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Generalized Green-Schwarz anomaly cancellation mechanism

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We show that the chiral anomaly can be canceled by a *single* antisymmetric tensor if the anomaly form factorizes into the product of two invariant polynomials of curvature two-forms, R and F. This generalizes the Green-Schwarz cancellation mechanism, where the factorization is of the form $(\mathrm{Tr}R^2-\mathrm{Tr}F^2)$ times an invariant polynomial. The Yang-Mills group constraints differ from those obtained by the Green-Schwarz mechanism if the anomaly form is factorized differently and thus new anomaly-free theories emerge. This generalized mechanism works for any theories in $D=4k$ dimensions and nonsupergravity theories in $D=4k-2$. However, for supergravity-type theories in $D=4k-2$ only the original Green-Schwarz factorization works. We explicitly construct many anomaly-free theories in $D=8$ and $D=10$, using the generalized mechanism.

I. INTRODUCTION

From the anomaly point of view, the most surprising fact about superstring theories is that the new anomaly-canceling mechanism is built-in. Green and Schwarz[1] showed that the two-index antisymmetric tensor field B can cancel the chiral anomaly if the anomaly form, which is a $(D+2)$-form for a space-time of dimension D, can be factorized in such a way that it is of the form $(\mathrm{Tr}R^2-\alpha\,\mathrm{Tr}F^2)X(F,R)$. Note that $X(F,R)$ is an invariant polynomial made of curvature two-forms F for the Yang-Mills part and R for the gravitation part and the coefficient α depends on the space-time dimension.[2] It is amazing that just one B field, which is a part of a supergravity multiplet, with the right transformation property is enough to cancel the entire anomaly. Eventually, Lerche, Nilsson, Schellekens, and Warner[3] succeeded in deriving this canceling term in the string one-loop amplitude. Baulieu[4] found that the anomaly of any theory can be canceled by introducing many antisymmetric tensors of different ranks, as long as the anomaly form of that theory is made of the product of traces of R and F. [We will call the $(D+2)$-form, which is related to the anomaly, the I-form. This form can be determined from the index theorem.[5]] However, we are interested in cases where just one new field can cancel the entire anomaly of the original anomalous theory.

In this paper, we show that the generalized Green-Schwarz mechanism exists: if the anomaly form can be factorized into a product of two invariant polynomials, $X_{2l}X_{D+2-2l}$, then the anomaly can be canceled by a *single* antisymmetric tensor B_{2l-2}. The reason why we can do this is the following.

Proposition. For the factorized I-form $I_{2n+2}=X_{2l}X_{2n+2-2l}$, the anomaly can always be written as

$$\omega_{2n}^1=\omega_{2l-2}^1X_{2n+2-2l}\ ,$$

where ω_{2l-2}^1 is the anomaly for X_{2l}.

Furthermore, the field strength dH of this antisymmetric tensor is uniquely fixed as

$$H=dB_{2l-2}+\omega_{2l-1}^0\ ,$$

where $X_{2l}=d\omega_{2l-1}^0$.

In order to derive the result just mentioned, we must investigate the ambiguities of fixing the anomaly of a theory, which have not been emphasized before. Let us look at what Green and Schwarz did:[1] in $D=10$, we *demand* that the anomaly form I_{12} factorizes as

$$I_{12}=(\mathrm{Tr}R^2-\mathrm{Tr}F^2)X_8(R,F)\ . \tag{1.1}$$

They find that the anomaly from this form is given by

$$\omega_{10}^1=\tfrac{2}{3}(\omega_{3L}^0-\omega_{3Y}^0)d\omega_6^1+\tfrac{1}{3}(\omega_{2L}^1-\omega_{2Y}^1)X_8(R,F)\ , \tag{1.2}$$

where we will give the precise definition of these forms in the next section. This anomaly can be canceled by adding to the action a term of the form

$$\Delta S=BX_8(R,F)-\tfrac{2}{3}(\omega_{3L}^0-\omega_{3Y}^0)\omega_7^0 \tag{1.3}$$

if the variation of B satisfies

$$\delta B=-(\omega_{2L}^1-\omega_{2Y}^1)\ , \tag{1.4}$$

since then $\omega_{10}^1+\delta\Delta S=0$.

However, one should note that there are ambiguities in defining the various forms that appear in the system of equations for forms, since any exact form is closed, i.e., $d^2\omega=0$. In the equations above, the coefficients $\frac{1}{3}$ and $\frac{2}{3}$ change into $(\frac{1}{3}-\beta)$ and $(\frac{2}{3}+\beta)$ if we add a term of the exact form, $\beta d((\omega_{3L}^0-\omega_{3Y}^0)\omega_7^0)$, to ω_{11}^0 (Ref. 6). (The consistency condition $\delta\omega_{10}^1=-d\omega_9^2$ is still satisfied.) However, the coefficient in front of $BX_8(R,F)$ is uniquely fixed (i.e., never changes), although it seems that the anomaly is not a well-defined object. In the next section we explain how ambiguities appear in the determination of the anomaly, using the system of equations for differential forms, and, utilizing these ambiguities, derive the generalized Green-Schwarz mechanism. In Sec. III we give new anomaly-free theories in $D=8$ and 10, using the generalized mechanism. We also show that supergravitylike theories in $D=4k-2$ cannot have the new anomaly-canceling mechanism, except the original Green-Schwarz one.

II. GENERALIZED GREEN-SCHWARZ MECHANISM

In this section we prove the proposition given in the previous section. We give a rather detailed exposition of how to get the anomaly by the system of equations for forms, since we are interested in the ambiguities for the anomaly.

Let us assume that the anomaly form is factorized as the product of two invariant polynomials:

$$I_{2n+2}=X_{2l}X_{2n+2-2l}\ , \tag{2.1}$$

where the space-time dimension is $2n$ and X_{2m} ($m=2l$ and $2n+2-2l$) are made of polynomials of traces of R and F. We call X_{2m} invariant, since its variation under the gauge transformation vanishes and it is closed:

$$\delta X_{2m}=0 \quad \text{and} \quad dX_{2m}=0\ . \tag{2.2}$$

For these two invariant polynomials, we can find the system of equations[7]

$$\begin{aligned} X_{2m}&=d\omega^0_{2m-1}\ ,\\ \delta\omega^0_{2m-1}&=-d\omega^1_{2m-2}\ ,\\ \delta\omega^1_{2m-2}&=-d\omega^2_{2m-3}\ , \end{aligned} \tag{2.3}$$

where $2m=2l$ and $2n+2-2l$. Therefore, the most general Chern-Simons form ω^0_{2n+1}, which satisfies $I_{2n+2}=d\omega^0_{2n+1}$, is given by

$$\begin{aligned} \omega^0_{2n+1}&=a\,\omega^0_{2l-1}X_{2n+2-2l}+(1-a)X_{2l}\omega^0_{2n+1-2l}+dA\\ &=a\,\omega^0_{2l-1}d\omega^0_{2n+1-2l}\\ &\quad+(1-a)d\omega^0_{2l-1}\omega^0_{2n+1-2l}+dA\ , \end{aligned} \tag{2.4}$$

where a is an arbitrary coefficient and A is an arbitrary function. We keep the arbitrariness as much as possible until the end. Note the exterior derivative chain rule for forms,

$$\begin{aligned} d(\omega^\alpha_n\omega^\beta_m)&=d\omega^\alpha_n\omega^\beta_m+(-)^n\omega^\alpha_n d\omega^\beta_m\ ,\\ \delta(\omega^\alpha_n\omega^\beta_m)&=\delta\omega^\alpha_n\omega^\beta_m+(-)^\alpha\omega^\alpha_n\delta\omega^\beta_m\ , \end{aligned} \tag{2.5}$$

where n (m) and α (β) denote the ranks of forms with respect to d and δ, and the anticommutation rule for the exterior derivative and the gauge variation

$$d\delta+\delta d=0\ . \tag{2.6}$$

The gauge variation of the Chern-Simons form is given by

$$\begin{aligned} \delta\omega^0_{2n+1}=&-a\,d\omega^1_{2l-2}d\omega^0_{2n+1-2l}\\ &-(1-a)d\omega^0_{2l-1}d\omega^1_{2n-2l}-d(\delta A)\ , \end{aligned}$$

where we have used $\delta\omega^0_{2m-1}=-d\omega^1_{2m-2}$. However,

$$\begin{aligned} d\omega^1_{2l}d\omega^0_{2n+1-2l}=d[\,&b\,\omega^1_{2l-2}d\omega^0_{2n+1-2l}\\ &+(b-1)d\omega^1_{2l-2}\omega^0_{2n+1-2l}]\ , \end{aligned} \tag{2.7}$$

$$\begin{aligned} d\omega^0_{2l-1}d\omega^1_{2n-2l}=d[\,&c\,\omega^0_{2l-1}d\omega^1_{2n-2l}\\ &+(1-c)d\omega^0_{2l-1}\omega^1_{2n-2l}]\ . \end{aligned} \tag{2.8}$$

Note that because of the chain rule above in one case we have $(b-1)$ and in the other $(1-c)$. Consequently, the gauge variation of ω^0_{2n+1} becomes

$$\begin{aligned} \delta\omega^0_{2n+1}=-d[\,&ab\,\omega^1_{2l-2}d\omega^0_{2n+1-2l}\\ &+a(b-1)d\omega^1_{2l-2}\omega^0_{2n+1-2l}\\ &+(1-a)c\,\omega^0_{2l-1}d\omega^1_{2n-2l}\\ &+(1-a)(1-c)d\omega^0_{2l-1}\omega^1_{2n-2l}+\delta A]\ . \end{aligned}$$

Therefore, the anomaly ω^1_{2n}, which satisfies $\delta\omega^0_{2n+1}=-d\omega^1_{2n}$, is given by

$$\begin{aligned} \omega^1_{2n}=\,&ab\,\omega^1_{2l-2}X_{2n+2-2l}+a(b-1)d\omega^1_{2l-2}\omega^0_{2n+1-2l}\\ &+(1-a)c\,\omega^0_{2l-1}d\omega^1_{2n-2l}\\ &+(1-a)(1-c)X_{2l}\omega^1_{2n-2l}+\delta A+dB\ , \end{aligned} \tag{2.9}$$

where the ambiguity appears as arbitrary coefficients a, b, and c and arbitrary functions A and B, since $\delta^2=0$ and $d^2=0$. Using

$$\begin{aligned} d(\omega^1_{2l-2}\omega^0_{2n+1-2l})&=d\omega^1_{2l-2}\omega^0_{2n+1-2l}+\omega^1_{2l-2}d\omega^0_{2n+1-2l}\\ &=d\omega^1_{2l-2}\omega^0_{2n+1-2l}+\omega^1_{2l-2}X_{2n+2-2l}\ ,\\ d(\omega^0_{2l-1}\omega^1_{2n-2l})&=d\omega^0_{2l-1}\omega^1_{2n-2l}-\omega^0_{2l-1}d\omega^1_{2n-2l}\\ &=X_{2l-2}\omega^1_{2n-2l}-\omega^0_{2l-1}d\omega^1_{2n-2l}\ , \end{aligned} \tag{2.10}$$

we get a symmetric form of the anomaly

$$\omega^1_{2n}=a\,\omega^1_{2l-2}X_{2n+2-2l}+(1-a)X_{2l}\omega^1_{2n-2l}+\delta A+dB'\ , \tag{2.11}$$

where B changed into B'. Now we choose the arbitrary functions A and B' to be

$$\begin{aligned} A&=\beta\omega^0_{2l-1}\omega^0_{2n+1-2l}\ ,\\ B'&=\beta(\omega^1_{2l-2}\omega^0_{2n+1}-\omega^0_{2l-1}\omega^1_{2n-2l})\ . \end{aligned} \tag{2.12}$$

Then we have

$$\begin{aligned} \delta A&=\beta\,\delta\omega^0_{2l-1}\omega^0_{2n+1-2l}+\beta\omega^0_{2l-1}\delta\omega^0_{2n+1-2l}\\ &=-\beta\,d\omega^1_{2l-1}\omega^0_{2n+1-2l}-\beta\omega^0_{2l-1}d\omega^1_{2n+1-2l}\\ &=\beta\omega^1_{2l-1}X_{2n+2-2l}-\beta X_{2l}\omega^1_{2n+1-2l}\\ &\quad-\beta\,d(\omega^1_{2l-2}\omega^0_{2n+1}-\omega^0_{2l-1}\omega^1_{2n-2l})\ , \end{aligned} \tag{2.13}$$

where in the last step we used Eq. (2.10). Therefore, by choosing $\beta=1-a$, we obtain the proposition in Sec. I

$$\omega^1_{2n}=\omega^1_{2l-2}X_{2n+2-2l}\ . \tag{2.14}$$

The choice of $\beta=-a$ leads to

$$\omega^1_{2n}=X_{2l}\omega^1_{2n-2l}\ . \tag{2.15}$$

Hereafter, we will take the anomaly of the first form. Because of the two alternative choices for the anomaly, the dual form of a canceling mechanism is always there.

In physical terms, the reason for the ambiguities in the choice of the form for the anomaly can be explained as follows: the ambiguity associated with the exact form dB' comes from the fact that the actual physical anomaly is the integral of the anomaly ω^1_{2n} over the (compact) space-time manifold without boundary. The ambiguity associated with δA is the fact that we can always add (polynomial) counterterms to the action. Thus, if the anomaly can be given as the gauge variation of something, then it is not a true anomaly. Therefore, Eqs. (2.11), (2.14), and (2.15) tell us that even though the leading anomaly terms are absent by the requirement of the factorization, there still remains a nontrivial piece of the anomaly. Because ω^1_{2l-2} is the anomaly for X_{2l}, what our proposition says is that the anomaly at dimension $2n$ can be "reduced" to the anomaly at $2l-2$ if the anomaly form is factorized into $X_{2l}X_{2n+2-2l}$. This is the reason why the anomaly can be canceled in such a simple way by adding the following term to the action:

$$\Delta S = B_{2l-2}X_{2n+2-2l} , \tag{2.16}$$

with

$$\delta B_{2l-2} = -\omega^1_{2l-2} . \tag{2.17}$$

It is easy to see this form of the anomaly still satisfies the Wess-Zumino consistency condition[7] $\delta\omega^1_{2n} = -d\omega^2_{2n-1}$, since

$$\delta\omega^1_{2n} = \delta\omega^1_{2l-2}X_{2n+2-2l} = -d\omega^2_{2l-3}X_{2n+2-2l} = -d(\omega^2_{2l-3}X_{2n+2-2l}) ,$$

using the fact that $X_{2n+2-2l}$ is an invariant polynomial.

Now we want to fix the field strength (curvature) for this antisymmetric tensor B_{2l-2}. If we demand that the new kinetic term for this antisymmetric tensor does not affect the anomaly cancellation, we must have

$$\delta H = 0 . \tag{2.18}$$

For the field strength of the form, for an unknown Y,

$$H = dB_{2l-2} + Y , \tag{2.19}$$

we have

$$\delta H = \delta dB_{2l-2} + \delta Y = -d(\delta B_{2l-2}) + \delta Y = +d(\omega^1_{2l-2}) + \delta Y = -\delta\omega^0_{2l-1} + \delta Y . \tag{2.20}$$

Therefore, we find that the solution is given by

$$Y = \omega^0_{2l-1} + \delta C , \tag{2.21}$$

where C is an arbitrary function. However, H should not contain any gauge variations and thus we find a unique field strength for B_{2l-2}:

$$H = dB_{2l-2} + \omega^0_{2l-1} . \tag{2.22}$$

Note that the Bianchi identity $d^2H=0$ is satisfied, since

$$d^2H = d(d\omega^0_{2l-1}) = dX_{2l} = 0 .$$

III. MORE ANOMALY-FREE THEORIES

As we have discussed in Ref. 8, the anomaly structure is completely different in dimensions $4k$ and $4k-2$: in $D=4k$, only Yang-Mills gauged Weyl fermions contribute to the anomaly, while in $D=4k-2$ all Weyl fermions contribute to the anomaly. Note that we only deal with the field-theory limit of some theory. Therefore, we discuss theories in $D=4k$ and $4k-2$ separately. Note that, for the original Green-Schwarz factorization, solutions are given in Refs. 8 and 9.

A. $D=4k$ theories

In dimension $D=4k$, the I-form (not the anomaly itself) is given by

$$I_{4k+2} = \hat{A}(R)\mathrm{Ch}(F) , \tag{3.1}$$

where we pick the $(4k+2)$-forms from the right-hand side. The explicit form of the Dirac genus $\hat{A}(R)$ and the Chern character $\mathrm{Ch}(F)$ are now well known:[9]

$$\mathrm{Ch}(F) = \mathrm{Tr}\exp(iF) , \qquad \hat{A}(R) = \exp\left[-\sum_{k=1}\frac{B_k}{4k(2k)!}\mathrm{Tr}R^{2k}\right] , \tag{3.2}$$

where we use the convention $2\pi=1$, and B_k's are Bernoulli numbers $B_1=\frac{1}{6}$, $B_2=\frac{1}{30}$, $B_3=\frac{1}{42}$, $B_4=\frac{1}{30}, \ldots$. The most trivial cancellation is to make $I_{4k+2}=0$, which requires that all odd-order traces of F vanish. Therefore, this trivial cancellation of the anomaly always occurs when the Yang-Mills gauge group is of noncomplex type: Sp(N), SO(odd), SO($4k$), SU(2), G_2, F_4, E_7, E_8. In order to have a nontrivial case, the representation must be complex and the gauge group must be SU(N) ($N\geq 3$), SO($4k+2$), or E_6, since SU(N) has nonvanishing odd-order indices up to order N, SO($4k+2$) has a $(2k+1)$th-order index, and E_6 has fifth- and ninth-order indices.[8] All other cases are trivial even for these complex groups.

In order for the anomaly to be canceled in a nontrivial but simple way, this form must be factorized into

$$I_{4k+2} = X_{2l}X_{4k+2-2l} .$$

Thus, one of the X's is always $4m$-form and the other is $(4n+2)$-form. In particular, since $\hat{A}(R)$ contains only $4m'$-forms while $\mathrm{Ch}(F)$ contains even-order forms, the only choice for X_2 is

$$X_2 = i\,\mathrm{Tr}F . \tag{3.3}$$

However, this factorization occurs only for U(1), since semisimple groups have always a vanishing TrF. Even if a theory has a U(1) factor, this factorization will not be allowed if Tr$F=0$. Thus, it is very unlikely to have a factorization of the form X_2X_{4k}. Now we discuss the cases in $D=4$ and 8.

1. $D=4$

In this dimension, the anomaly form is given by

$$I_6 = \hat{A}(R)_4 i\,\mathrm{Tr}F + i^3\mathrm{Tr}F^3 , \tag{3.4}$$

where $\hat{A}(R)_4$ denotes the fact that a four-form is taken from the Dirac genus. Thus, the only allowed factorization is of the form X_2X_4, which demands that $\mathrm{Tr}F$ does not vanish. However, in the standard model, $SU(3)_c\times SU(2)_L\times U(1)_Y$, all possible U(1) factors have vanishing traces. Actually, in various situations we have $\mathrm{Tr}F=0$ automatically and the factorization does not work: One case exists when the U(1) operator Y is contained in a vector charge operator Q, since then

$$0=\mathrm{Tr}(Q_L+Q_R)=\mathrm{Tr}[\text{non-U(1) factors}]+\mathrm{Tr}(Y_L+Y_R)$$
$$=\mathrm{Tr}(Y_L+Y_R)\ .$$

Another case exists when the U(1) operator comes from a simple or semisimple grand unified theory (GUT). Therefore, it is very unlikely to have new anomaly-free theories in $D=4$. We hereafter disregard the factorization of X_2X_{4k} type.

2. $D=8$

Since we discard the possibility of X_2X_{4k} type, the only allowed factorization is of the form X_4X_6 in this dimension. Thus, the possible factorization is in the form

$$I_{10}=[c\hat{A}(R)_4+\alpha_0^4](r_4^2+r_0^6)\ , \tag{3.5}$$

where the superscript (subscript) denotes the Yang-Mills degrees (the gravitational degrees). By comparing with Eq. (3.1), we obtain

$$\begin{aligned}&i\hat{A}_8\mathrm{Tr}F=c\hat{A}_4r_4^2\ ,\\&i^3\hat{A}_4\mathrm{Tr}F^3=c\hat{A}_4r_0^6+r_4^2\alpha_0^4\ ,\\&i^5\mathrm{Tr}F^5=\alpha_0^4r_0^6\ .\end{aligned} \tag{3.6}$$

We must require the independence of $\mathrm{Tr}R^{2k}$, since otherwise the topology of space-time must be fixed. Thus, from the first equation, we have

$$\mathrm{Tr}F=0\ . \tag{3.7}$$

Then we have $cr_4^2=0$, which leads to two solutions. One solution is for $c\neq 0$ and $r_4^2=0$:

$$\begin{aligned}&I_{10}=(\hat{A}_4+\alpha_0^4)\mathrm{Tr}F^3\ ,\\&\mathrm{Tr}F=0\ ,\\&\mathrm{Tr}F^5=c\,\mathrm{Tr}F^3\ ,\end{aligned} \tag{3.8}$$

which is the Green-Schwarz factorization. Because $\mathrm{Tr}F^3\neq 0$, the gauge group must contain $SU(N)$ $(N\geq 3)$. The solutions belonging to this case are given in Ref. 8.

The new nontrivial solution is, for $c=0$ and $r_4^2=0$,

$$\begin{aligned}&I_{10}=\alpha_0^4r_0^6\ ,\\&\mathrm{Tr}F=\mathrm{Tr}F^3=0\ ,\\&i^5\mathrm{Tr}F^5=\alpha_0^4r_0^6\ .\end{aligned} \tag{3.9}$$

Note that in this case δB contains only the Yang-Mills Chern-Simons form, while in the first case δB contains both the Lorentz and Yang-Mills pieces. (For the case where $c=0$ but $r_4^2\neq 0$, we have $r_4^2\propto\hat{A}_4\alpha_0^2$. However, since $\alpha_0^2\propto\mathrm{Tr}F=0$, this will not happen.) Since E_6 and SO(10) have no third-order Casimir invariants, the last equation for these groups means $\mathrm{Tr}F^5=0$, which is the complete cancellation of the anomaly. [For these two groups, it is very easy to find *infinitely* many solutions for the complete cancellation of the anomaly, using *two* irreducible representations (irreps). Two solutions are given in Ref. 8.] Thus, in order to get the nontrivial factorization, the gauge group must contain $SU(N)$ $(N\geq 3)$.

Now we try to look for new nontrivially factorizing solutions for Eq. (3.9) in $SU(N)$ $(N\geq 3)$. We have[8], for a representation $\Lambda=\sum\Lambda_j$,

$$\begin{aligned}&\mathrm{Tr}F^3\left[\sum\Lambda_j\right]=\sum Q_3(\Lambda_j)\mathrm{Tr}f^3\ ,\\&\mathrm{Tr}F^5\left[\sum\Lambda_j\right]=\sum Q_5(\Lambda_j)\mathrm{Tr}f^5\\&\qquad+\sum\frac{10}{6+d_0}\left[\frac{d_0}{d(\Lambda_j)}-\frac{Q_2^0}{Q_2(\Lambda_j)}\right]\\&\qquad\times Q_2(\Lambda_j)Q_3(\Lambda_j)\mathrm{Tr}f^2\mathrm{Tr}f^3\ ,\end{aligned} \tag{3.10}$$

where d_0 and Q_2^0 are the dimension and the second-order index of the adjoint rep, $Q_m(\Lambda_j)$ the mth-order index, and f the fundamental rep (**N**). Thus, Eq. (3.9) demands that

$$\sum Q_3(\Lambda_j)=0,\quad \sum Q_5(\Lambda_j)=0\ , \tag{3.11}$$

and

$$\mathrm{Tr}F^5\left[\sum\Lambda_j\right]=\frac{10d_0}{6+d_0}\sum\frac{Q_2(\Lambda_j)Q_3(\Lambda_j)}{d(\Lambda_j)}\mathrm{Tr}f^2\mathrm{Tr}f^3\ . \tag{3.12}$$

Note that the right-hand side is in proportion not to $\mathrm{Tr}F^3$ but to $\mathrm{Tr}f^3$. This way of factorizing is in close resemblance to the anomaly-free solution for the $D=10$ nonsupergravity theory.[10] For a single irrep solution, it is extremely rare for a *complex* rep to satisfy the absence of a cubic trace, although we know at least two irreps with very high dimensions ($\sim 10^{11}$) which do so.[11] Thus we look for solutions with two irreps: $\Lambda=m_1\Lambda_1+m_2\Lambda_2$. Using the fact that

$$Q_n\left[\sum\Lambda_j\right]=\sum Q_n(\Lambda_j)\ ,$$

we must have, in order to get the nonvanishing m_j's,

$$Q_3(\Lambda_1)Q_5(\Lambda_2)-Q_3(\Lambda_2)Q_5(\Lambda_1)=0\ . \tag{3.13}$$

Because $Q_5=0$ for any irrep of SU(3) and SU(4), it is easy to find infinitely many solutions. For $SU(N)$ $(N\geq 5)$, the solutions are rare. For the notation of the irrep, we use the Young-tableau notation $[f_1,f_2,\ldots,f_n]$, where f_j denotes the number of boxes at the jth row. After searching solutions with irreps having up to six Young-tableau boxes with $N\leq 20$ for $SU(N)$,[8] we have found four solutions.

(1) For SU(10),

$$\Lambda=105\{\mathbf{210}\}+\{\mathbf{4950^*}\}\ , \tag{3.14}$$

where the first (second) irrep is of type [1,1,1,1] ([2,2,2]).
(2) For SU(16),

$$\Lambda = 116\{\mathbf{120}\} + \{\mathbf{7140^*}\} , \tag{3.15}$$

where the first (second) is of type [1,1] ([2,1,1]).
(3) For SU(17),

$$\Lambda = 8211\{\mathbf{153}\} + \{\mathbf{395353^*}\} , \tag{3.16}$$

where the first (second) is of type [2] ([4,2]).
(4) For SU(17),

$$\Lambda = 476\{\mathbf{680}\} + \{\mathbf{138720^*}\} , \tag{3.17}$$

where the first (second) is of type [1,1,1] ([2,2,2]). The asterisk denotes complex conjugation.

B. $D=4k-2$ theories

In $D=4k-2$ dimensions, not only the Yang-Mills but also the gravitational particles contribute to the anomaly and thus we discuss both the supergravity-type theories and nonsupergravity theories.

1. Supergravitylike theories

First, we look at $D=10$. For theories with a gravitino and Yang-Mills particles, the I-form can be written as[9]

$$I_{12} = \overline{R_{12}} + \overline{R_8}\mathrm{Tr}F^2 + \overline{R_4}\mathrm{Tr}F^4 + \overline{R_0}\mathrm{Tr}F^6 , \tag{3.18}$$

where

$$\begin{aligned}
\overline{R_{12}} &= \left[n - \frac{12}{B_3} \right] C_3 \mathrm{Tr}R^6 \\
&\quad + \left[n + \frac{8}{B_2} - \frac{4}{B_1} \right] C_1 C_2 \mathrm{Tr}R^2 \mathrm{Tr}R^4 \\
&\quad + \left[\frac{n}{3!} - \frac{1}{2}\frac{4}{B_1} \right] C_1^3 (\mathrm{Tr}R^2)^3 , \\
\overline{R_8} &= -\tfrac{1}{2}[C_2 \mathrm{Tr}R^4 + \tfrac{1}{2} C_1^2 (\mathrm{Tr}R^2)^2] , \\
\overline{R_4} &= \frac{1}{4!} C_1 \mathrm{Tr}R^2 , \\
\overline{R_0} &= -\frac{1}{6!} ,
\end{aligned} \tag{3.19}$$

with

$$C_k = \frac{B_k}{4k(2k)!} \quad \text{and} \quad n = n_G + l + 9 . \tag{3.20}$$

The number of Yang-Mills particles is n_G and the number of spin-$\frac{1}{2}$ singlets is l. One immediate consequence for the factorization is that

$$n - \frac{12}{B_3} = n - 504 = 0 \tag{3.21}$$

since we demand that $\mathrm{Tr}R^6$ is independent of $\mathrm{Tr}R^4$ and $\mathrm{Tr}R^2$. It is important to notice that for this value of n the coefficients for $\mathrm{Tr}R^4\mathrm{Tr}R^2$ and $(\mathrm{Tr}R^2)^3$ do not vanish in $\overline{R_{12}}$.

Now, the generalized Green-Schwarz factorization takes one of three forms:

$$I_{12} = X_2 X_{10}, \quad X_4 X_8, \quad \text{or} \quad X_6 X_6' . \tag{3.22}$$

However, for factorizations of types X_2X_{10} or X_6X_6' we must have

$$\overline{R_{12}} = 0 \tag{3.23}$$

since X_2 or X_6 must contain Yang-Mills two-forms and thus we cannot get any terms containing gravitational 12-forms. Therefore this possibility cannot be realized, since this equation says that $\mathrm{Tr}R^4$ is no longer independent of $\mathrm{Tr}R^2$, i.e., the space-time topology must be fixed. Thus we have only one possible way of factorizing I_{12}.

For the factorization of the type X_4X_{12},

$$I_{12} = (c\,\mathrm{Tr}R^2 + \alpha_0^4)(r_8^0 + r_4^4 + r_0^8) , \tag{3.24}$$

we have

$$\begin{aligned}
\overline{R_{12}} &= c\,\mathrm{Tr}R^2 r_8^0 , \\
\overline{R_8}\mathrm{Tr}F^2 &= c\,\mathrm{Tr}R^2 r_4^4 + \alpha_0^4 r_8^0 , \\
\overline{R_4}\mathrm{Tr}F^4 &= c\,\mathrm{Tr}R^2 r_0^8 + \alpha_0^4 r_4^4 , \\
\overline{R_0}\mathrm{Tr}F^6 &= \alpha_0^4 r_0^8 .
\end{aligned} \tag{3.25}$$

First of all, $c \neq 0$, since otherwise we get a nontrivial relation between $\mathrm{Tr}R^4$ and $(\mathrm{Tr}R^2)^2$ by having $\overline{R_{12}}=0$. Then the rest of the coefficients (r_8^0, r_4^4, r_0^8, α_0^4) are completely fixed in terms of c. Thus, we get *only* the famous Green-Schwarz factorization. In $D=8$, we had a choice of two factorizations by the choice of either $c=0$ or $c\neq 0$, but not here.

Therefore, we have shown that for the supergravity-type matter content in $D=10$ the Green-Schwarz factorization is the only way to cancel the anomaly, using a *single* antisymmetric tensor. It is interesting to note that the modular invariance for heterotic-type string theories requires the Green-Schwarz factorization of the anomaly as shown by Schellekens and Warner.[12]

We can generalize the argument to higher dimensions of $D=4k-2$. In the case of supergravitylike theories we have

$$I_{4k} = \overline{R_{4k}} + \overline{R_{4(k-1)}}\mathrm{Tr}F^2 + \cdots + \overline{R_0}\mathrm{Tr}F^{2k} . \tag{3.26}$$

For the factorization of the type $X_{4l-2}X_{4k+2-4l}$ $(1 \le l \le k)$, we always get the constraint

$$\overline{R_{4k}} = 0 . \tag{3.27}$$

Therefore the only allowed factorization in supergravitylike theories is of the type

$$I_{4k} = X_{4l} X_{4k-4l} \quad (1 \le l \le k-1) . \tag{3.28}$$

In the case of $D=10$ supergravitylike theories, we have only one factorization possible of the type X_4X_8 with X_4 containing $\mathrm{Tr}R^2$ (i.e., $c\neq 0$). In higher dimensions of $D=4k-2$, the pure gravitational I-form $\overline{R_{4k}}$ is of the type

$$c_0 \mathrm{Tr} R^{2k} + c_1 \mathrm{Tr} R^2 \mathrm{Tr} R^{2k-2} + c_2 \mathrm{Tr} R^4 \mathrm{Tr} R^{2k-4} + \cdots , \tag{3.29}$$

where c_j are all independent in usual cases. (For nonsupergravity theories, c_j are all in proportion to one single constant. See the next subsection.) Thus, for the factorization of $l > 2$, one gets the equation solely made of traces of space-time curvature two-forms. This means that the space-time topology must be fixed for a theory to become anomaly-free. Consequently, for supergravitylike theories in $D = 4k - 2$, only the original Green-Schwarz mechanism can cancel the anomaly.

2. *Nonsupergravity theories in D = 10*

For spin-$\frac{1}{2}$ Weyl n_G Yang-Mills particles and l singlets we have

$$I_{12} = (n_G + l)\hat{A}(R)_{12} - \tfrac{1}{2}\hat{A}(R)_8 \mathrm{Tr} F^2 + \frac{1}{4!}\hat{A}(R)_4 \mathrm{Tr} F^4 - \frac{1}{6!}\mathrm{Tr} F^6 . \tag{3.30}$$

The factorization requires that

$$n_G + l = 0 , \tag{3.31}$$

which can be satisfied easily by adjusting l, or having different chirality particles.[10] (However, in general we need a certain number of singlets. This is the price we pay for nonsupergravity theories in $D = 4k - 2$. Note that we do not need singlets to cancel anomalies in $D = 4k$.) Doing this, the pure-gravity 12-form part goes away, in contrast with the case in supergravity-type theories. Hence we can have *three* possible factorizations: $X_2 X_{10}$, $X_4 X_8$, and $X_6 X_6'$. However, for $X_2 X_{10}$ we need $\mathrm{Tr} F \neq 0$. Now, we discuss the factorization of the types $X_4 X_8$.

For the factorization $X_4 X_8$,

$$X_4 = c\, \mathrm{Tr} R^2 + \alpha_0^4, \quad X_8 = r_8^0 + r_4^4 + r_0^8 , \tag{3.32}$$

we have

$$\begin{aligned} 0 &= c\, \mathrm{Tr} R^2 r_8^0 , \\ -\tfrac{1}{2}\hat{A}_8 \mathrm{Tr} F^2 &= \alpha_0^4 r_8^0 + c\, \mathrm{Tr} R^2 r_4^4 , \\ +\frac{1}{4!}\hat{A}_4 \mathrm{Tr} F^4 &= \alpha_0^4 r_0^8 + c\, \mathrm{Tr} R^2 r_0^8 , \\ -\frac{1}{6!}\mathrm{Tr} F^6 &= \alpha_0^4 r_0^8 . \end{aligned} \tag{3.33}$$

We have three choices: ($c \neq 0$, $r_8^0 = 0$), ($c = 0$, $r_8^0 \neq 0$), or ($c = 0$, $r_8^0 = 0$). In the first case, we have the Green-Schwarz factorization and

$$\mathrm{Tr} F^2 = 0, \quad \mathrm{Tr} F^6 \propto \mathrm{Tr} F^4 , \tag{3.34}$$

which means, in group-theory indexwise,

$$Q_2 = 0, \quad Q_3 = 0, \quad Q_4 \neq 0, \quad Q_6 = 0 . \tag{3.35}$$

We gave a general solution for SO(N) for any N in Ref. 9: $\Lambda = (m \text{ spinors}) - m 2^{(N-7)/2}$ vectors. The famous SO(16)×SO(16) solution is given by[10] $\Lambda = (16,16) - (128,1) - (1,128)$.

In the second case, which is a new anomaly-free solution, we have

$$\mathrm{Tr} F^4 \propto \mathrm{Tr} F^2, \quad \mathrm{Tr} F^6 \propto \mathrm{Tr} F^2 , \tag{3.36}$$

which means

$$Q_2 \neq 0, \quad Q_3 = 0, \quad Q_4 = 0, \quad Q_6 = 0 . \tag{3.37}$$

Note that these index constraints for the Yang-Mills group are the same as those of $D = 18$ supergravity-type theories, except the traces are more stringent for supergravity-type theories. Thus all the solutions we have found for $D = 18$ in Ref. 9 apply here in $D = 10$. For example, the adjoint rep of E_8 is a solution with

$$I_{12} = \mathrm{Tr} F^2 \left[-\frac{1}{2}\hat{A}_8 + \frac{1}{4!100}\hat{A}_4 \mathrm{Tr} F^2 - \frac{1}{6!7200}(\mathrm{Tr} F^2)^2 \right] . \tag{3.38}$$

In the last case we have

$$\mathrm{Tr} F^2 = 0, \quad \mathrm{Tr} F^4 \propto \alpha_0^4, \quad \mathrm{Tr} F^6 \propto \alpha_0^4 , \tag{3.39}$$

which means

$$Q_2 = 0, \quad Q_3 = 0, \quad Q_4 = 0, \quad Q_6 = 0 . \tag{3.40}$$

These are more stringent than the last two cases and so far we could not find solutions, except the trivial vector-like theories.

For the factorization of the type $X_6 X_6'$, with $\mathrm{Tr} F = 0$, we have only $\mathrm{Tr} F^3$ available for six-forms and thus the groups are limited to only SU(N) ($N \geq 3$) and U(1). In this case, index constraints are

$$Q_2 = 0, \quad Q_3 \neq 0, \quad Q_4 = 0, \quad Q_6 = 0 . \tag{3.41}$$

We find a solution for any N of SU(N):

$$\Lambda = 2N\{\mathbf{N}\} - (\text{adjoint}) - (N^2+1)(\text{singlets}) , \tag{3.42}$$

using the fact that[13]

$$\begin{aligned} Q_m(\mathbf{N}) &= Q_3(\mathbf{N}) = 1 , \\ Q_m(\text{adjoint}) &= 2N, \quad Q_3(\text{adjoint}) = 0 \end{aligned} \tag{3.43}$$

for even m. Hence, in $D = 10$ we have new anomaly-free theories, using the generalized Green-Schwarz mechanism introduced in Sec. II.

IV. CONCLUSIONS

We have shown that the chiral anomaly can be canceled by a single antisymmetric tensor if the I-form factorizes as the product of *any* two invariant polynomials.

In the original Green-Schwarz case, one of the invariant polynomials is $\mathrm{Tr}R^2-\alpha\,\mathrm{Tr}F^2$. However, a different choice of invariant polynomials does not allow factorization for $D=4k-2$ supergravity-type theories. Only theories in $D=4k$ and non-supergravity-type theories in $D=4k-2$ have the alternative factorization available. We have found new anomaly-free solutions for them in $D=8$ and 10, using the alternative factorization.

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