Nonlocal Noether currents and conformal invariance for super chiral fields

Wei Li
*Department of Mathematics and Computer Science, Clarkson University, Potsdam, New York 13676**
and Institute of Modern Physics, Northwest University, Xian, China

Bo-yu Hou
Center of Theoretical Physics, Chinese Center of Advanced Science and Technology (World Laboratory), Beijing, China
and Institute of Modern Physics, Northwest University, Xian, China

San-ruo Hao
Institute of Modern Physics, Northwest University, Xian, China
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We find an infinitesimal transformation relating the parametric-conformal symmetry in super chiral fields. This allows us to derive an infinite set of nonlocal Noether currents and to form an infinite-dimensional conformal algebra.

I. INTRODUCTION

The concept of an infinite set of conserved Noether currents in $(1+1)$-dimensional field theories has received considerable attention. Such sets have been found both for quantum and classical theories such as the principal chiral model[1] and the Heisenberg model.[2,3] The existence of an infinite set of conserved Noether currents is closely connected to the existence of an infinite-dimensional Lie algebra, the well-known Kac-Moody algebra which acts on the solution space of a nonlinear system.[4] The explicit expressions of the infinitesimal transformations for this type of hidden symmetry, which change the Lagrangian density by a total divergence and give rise to the desired nonlocal Noether currents, have been well developed.

However, it is interesting to notice that there exists another type of hidden symmetry which is known to allow for another infinite-dimensional Lie group, the so-called Virasoro group in some integrable systems.[5] This means that the algebra is enlarged by a Virasoro algebra forming a semidirect product with the Kac-Moody algebra. In this paper we show that parallel developments can be made for super chiral fields. To make sure of the existence of another set of nonlocal conserved currents, we propose new infinitesimal transformations and show that they constitute symmetries of the field equations. We also hope to be able to extend the hidden-symmetry algebra for super chiral fields to the semidirect product with the Kac-Moody and Virasoro algebras.

Here we would like to point out that there exist two different types of Virasoro symmetries as well as those of the Kac-Moody symmetries in two-dimensional chiral models. One kind is involved in quantum field theory, the other is considered in the classical case. According to the pioneer work,[6] the model with the Wess-Zumino term possesses the Kac-Moody and Virasoro symmetries on the two light-cone components. The algebra of generators corresponding to these symmetries, constructed from the quantum commutators of currents and the energy-momentum tensors, respectively, is calculated to be the semidirect product algebra. Since there is a construction of Virasoro algebra in terms of bilinears in the Kac-Moody generators due to a connection between the energy-momentum tensors and currents, both algebras are thought of being relevant. At the classical level, the existences of hidden symmetries under consideration are independent of whether the model has the Wess-Zumino term. In fact, hidden symmetries represent the symmetries of the gauge transformation and the conformal transformation of the spectrum parameter which appear in the linearization equations of the model. The Kac-Moody and Virasoro algebras are, respectively, related with the infinitesimal Riemann-Hilbert transformations and have irrelevant structures. Moreover, unlike the quantum Wess-Zumino model, the hidden-symmetry algebras lack the central extensions and their representations are shown to be nonunitary and nonhighest weight. Therefore, these different symmetries are used to describe the different features of the chiral model at the quantum and classical levels. So far the relationships between them have not made clear yet.

II. NOTATION

We first introduce notations which are in common use in many references. Considering a super chiral field $\hat{g}(\xi,\eta,\theta_1,\theta_2)$ where ξ and η are light-cone coordinates and θ_1 and θ_2 are anticommuting coordinates, the superpotential $\hat{A}_i$ can be defined as

$$\hat{A}_i=\hat{g}^{\dagger}\hat{d}_i\hat{g} \quad (i=1,2)\ , \tag{1}$$

where $\hat{g}$ is unitary and commutes with θ_i:

$$[\theta_i,\hat{g}]=0\ . \tag{2}$$

With this notation the action of the super chiral fields is

$$S=\int d\xi\, d\eta\, d\theta_1 d\theta_2 \mathcal{L}(\xi,\eta,\theta_1,\theta_2)\ , \tag{3}$$

where

$$\mathcal{L}(\xi,\eta,\theta_1,\theta_2)=\mathrm{tr}(\hat{d}_1\hat{g}^{\dagger}\hat{d}_2\hat{g}) \tag{4}$$

and

$$\hat{d}_1=\frac{\partial}{\partial\theta_2}-i\theta_2\frac{\partial}{\partial\xi}\ , \tag{5}$$

$$\hat{d}_2=-\frac{\partial}{\partial\theta_1}+i\theta_1\frac{\partial}{\partial\eta}\ . \tag{6}$$

Thus we can derive the equation of motion from the Lagrangian density

$$\hat{d}_1\hat{A}_2-\hat{d}_2\hat{A}_1=0\ . \tag{7}$$

According to the definition of superpotentials, they satisfy the curvature-free condition

$$\hat{d}_1\hat{A}_2+\hat{d}_2\hat{A}_1+\{\hat{A}_1,\hat{A}_2\}=0 \tag{8}$$

and

$$\hat{d}_1\hat{d}_2+\hat{d}_2\hat{d}_1=0\ , \tag{9}$$

where { , } denotes the anticommutations.

It is not difficult to prove that Eq. (7) together with Eqs. (8) and (9) is the integrability condition for the following linearization equations:

$$\hat{d}_1\hat{\phi}(\lambda)=-\frac{\lambda}{1+\lambda}\hat{A}_1\hat{\phi}(\lambda)\ , \tag{10a}$$

$$\hat{d}_2\hat{\phi}(\lambda)=\frac{\lambda}{1-\lambda}\hat{A}_2\hat{\phi}(\lambda)\ , \tag{10b}$$

where λ is a real parameter and

$$[\hat{\phi},\theta_i]=0\ . \tag{11}$$

III. THE PARAMETRIC CONSERVED NOETHER CURRENTS

Now we consider the infinitesimal transformation

$$\hat{g}^{\dagger}\delta\hat{g}=-\frac{1}{2\pi i}\int_{C_{0,\lambda'}}\frac{(1-\gamma^2)^2}{\gamma^4(\gamma-\lambda')}\dot{\hat{\phi}}(\gamma)\hat{\phi}^{-1}(\gamma)d\gamma\ , \tag{12}$$

where an infinitesimal constant is omitted, and $C_{0,\lambda'}$ denotes a circle surrounding $\gamma=0,\lambda'$ in the complex γ plane, and $\hat{\phi}(\gamma)$ satisfies the linearization equations (10) with the parameter γ. The overdot stands for differentiation with respect to the parameter. We can verify that the Lagrangian density (3) is changed by a total divergence under the transformation (12).

In fact, from Eqs. (11) and (12), we find that

$$\delta\mathcal{L}=-\int \mathrm{tr}\{\hat{d}_1[\dot{\hat{\phi}}(\gamma)\hat{\phi}^{-1}(\gamma)]\hat{A}_2 -\hat{d}_2[\dot{\hat{\phi}}(\gamma)\hat{\phi}^{-1}(\gamma)]\hat{A}_1\}\ , \tag{13}$$

where

$$\int=-\frac{1}{2\pi i}\int_{C_{0,\lambda'}}d\gamma\frac{(1-\gamma^2)^2}{\gamma^4(\gamma-\lambda')}\ . \tag{14}$$

With the help of the linearization equations (10), we can obtain the identities

$$\begin{aligned}\int \mathrm{tr}\{\hat{d}_1[\dot{\hat{\phi}}(\gamma)\hat{\phi}^{-1}(\gamma)]\}\hat{A}_2 &=-\int\frac{1-\gamma}{\gamma}\mathrm{tr}(\hat{d}_1\dot{\hat{\phi}}\hat{d}_2\hat{\phi}^{-1}+\hat{d}_2\dot{\hat{\phi}}\hat{d}_1\hat{\phi}^{-1}) +\int\frac{-1}{\gamma(1-\gamma)}\hat{A}_2\\ &=-\int\frac{1-\gamma}{\gamma}\mathrm{tr}[\hat{d}_1(\dot{\hat{\phi}}\hat{d}_2\hat{\phi}^{-1})+\hat{d}_2(\dot{\hat{\phi}}\hat{d}_1\hat{\phi}^{-1})]\end{aligned} \tag{15}$$

and

$$\int \mathrm{tr}(\hat{d}_2\dot{\hat{\phi}}\hat{\phi}^{-1})\hat{A}_2=\int\frac{1+\gamma}{\gamma}\mathrm{tr}[\hat{d}_1(\dot{\hat{\phi}}d_2\hat{\phi}^{-1}) +\hat{d}_2(\dot{\hat{\phi}}\hat{d}_1\hat{\phi}^{-1})]\ , \tag{16}$$

where we used the fact that

$$\int_{C_{0,\lambda'}}\frac{1}{\gamma^{k+1}(\gamma-\lambda')}d\gamma=0 \quad \text{for } k\geq 0\ .$$

Upon substituting Eqs. (15) and (16) into Eq. (13), $\delta\mathcal{L}$ can be expressed as a total divergence:

$$\delta\mathcal{L}=\int\frac{2}{\gamma}\mathrm{tr}[\hat{d}_1(\dot{\hat{\phi}}\hat{d}_2\hat{\phi}^{-1})+\hat{d}_2(\dot{\hat{\phi}}\hat{d}_1\hat{\phi}^{-1})]\ . \tag{17}$$

This means that the action is invariant under the transformation (12). We should bear in mind that integration is only available to the parameter, not to the coordinates.

The derivation of nonlocal conserved currents from this new hidden symmetry is straightforward. Using the equation of motion, we easily get another expression for

$$\delta\mathcal{L}=-\int \mathrm{tr}[\hat{d}_1(\dot{\hat{\phi}}\hat{\phi}^{-1}\hat{A}_2)-\hat{d}_2(\dot{\hat{\phi}}\hat{\phi}^{-1}\hat{A}_1)]\ . \tag{18}$$

Combining these two expressions of the variation of the Lagrangian density, we obtain the conversation law

$$\hat{d}_1\hat{J}_2+\hat{d}_2\hat{J}_1=0\ , \tag{19}$$

where the parametric conserved currents J are defined by

$$\begin{aligned}\hat{J}_1&=\int\frac{2}{\gamma}\mathrm{tr}(\dot{\hat{\phi}}\hat{d}_1\hat{\phi}^{-1})-\int \mathrm{tr}(\dot{\hat{\phi}}\hat{\phi}^{-1}\hat{A}_1)\\ &=\int\frac{1-\gamma}{1+\gamma}\mathrm{tr}(\dot{\hat{\phi}}\hat{\phi}^{-1}\hat{A}_1)\ ,\end{aligned} \tag{20a}$$

$$\begin{aligned}\hat{J}_2&=\int\frac{2}{\gamma}\mathrm{tr}(\dot{\hat{\phi}}d_2\hat{\phi}^{-1})+\int \mathrm{tr}(\dot{\hat{\phi}}\hat{\phi}^{-1}\hat{A}_2)\\ &=-\int\frac{1+\gamma}{1-\gamma}\mathrm{tr}(\dot{\hat{\phi}}\hat{\phi}^{-1}\hat{A}_2)\ .\end{aligned} \tag{20b}$$

Expanding $\hat{J}_1$ and $\hat{J}_2$ in powers of λ' at $\lambda'=0$, we can obtain the infinite set of nonlocal conserved Noether currents as one did for the Kac-Moody symmetry.

In addition to the invariance of the action, our transformation yields the invariance of the equation of motion. The proof follows from the fact that

$$\delta\hat{A}_1=-\int\frac{1}{\gamma}\hat{d}_1(\dot{\hat{\phi}}\hat{\phi}^{-1})\ , \tag{21a}$$

$$\delta\hat{A}_2=\int\frac{1}{\gamma}\hat{d}_2(\dot{\hat{\phi}}\hat{\phi}^{-1})\ . \tag{21b}$$

In a similar way, we can also prove that our transformation makes the variation of the energy-momentum density vanish.

IV. ALGEBRAIC STRUCTURE OF THE SYMMETRY TRANSFORMATION

In this section let us consider the algebraic structure of the new infinitesimal transformation. To do it we need to expand into the form $\hat{\delta}=\hat{\delta}^{(k)}\lambda'$ and then calculate the commutator relations between $\hat{\delta}^{(k)}$ and $\hat{\delta}^{(l)}$ $(k,l\geq 0)$ to make sure what algebraic structure they are.

In calculation we deal with the explicit expression of the operator $\hat{\delta}^{(k)}$ acting on the generating function $\hat{\phi}_{(\lambda)}$. From the linearization equations, we know that corresponding to the transformation (12), $\hat{\delta}\hat{\phi}(\lambda)$ should satisfy the equations

$$d_1\hat{\delta}\hat{\phi}(\lambda)=-\frac{\lambda}{1+\lambda}\hat{\delta}\hat{A}_1\hat{\phi}(\lambda)-\frac{\lambda}{1+\lambda}\hat{A}_1\hat{\delta}\hat{\phi}(\lambda)\ , \tag{22a}$$

$$d_2\hat{\delta}\hat{\phi}(\lambda)=\frac{\lambda}{1-\lambda}\hat{\delta}\hat{A}_2\hat{\phi}(\lambda)+\frac{\lambda}{1-\lambda}\hat{A}_2\hat{\delta}\hat{\phi}(\lambda)\ . \tag{22b}$$

We are led to

$$\hat{\delta}\hat{\phi}(\lambda)=-\frac{\lambda}{2\pi i}\int_{C_{0,\lambda,\lambda'}}d\gamma\frac{(1-\gamma^2)^2}{\gamma^3(\gamma-\lambda')(\gamma-\lambda)}\times\dot{\hat{\phi}}(\gamma)\hat{\phi}(\gamma)^{-1}\hat{\phi}(\lambda) \tag{23}$$

by directly solving Eqs. (22). Then expanding it in powers of λ' at $\lambda'=0$, we get

$$\hat{\delta}^{(k)}\hat{\phi}(\lambda)=-\frac{\lambda}{2\pi i}\int_{C_{0,\lambda}}d\gamma\frac{(1-\gamma^2)^2}{\gamma^{k+4}(\gamma-\lambda)}\times\dot{\hat{\phi}}(\gamma)\hat{\phi}^{-1}(\gamma)\phi(\lambda)\ . \tag{24}$$

For the convenience of discussion, we introduce a new set of operators L_k:

$$\hat{L}_k\hat{\phi}(\lambda)=-\frac{\lambda}{2\pi i}\int_{C_{0,\lambda}}d\gamma\frac{\gamma^{-k}}{\gamma-\lambda}\dot{\hat{\phi}}(\gamma)\hat{\phi}^{-1}(\gamma)\hat{\phi}(\lambda)\ . \tag{25}$$

Using the previous results given in Refs. 5, we can easily give the commutator relations

$$[\hat{L}_k,\hat{L}_l]\hat{\phi}(\lambda)=(k-1)\hat{L}_{k+l}\hat{\phi}(\lambda)\ . \tag{26}$$

It is obvious to observe that these operators $\{L_k\}$ provide a representation of the Virasoro algebra without the central extension. However, we are interested in the commutations of the operators $\{\hat{\delta}_k\}$ rather than $\{L_k\}$ because the latter is not the symmetry transformation for the fields.

We can express our transformations in terms of the operators L:

$$\hat{\delta}^{(k)}\hat{\phi}(\lambda)=(\hat{L}_{k+4}-2\hat{L}_{k+2}+\hat{L}_k)\hat{\phi}(\lambda)\ . \tag{27}$$

By using Eq. (26), it is easy to give the commutator relations between $\hat{\delta}^{(k)}$ and $\hat{\delta}^{(l)}$:

$$[\hat{\delta}^{(k)},\hat{\delta}^{(l)}]\hat{\phi}(\lambda)=(k-l)(\hat{\delta}^{(k+l+4)}-2\hat{\delta}^{(k+l+2)}+\hat{\delta}^{(k+l)})\hat{\phi}(\lambda) \tag{28}$$

for $k,l\geq 0$. Although these commutator realizations seem to be strange and complicated, not so neat and elegant as the usual forms of the commutator relations of the Virasoro algebra, they certainly establish a representation of an infinite-dimensional Lie algebra, which is related to the conformal algebra.

It is well known that there exists the Kac-Moody symmetry for the super chiral fields which corresponds to the infinitesimal transformation

$$\hat{T}_a^{(k)}\hat{g}=-\frac{\hat{g}}{2\pi i}\int d\gamma\,\gamma^{-k-1}\hat{\phi}(\gamma)T_a\hat{\phi}^{-1}(\gamma)\ , \tag{29}$$

$$\hat{T}_a^{(k)}\hat{\phi}(\lambda)=-\frac{\lambda}{2\pi i}\int d\gamma\frac{\gamma^{-k}}{\gamma-\lambda}\hat{\phi}(\gamma)T_a\hat{\phi}(\gamma)^{-1}\hat{\phi}(\lambda)\ . \tag{30}$$

The commutator relation between $\hat{T}^{(k)}$ and $\hat{T}^{(l)}$ is calculated to be

$$[\hat{T}_a^{(k)},\hat{T}_b^{(l)}]\hat{\phi}(\lambda)=f_{ab}^c\hat{T}_c^{(k+l)}\phi(\lambda)\ . \tag{31}$$

It is also convenient to discuss the relations of the different types of the symmetry transformations. Using integration forms of both transformations we find that

$$[\hat{\delta}^{(k)},\hat{T}_b^{(l)}]\hat{\phi}(\lambda)=-l(\hat{T}_b^{(k+l+4)}-2\hat{T}_b^{(k+l+2)}+\hat{T}_b^{(k+l)})\hat{\phi}(\lambda)\ . \tag{32}$$

So we succeed in extending the hidden-symmetry algebra into the semidirect product of the infinite-dimensional loop algebra and conformal algebra.

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*Present address.

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