon})=-\epsilon g_n(\epsilon,\lambda_0\mu^{-\epsilon})\ , \qquad (5.4)$$

where now $g_n(\epsilon,\lambda_0\mu^{-\epsilon})$ may also contain $1/\epsilon$ terms when expanded in powers of ϵ and $(\lambda_0\mu^{-\epsilon})$. We thus have

$$\langle\theta^{\mathrm{imp}\,\mu}_\mu\rangle=\text{finite}-\epsilon X\langle\partial^2(\phi^T\phi)\rangle^R\ , \qquad (5.5)$$

where

$$X=Z_{18}+g(\epsilon,e_0^2\mu^{-\epsilon},\lambda_0\mu^{-\epsilon})Z_m^{-1}\ . \qquad (5.6)$$

Thus, to obtain a finite energy-momentum tensor, one must find a $g(\epsilon,e_0^2\mu^{-\epsilon},\lambda_0\mu^{-\epsilon})$ such that X given above does not contain worse than simple poles. We shall show, in what follows, that it is not possible to do so consistently except at $e=0$.

For this purpose, we shall use the renormalization group (RG) equation satisfied by Z_{18} which can be derived straightforwardly from Eqs. (2.18) and (2.19), following the procedure of Appendix C of I. It is

$$(-\lambda\epsilon+\beta^\lambda)\frac{\partial Z_{18}}{\partial\lambda}+\left[-\frac{e\epsilon}{2}+\beta^e\right]\frac{\partial Z_{18}}{\partial e}-2\gamma_mZ_{18}=Z_{11}\gamma_{18}+Z_{16}\gamma_{68}\ . \qquad (5.7)$$

Substituting from Eq. (5.6) for Z_{18} and using

$$\mu\frac{\partial}{\partial\mu}[g(\epsilon,e_0^2\mu^{-\epsilon},\lambda_0\mu^{-\epsilon})Z_m^{-1}]=2\gamma_m g(\epsilon,e_0^2\mu^{-\epsilon},\lambda_0\mu^{-\epsilon})Z_m^{-1}+Z_m^{-1}\mu\frac{\partial}{\partial\mu}g(\epsilon,e_0^2\mu^{-\epsilon},\lambda_0\mu^{-\epsilon})$$
$$=2\gamma_m g(\epsilon,e_0^2\mu^{-\epsilon},\lambda_0\mu^{-\epsilon})Z_m^{-1}+Z_m^{-1}\mu\frac{\partial}{\partial\mu}\sum_{n=0}^{\infty}g_{2n}(\epsilon,\lambda_0\mu^{-\epsilon})(e_0^2\mu^{-\epsilon})^n \tag{5.8}$$

one obtains an equation satisfied by X:

$$(-\lambda\epsilon+\beta^\lambda)\frac{\partial X}{\partial\lambda}+\left[-\frac{e\epsilon}{2}+\beta^e\right]\frac{\partial X}{\partial e}-2\gamma_m X-Z_{11}\gamma_{18}-Z_{16}\gamma_{68}=-\epsilon\sum_{n=0}^{\infty}(e_0^2\mu^{-\epsilon})^n\left[ng_{2n}+\frac{\partial g_{2n}}{\partial(\lambda_0\mu^{-\epsilon})},\lambda_0\mu^{-\epsilon}\right]Z_m^{-1} . \tag{5.9}$$

Now, suppose it were possible to choose g_n's such that X has no worse than simple poles (which would imply the existence of a finite energy-momentum tensor). Then, as Z_{11} and Z_{16} have only simple poles, the left-hand side of Eq. (5.9) has at worst simple poles, and hence so does the right-hand side. Hence,

$$\epsilon^2\left[\sum_{n=0}^{\infty}(e_0^2\mu^{-\epsilon})^n\left[ng_{2n}+\frac{\partial g_{2n}}{\partial(\lambda_0\mu^{-\epsilon})}\lambda_0\mu^{-\epsilon}\right]Z_m^{-1}\right]=\text{finite} . \tag{5.10}$$

In I we solved the same equation in the context of scalar QED using the result of Eq. (2.21) and it was shown that Eq. (5.10), when considered up to $O(e^2)$, implies that

$$g_2(\epsilon,\lambda_0\mu^{-\epsilon})=0 . \tag{5.11}$$

This implies that the improvement term in Eq. (5.1) is consistent with the finiteness of $\langle\theta_\mu^{\text{imp}\,\mu}\rangle$ in $O(e^2\lambda^n)$ only if

$$g(\epsilon,e_0^2\mu^{-\epsilon},\lambda_0\mu^{-\epsilon})=g_0(\epsilon)+O(e^4) , \tag{5.12}$$

i.e., the improvement term obtained to $O(e^0)$ is sufficient even to $O(e^2)$. But this contradicts the result in Appendix A that an additional improvement term is necessarily needed to make $\theta_{\mu\nu}$ finite in $O(\lambda^2e^2)$. Hence, we conclude that it is *not* possible to find an improved energy-momentum tensor of the form given in Eq. (5.1) which may be finite even to $O(e^2\lambda^n)$.

VI. IMPROVEMENT TERM DEPENDENCE OF THE FORM $\bar{g}(\epsilon,e^2,\lambda)$

In this section we shall consider an improved energy-momentum tensor of the form

$$\theta_{\mu\nu}^{\text{imp}}=\theta_{\mu\nu}+\left[\frac{n-2}{4(1-n)}+\frac{\bar{g}(\epsilon,e^2,\lambda)}{1-n}\right](\partial_\mu\partial_\nu-\partial^2g_{\mu\nu})\phi^T\phi , \tag{6.1}$$

where $\bar{g}(\epsilon,e^2,\lambda)$ is finite for finite e^2 and λ. Retracing the steps of the previous section, we obtain

$$\langle\theta_\mu^{\text{imp}\,\mu}\rangle=\text{finite}-\epsilon X\langle\partial^2(\phi^T\phi)\rangle , \tag{6.2}$$

where

$$X=Z_{18}+g(\epsilon,e^2,\lambda)Z_m^{-1}$$

and

$$g(\epsilon,e^2,\lambda)=-\frac{1}{\epsilon}\bar{g}(\epsilon,e^2,\lambda)=\sum_{n=0}^{\infty}e^{2n}g_{2n}(\epsilon,\lambda) . \tag{6.3}$$

As in the previous section, the RG equation satisfied by X can be obtained using

$$\mu\frac{\partial}{\partial\mu}g(\epsilon,e^2,\lambda)=(-\lambda\epsilon+\beta^\lambda)\frac{\partial g}{\partial\lambda}+\left[-\frac{e\epsilon}{2}+\beta^e\right]\frac{\partial g}{\partial e} \tag{6.4}$$

and reads

$$(-\lambda\epsilon+\beta^\lambda)\frac{\partial X}{\partial\lambda}+\left[-\frac{e\epsilon}{2}+\beta^e\right]\frac{\partial X}{\partial e}-2\gamma_m X-Z_{11}\gamma_{18}-Z_{16}\gamma_{68}=+\left[(-\lambda\epsilon+\beta^\lambda)\frac{\partial g}{\partial\lambda}+\left[-\frac{e\epsilon}{2}+\beta^e\right]\frac{\partial g}{\partial e}\right]Z_m^{-1} . \tag{6.5}$$

As before existence of finite $\langle\theta_\mu^{\text{imp}\,\mu}\rangle$ requires that X has at worst simple poles in ϵ. If it were possible to choose $g(\epsilon,e^2,\lambda)$ such that X has no worse than simple pole, then as in the previous section, Eq. (6.5) implies that

$$\epsilon\left[(-\lambda\epsilon+\beta^\lambda)\frac{\partial g}{\partial\lambda}+\left[-\frac{e\epsilon}{2}+\beta^e\right]\frac{\partial g}{\partial e}\right]Z_m^{-1}=\text{finite} . \tag{6.6}$$

We considered Eq. (6.6) in I, in the context of scalar QED with help of the result of Eq. (2.20). Following the same procedure one obtains, in the present case also,

$$g(\epsilon,e^2,\lambda)=g_0(\epsilon)+O(e^4) . \tag{6.7}$$

This is exactly analogous to Eq. (5.12) and from the discussion below Eq. (5.12), it follows that it is not possible to find an improved energy-momentum tensor of the form given in Eq. (6.1) which may be finite even to $O(e^2\lambda^n)$.

APPENDIX A

It was shown in both Secs. V and VI that, in order that X has no worse than simple poles to $O(e^2)$, it was *necessary* (but not sufficient) that $\tilde{g}$ be a function of ϵ only. Now we show explicitly that even in this case X does have double poles to $O(\lambda^2 e^2)$. This proves that for no choice of $\tilde{g}$ of either forms chosen in Secs. V and VI does X have no poles worse than simple poles. The proof proceeds exactly analogous to Appendix A of I.

The double poles in $X=Z_{18}-g_0(\epsilon)Z_m^{-1}$ arise entirely from those in Z_{18} in $O(\lambda^2 e^2)$, since $g_0(\epsilon)$ begins as $O(\epsilon^2)$ (Ref. 7). Direct calculation shows that Z_{18} has no worse than simple poles in $O(\lambda e^2)$ and the simple-pole term is nonvanishing. The double-pole term in Z_{18} in $O(\lambda^2 e^2)$ is obtained in terms of simple-pole term in Z_{18} in $O(\lambda e^2)$ via the renormalization-group equation satisfied by Z_{18}:

$$(-\lambda\epsilon+\beta^\lambda)\frac{\partial Z_{18}}{\partial\lambda}+\left[-\frac{e\epsilon}{2}+\beta^e\right]\frac{\partial Z_{18}}{\partial e}-2\gamma_m Z_{18}=\left[1-\frac{\beta^\lambda}{\lambda\epsilon}\right]\gamma_{18}+\left[\frac{\beta^\lambda}{\lambda\epsilon}-2\frac{\beta^e}{e\epsilon}\right]\gamma_{68}$$
$$=\left[1-\frac{\beta^\lambda}{\lambda\epsilon}\right]\left[-\lambda\frac{\partial}{\partial\lambda}Z_{18}^{(1)}-\frac{e}{2}\frac{\partial}{\partial e}Z_{18}^{(1)}\right]+\left[\frac{\beta^\lambda}{\lambda\epsilon}-2\frac{\beta^e}{e\epsilon}\right]\left[-\lambda\frac{\partial}{\partial\lambda}Z_{68}^{(1)}-\frac{e}{2}\frac{\partial}{\partial e}Z_{68}^{(1)}\right]. \quad \text{(A1)}$$

Following the same procedure as in Appendix A of I and using (a) Z_{18} and Z_{68} vanish to $O(e^2)$, (b) Z_{68} vanishes at $e=0$, (c) at $e=0$, Z_{18} begins as λ^3, (d) $Z_{18}-Z_{68}$ has no poles in $O(\lambda e^2)$ because $O_1-O_6=-\lambda_0(\phi^T\phi)^2/4!$ needs no counterterms proportional to O_8 in this order, (e) β^e has no term of $O(\lambda e)$ as verified by direct calculation, one obtains

$$Z_{18(2,1)}^{(2)}=-\tfrac{1}{3}(\beta_2-2\gamma_m^{(1)})Z_{18(1,1)}^{(1)}\neq 0$$

as neither factors vanish. Here $Z_{18(m,n)}^{(r)}$ is the coefficient of $\lambda^m e^{2n}/\epsilon^r$ in Z_{18}.

Hence, Z_{18} and therefore X does have double poles in $O(\lambda^2 e^2)$. Therefore, the improvement coefficient $g_0(\epsilon)$ obtained from $O(e^0)$ calculation does not suffice in $O(\lambda^2 e^2)$ to make $\theta_\mu^{\text{imp}\,\mu}$ finite to this order.

APPENDIX B

We consider the operator

$$O=-\tfrac{1}{2}\xi_0\sum(\partial\cdot A^a)^2+\bar{C}_a\partial^\mu D_\mu^{ab}C_b .$$

It is easy to show that O is invariant under Becchi-Rouet-Stora (BRS) transformations. If one considers an action with a source term added that couples to O,

$$S'=S+\int O(x)N(x)d^n x ,$$

then S' is also BRS invariant. From this fact it is easy to show that the Ward-Takahashi (WT) identity satisfied by the divergent part of the generating functional for proper vertices with one insertion of $O(x)$ is identical to that satisfied by the corresponding generating functional for a gauge-invariant operator. Hence it can only mix with those operators that mix with a dimension-four Lorentz-scalar operator. These operators are $(O_1\text{-}O_7)$, O_2, O_3, $(O_6\text{-}O_7)$, and O_8. Now from equations of motion of the antighost field it is easy to show that $\bar{C}_a\partial^\mu D_\mu^{ab}C_b$ is a finite operator. Furthermore

$$\left\langle -\tfrac{1}{2}\xi_0\sum(\partial\cdot A^a)^2\right\rangle$$

satisfies the WT identity

$$\left\langle -\frac{1}{2}\xi_0\sum[\partial\cdot A^a(x)]^2+\tfrac{1}{2}\bar{C}^a\partial^\mu D_\mu^{ab}C_b(x)\right\rangle=\frac{1}{2}\left\langle \bar{C}_a(x)\partial\cdot A^a(x)\int d^n x'\sum J_\mu^b(x')D_\mu^{bd}C_d(x')+J^b(x')f_{bde}\phi_d(x')C_e(x')-\bar{\eta}_b(x')\tfrac{1}{2}e_0 f_{abc}C_a(x')C_c(x')-\tfrac{1}{2}\xi_0[\partial\cdot A^b(x')\eta_b(x')]\right\rangle . \quad \text{(B1)}$$

Note that each term on the right-hand side is proportional to a source. Hence the only operators from the above set consistent with Eq. (B1) are O_3,O_4,O_5. Hence only Z_{73},Z_{74},Z_{75} are nonzero while Z_{77} is 1.

The operator O_4 is a class-I gauge-variant operator[14] and can only mix with other class-I operators (here O_3 and O_4) and with class-II operators (here O_5). Next we comment on the closure of the set O_i ($i=1,\ldots,8$) under renormalization. $(O_1\text{-}O_7)$, O_2, $(O_6\text{-}O_7)$, and O_8 are gauge-invariant operators which do not vanish by equations of motion and from the results on the renormalization of gauge-invariant operators[15] they can only mix with themselves and operators O_3,O_4,O_5. Hence the set O_i ($i=1,2,\ldots,8$) is closed under renormalization.

Next, we shall show that the operator

$$O'=\xi_0 A_\mu^a(\partial\cdot A^a)-\bar{C}^a D_\mu^{ab}C_b$$

is a finite operator. This follows from the WT identity that

$$\langle \xi_0 A_\mu^a(x)\partial\cdot A^a(x) - \bar{C}^a(x) D_\mu^{ab} C_b(x) \rangle = i \Big\langle \bar{C}^a(x) A_\mu^a(x) \int d^n x' [J_\mu^b(x') D_\mu^{bd} C_d(x') + J^b(x') f_{bde} \phi_d(x') C_e(x') - \bar{\eta}_b(x') \tfrac{1}{2} e_0 f_{bde} C_0(x') - \tfrac{1}{2}\xi_0(\partial\cdot A^b)\eta_b(x')] \Big\rangle \;. \tag{B2}$$

This WT identity is very similar to that satisfied by $A_\mu(x)\partial\cdot A(x)$ given in Appendix B of I for the case of scalar QED, in that each term on the right-hand side is proportional to a source. The argument given there following Eq. (B8) based on dimensions and global gauge invariance applies here also and the right-hand side of Eq. (B2) is finite, proving the finiteness of O'.

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