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Covariant Virasoro operators and free bosonic string field theory

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A new formulation of the free bosonic gauge-covariant string field theory is presented. The action functional as well as the gauge transformation laws are constructed generalizing a procedure given by Neveu and West. A new type of mathematical object, called the "covariant Virasoro operator," is defined and used to cast the results into an extremely elegant form.

I. INTRODUCTION

The aim of this paper is to present a new formulation of the free bosonic string field theory, including both the open- and the closed-string case. It is based upon the Lorentz-covariant approach to string theory and the traditional operator formalism, working in an appropriate Fock space.[1,2]

One crucial point in gauge-covariant field theory is the necessity of introducing supplementary fields in addition to the original string field. There are a lot of approaches to this subject in the literature.[3–20] In the Becchi-Rouet-Stora-Tyutin- (BRST-)motivated versions, the supplementary fields arise from the expansion of a string functional $\psi[x(\sigma),c(\sigma),\bar{c}(\sigma)]$ with respect to the ghost variables (see, e.g., Ref. 18). In some of the older versions, however, the supplementary fields are written down as objects of their own. The starting point for this study is the formalism given by Neveu and West in Ref. 14 and the analysis of the pure string field action contained in Ref. 21. Neveu and West introduced the supplementary fields and constructed their action functional step by step for the first few mass levels. However, the way in which additional fields arise at each level, as well as the form they are coupled in the action, are not very transparent. This seems to be one of the reasons why the further development of string field theory in the operator formalism preferred different approaches (see, e.g., Refs. 15–20).

The original formulation of Neveu and West is singled out by the feature that, upon putting all supplementary fields to zero, one obtains a local, pure string field action which still describes physical string states by means of a reduced gauge invariance. This was studied in Ref. 21.

In this work, the supplementary fields are reintroduced in this picture *at once for all levels.* Because of an appropriate identity between the Virasoro operators, the structure of the supplementary fields becomes more transparent than in Ref. 14. The complete action functional for the open-string case is exhibited in Sec. II. The numerical form of the couplings is still given by rather lengthy expressions. In Sec. III the results obtained so far are reexpressed in an extremely simple notation. We introduce some operations that are constructed like covariant derivatives, the connection coefficients being extracted from certain structure constants of the Virasoro algebra, and the Virasoro operators playing the role of the partial derivatives. These objects are called "covariant Virasoro operators" and obey a simple algebra, thereby exhibiting a kind of "curvature tensor." All connection coefficients disappear in the final form of the action and the gauge transformation laws, thus giving rise to a notation one would call "covariant" in some sense.

In Sec. IV the formalism is carried over to the closed-string case without further computation by means of introducing a new index type, unifying left- and right-moving quantities. In formal structure, the results are identical to those of the open-string theory. In the summary, some concluding remarks are made. An appendix provides some additional material to the formalism of Secs. III and IV. The conventions used are those of Ref. 21 with the exception that a scalar product $\langle \psi|\psi'\rangle$ is automatically accompanied by an integration over D-dimensional Minkowski space.

II. CONSTRUCTION OF THE OPEN-STRING FIELD ACTION

We begin with the field theory of open strings. The oscillator variables are normalized as

$$[\alpha_m{}^\mu,\alpha_n{}^\nu]=\eta^{\mu\nu} m\,\delta_{m+n,0}\ , \tag{2.1}$$

$$(\alpha_n{}^\mu)^\dagger=\alpha_{-n}{}^\mu \tag{2.2}$$

$(m,n \in Z)$, with

$$\alpha_0{}^\mu = p^\mu \equiv -i\partial^\mu\ . \tag{2.3}$$

The string field (or string functional) is an element of the corresponding Fock space[1,2]

$$|\psi\rangle=[\phi(x)-i\alpha_{-1}{}^\mu A_\mu(x)+\cdots]|0\rangle\equiv\psi[x(\sigma)]\ . \tag{2.4}$$

It describes the on-shell open-string states if

$$(L_0-1)|\psi\rangle=L_n|\psi\rangle=0 \tag{2.5}$$

for all $n\geq 1$, where the Virasoro operators are defined for every $n\in Z$ by

$$L_n=\tfrac{1}{2}\sum_{p=-\infty}^{\infty} :\alpha_{n-p}{}^\mu\alpha_{p\mu}: = L_{-n}{}^\dagger \tag{2.6}$$

and satisfy the Virasoro algebra with central extension

$$[L_m,L_n]=(m-n)L_{m+n}+\tfrac{1}{12}Dm(m^2-1)\delta_{m+n,0}\ , \quad (2.7)$$

$D=\eta^\mu_\mu$ being the dimension of Minkowski space. We will generally work in $D=26$, but it is instructive to keep D arbitrary in some formulas.

The bra corresponding to $|\psi\rangle$ is taken as

$$\langle\psi|=\langle 0|[\phi^*(x)+i\alpha_1{}^\mu A^*_\mu(x)+\cdots]\ . \quad (2.8)$$

Whenever a bra and a ket are glued together to form a scalar product, one first reshuffles the α's using (2.1) and

$$\alpha_n{}^\mu|0\rangle=0 \quad (2.9)$$

for $n\geq 0$. In the resulting expression, one sets

$$\langle 0|0\rangle=1 \quad (2.10)$$

and carries out an integration over D-dimensional Minkowski space which is not indicated in the abstract notation, i.e.,

$$\langle\psi|\psi'\rangle=\int d^D(x)[\phi^*(x)\phi'(x)+A^*_\mu(x)A^{\mu'}(x)+\cdots]\ . \quad (2.11)$$

In string field theory, it is usual to impose reality conditions upon the string field $|\psi\rangle$ and the supplementary fields (to be introduced below). However, the formalism presented here will work also in the general (complex) case. Therefore, we will leave all fields unconstrained. The technical consequence is to treat bras and kets as independent variables in the variation of the action functional. One may supplement the formalism by the conditions

$$\psi^*[x(\sigma)]=\psi[x(\sigma)] \quad (2.12)$$

or

$$\psi^*[x(\sigma)]=\psi[x(\pi-\sigma)] \quad (2.13)$$

(and appropriate ones for the supplementary fields and gauge parameters) without changing any of the relevant expressions.

The aim of free string field theory is to establish (2.5) as the field equations in a special gauge. Thereby, the action functional shall be local (in the derivatives) and invariant under the gauge transformations

$$\delta|\psi\rangle=\sum_{p=1}^{\infty}L_{-p}|\tau_p\rangle \quad (2.14a)$$

or equivalently

$$\delta\langle\psi|=\sum_{p=1}^{\infty}\langle\tau_p|L_p \quad (2.14b)$$

with arbitrary Fock-space elements $|\tau_p\rangle$ as gauge parameters. As is well known, these assumptions require the introduction of supplementary fields. In Ref. 14, Neveu and West constructed an action functional up to the first few mass levels. The part of their action which involves only the string field is given by

$$S_{\psi\psi}=-\tfrac{1}{2}\langle\psi|\mathcal{K}|\psi\rangle \quad (2.15)$$

with

$$\mathcal{K}=2(L_0-1)-\sum_{n=1}^{\infty}\frac{1}{n}L_{-n}L_n\ . \quad (2.16)$$

In Ref. 21 it was argued that this action is sufficient to derive (2.5), but at the price of giving up the general transformation law (2.14). It is replaced by

$$\delta|\psi\rangle=\sum_{p=1}^{\infty}\mathcal{S}_{-p}|\chi_p\rangle\ , \quad (2.17)$$

where $\mathcal{S}_p$ are certain combinations of Virasoro operators and the $|\chi_p\rangle$ are annihilated by all L_n $(n\geq 1)$. In order to recover a gauge-covariant string field theory, i.e., an action to which (2.14) applies, we take (2.15) as our starting point. In contrast with Ref. 14, we will introduce *all* supplementary fields *at once.*

The key identity is obtained by applying a gauge transformation (2.14) to $S_{\psi\psi}$. Upon some computations using the Virasoro algebra (2.7) one finds

$$-L_p\mathcal{K}=\sum_{m,n=1}^{\infty}C^p_{mn}\mathcal{A}_{mn}+\tfrac{1}{12}(D-26)(p^2-1)L_p\ , \quad (2.18)$$

where

$$C^p_{mn}=\delta_{m+n,p}\left[1+\frac{p^2}{2mn}\right]+\delta_{pm}\frac{L_{-n}}{2n}+\delta_{pn}\frac{L_{-m}}{2m}$$
$$\equiv C^p_{nm} \quad (2.19)$$

and

$$\mathcal{A}_{mn}=L_mL_n+\frac{n(n+2m)}{n+m}L_{m+n} \quad (2.20)$$

for $p,m,n\geq 1$. Surprisingly,

$$\mathcal{A}_{mn}=\mathcal{A}_{nm}\ , \quad (2.21)$$

which determines the structure of the supplementary fields. The identity (2.18) is contained in Ref. 14 for $p=1$ and 2, with all Virasoro operators written down explicitly (and thus without exhibiting the general structure of the $\mathcal{A}_{mn}$).

Multiplying (2.18) by $\langle\tau_p|$ (or its Hermitian conjugate by $|\tau_p\rangle$), one immediately finds that in $D=26$, the combination

$$\mathcal{K}|\psi\rangle+\sum_{m,n=1}^{\infty}\mathcal{A}_{mn}{}^\dagger|\lambda_{mn}\rangle \quad (2.22)$$

is invariant under (2.14) if

$$\delta|\lambda_{mn}\rangle=\sum_{p=1}^{\infty}(C^p_{mn})^\dagger|\tau_p\rangle\ . \quad (2.23)$$

Thus, we introduce a set of supplementary fields

$$|\lambda_{mn}\rangle=|\lambda_{nm}\rangle \quad (2.24)$$

$(m,n\geq 1)$ and postulate (2.23) as their gauge transformation law. The couplings of the original string field to the supplementary fields are given by

$$S_{\psi\lambda}=-\tfrac{1}{2}\sum_{m,n=1}^{\infty}\langle\psi|\mathcal{A}_{mn}{}^\dagger|\lambda_{mn}\rangle \quad (2.25)$$

and its complex conjugate

$$S_{\lambda\psi}=-\tfrac{1}{2}\sum_{m,n=1}^{\infty}\langle\lambda_{mn}|\mathcal{A}_{mn}|\psi\rangle\ , \tag{2.26}$$

which have to be added to $S_{\psi\psi}$. Varying the ψ's in these two contributions, one obtains expressions containing only the gauge parameters and the λ's. This amounts to completing the action by adding a fourth term involving only the supplementary fields:

$$S_{\lambda\lambda}=-\tfrac{1}{2}\sum_{m,n,r,s=1}^{\infty}\langle\lambda_{mn}|\mathcal{C}_{mnrs}|\lambda_{rs}\rangle\ , \tag{2.27}$$

where the $\mathcal{C}$'s are built up locally by the Virasoro operators and obey the obvious relations

$$\mathcal{C}_{mnrs}=\mathcal{C}_{nmrs}=\mathcal{C}_{mnsr}\ , \tag{2.28}$$

$$\mathcal{C}_{mnrs}{}^{\dagger}=\mathcal{C}_{rsmn}\ , \tag{2.29}$$

the last one ensuring reality of $S_{\lambda\lambda}$. Varying the ψ's in (2.25) and (2.26) as well as the λ's in (2.27), one finds an equation for $\mathcal{C}$ which is necessary and sufficient for the complete action to be gauge invariant:

$$\mathcal{A}_{mn}L_{-p}+\sum_{r,s=1}^{\infty}\mathcal{C}_{mnrs}(C^p_{rs})^{\dagger}=0 \tag{2.30}$$

for all $m,n\geq 1$ (together with its Hermitian conjugate, which then follows automatically).

It is not hard to show that the solution to this equation is not unique. However, since we are interested in a theory which may be viewed as an ordinary free field theory for an infinite set of space-time fields $(\phi(x),A_\mu(x),\ldots)$, we restrict $\mathcal{C}$ to contain space-time derivatives ∂_μ only up to the second order. These derivatives enter $\mathcal{C}$ through the Virasoro operators by virtue of (2.3) and (2.6); hence we write down the ansatz

$$\mathcal{C}_{mnrs}=c_{mnrs}+\sum_{q=-\infty}^{\infty}c^q_{mnrs}L_q+\sum_{p,q=1}^{\infty}c^{pq}_{mnrs}L_{-p}L_q$$
$$+\sum_{p,q=1}^{\infty}(d^{pq}_{mnrs}L_pL_q+f^{pq}_{mnrs}L_{-p}L_{-q})\ , \tag{2.31}$$

where the coefficients are real numbers satisfying appropriate symmetry relations such as

$$c^q_{mnrs}=c^{-q}_{rsmn}\ , \tag{2.32}$$

$$c^{pq}_{mnrs}=c^{qp}_{rsmn}\ , \tag{2.33}$$

etc., in order to ensure (2.28) and (2.29). Note that L_0, which contains $p^\mu p_\mu$, only appears linearly in (2.31).

Upon inserting this ansatz into (2.30), one obtains a set of equations for the various coefficients. The most interesting fact is that there exists a *unique* solution. To see this, one has to arrange the Virasoro operators with respect to their natural order and to read off the coefficients, beginning with the cubic terms. For the first few values of the indices, one recovers that the couplings of the supplementary fields in $S_{\lambda\lambda}$ are identical to those given in Ref. 14. Thereby, one has to identify the fields $(\lambda_{11},\lambda_{12},\lambda_{22},\lambda_{13},\lambda_{1n},\lambda_{2n};n\geq 3)$ with the variables $(\tfrac{1}{2}\phi^{(2)},\phi^{(3)},\tfrac{1}{2}\chi^{(4)},\phi^{(4)},\phi^{(n+1)},\chi^{(n+2)})$ introduced by Neveu and West. This comparison shows the way in which the additional supplementary fields appear at each level in the procedure of Ref. 14. However, the notation of the supplementary field structure as in (2.24) seems to be more adequate because it is simpler and governs the fields at all levels.

The computation of $\mathcal{C}_{mnrs}$ to all orders from (2.30) and (2.31) is rather tedious but straightforward. The result is given by

$$d^{pq}_{mnrs}=f^{pq}_{mnrs}=0\ , \tag{2.34}$$

$$c^{pq}_{mnrs}=\left[\frac{mn}{2p}\delta_{pq}\delta_{mr}\delta_{ns}-\left[\frac{s}{2}\delta_{pr}\delta_{qm}\delta_{sn}+(r\leftrightarrow s)\right]\right]$$
$$+(m\leftrightarrow n)\ , \tag{2.35}$$

$$c^0_{mnrs}=-2mn(\delta_{mr}\delta_{ns}+\delta_{nr}\delta_{ms})\ , \tag{2.36}$$

$$c^q_{mnrs}=\left\{-\left[\frac{m^2}{2}+rs\right]\delta_{r+s,m}\delta_{qn}\right.$$
$$\left.+\left[mn\left[\frac{m}{2q}-1\right]\delta_{q+s,m}\delta_{rn}+(r\leftrightarrow s)\right]\right\}$$
$$+(m\leftrightarrow n) \tag{2.37}$$

for $q\geq 1$ [recall (2.32)]. In order to write down the c_{mnrs}, we define

$$\delta^p_{mnrs}=[\delta_{m+p,r}\delta_{s+p,n}+(m\leftrightarrow n)]+(r\leftrightarrow s) \tag{2.38}$$

and

$$c_m=\sum_{p,q=1}^{\infty}\frac{\delta_{p+q,m}}{pq}\equiv\frac{2}{m}\sum_{p=1}^{m-1}\frac{1}{p}\overset{m\to\infty}{\sim}\frac{2\ln m}{m}\ . \tag{2.39}$$

Then

$$c_{mnrs}=(\delta_{mr}\delta_{ns}+\delta_{nr}\delta_{ms})\left[\frac{mn}{4}(m^3c_m+n^3c_n)-mn\left[m^2+n^2+\frac{m+n}{2}-2\right]\right]$$
$$-\delta_{m+n,r+s}\left[\frac{mnrs}{C}+\frac{mn+rs}{2}C+\tfrac{1}{4}C^3\right]+\sum_{p=1}^{\infty}\delta^p_{mnrs}\left[\frac{mnrs}{2p}-\frac{p}{4}[C^2+2(mn+rs)]\right]\ , \tag{2.40}$$

where C stands for $m+n$, or equivalently for $r+s$, because c_{mnrs} is only nonzero if $m+n=r+s$.

Thus, we have arrived at a closed (although complicated) form of an action functional

$$S_{\rm open}=S_{\psi\psi}+S_{\psi\lambda}+S_{\lambda\psi}+S_{\lambda\lambda}\ , \tag{2.41}$$

which is invariant under (2.14) and (2.23) if $D=26$. The field equations are

$$\mathcal{K}|\psi\rangle+\sum_{m,n=1}^{\infty}A_{mn}{}^{\dagger}|\lambda_{mn}\rangle=0\ , \tag{2.42}$$

$$\mathcal{A}_{mn}|\psi\rangle+\sum_{r,s=1}^{\infty}\mathcal{C}_{mnrs}|\lambda_{rs}\rangle=0\ . \tag{2.43}$$

In order to reproduce the correct spectrum of states, it must be possible to gauge transform any solution of these equations to a solution of (2.5) together with vanishing supplementary fields. We have not proven this point in detail, but from the comparison with the work of Neveu and West, and by explicit manipulations at the lowest levels, it seems very likely to hold. However, if one likes, it still has the status of a conjecture.

There is an interesting feature connected with the gauge transformation law. It is possible (at least at the first few levels) to choose the gauge parameters $|\tau_p\rangle$ such that the $|\lambda_{mn}\rangle$ are invariant, and the transformation of $|\psi\rangle$ reduces to (2.17). In this sense, the formalism described in Ref. 21 is a partially gauge-fixed version of the present one.

It is clear that the numerical form of the action, especially Eqs. (2.35)–(2.40), is too complicated to promise fruitful application to the full (interacting) string theory. However, from the point of view of its derivation, the action seems to be rather natural. The essential input is provided by the operator $\mathcal{K}$ and the choice of the supplementary fields according to the identity (2.18). The choice of $\mathcal{K}$ is suggested by the results of Refs. 14 and 21 and is *unavoidable* as long as one envisages an action that may be partially gauge fixed to an $S_{\psi\psi}$ by just putting the supplementary fields to zero. This would not be possible, e.g., in the case of the formalism used in Ref. 15, where the pure $O(\psi^2)$ contribution is only $\langle\psi|L_0-1|\psi\rangle$. Also, the choice of the supplementary fields seems natural. One could reduce them in number using the fact that every $\mathcal{A}_{mn}$ is of the form

$$\mathcal{A}_{mn}=\sum_{p=1}^{\infty}\mathcal{F}^{p}_{mn}\mathcal{A}_{p1}=\sum_{p=1}^{3}\mathcal{G}^{p}_{mn}\mathcal{A}_{p1} \tag{2.44}$$

with appropriate operators $\mathcal{F}$ and $\mathcal{G}$ containing the L_q ($q\geq 1$). Inserting one of these identities into (2.18), one could choose supplementary fields $|\lambda_p\rangle$ with $p\geq 1$ or $p=1,2,3$ [or any mixture between these two cases: In fact, also Neveu and West offer several alternative versions of the supplementary field structure, and this is due to relations such as (2.44)]. But then the formulas for the gauge transformations and the λ terms of the action would become dramatically more complicated, e.g., involving ugly expressions for $\mathcal{F}^{p}_{mn}$. On the other hand, if one demands the gauge transformation law for the supplementary fields to contain the Virasoro operators only up to the first order, the previous choice is singled out. Thus, by means of some requirements which do not look too unnatural, the *whole* action is unique.

In this way one may get the feeling that the action functional is based upon a comparably simple structure, and that the notation used in this section is inadequate. After some trials it was indeed possible to reformulate the action in a much simpler notation, exhibiting a kind of formal "covariance" connected with the Virasoro algebra. This is presented in a more deductive way in the following section.

III. OPEN-STRING FIELD ACTION IN THE COVARIANT FORMALISM

In order to establish the framework for a more elegant notation appropriate to the action functional found in Sec. II, we first have to introduce the following index convention: All indices from the middle of the alphabet ($p,q,m,n,\ldots$) run over all positive integers. Furthermore, we will distinguish between upper and lower indices, and the Einstein summation convention for repeated indices is understood.

The string field $|\psi\rangle$ is kept as in the previous sections, whereas the supplementary fields as well as the gauge parameters are redefined as

$$|\phi^{mn}\rangle=2\sqrt{mn}\,|\lambda_{mn}\rangle=|\phi^{nm}\rangle\ , \tag{3.1}$$

$$|\xi^{m}\rangle=\sqrt{2m}\,|\tau_{m}\rangle\ . \tag{3.2}$$

The bra corresponding to $|\phi^{mn}\rangle$ is denoted by

$$\langle\phi_{mn}|=2\sqrt{mn}\,\langle\lambda_{mn}| \tag{3.3}$$

with subscript indices (similarly $\langle\xi_m|$). In order to present the formalism, we will encounter elements of the Fock space carrying indices of both positions, $|\chi^{mn\cdots}{}_{rs\cdots}\rangle$, and playing the role of "tensor fields" in what follows. For simplicity, we will call such objects "covariant string fields."

The next step is to replace the Virasoro operators by

$$l^{n}=\frac{1}{\sqrt{2n}}L_n\ , \tag{3.4}$$

$$l_{n}=\frac{1}{\sqrt{2n}}L_{-n}=(l^{n})^{\dagger}\ . \tag{3.5}$$

The Virasoro algebra then takes the form

$$\begin{aligned}
&[l^m,l^n]=\mathcal{T}_p{}^{nm}l^p\ ,\\
&[l_m,l_n]=\mathcal{T}^p{}_{mn}l_p\ ,\\
&[l^m,l_n]=\omega^m{}_{np}l^p+\omega_n{}^{mp}l_p+\Delta^m{}_nL_0-\Omega^m{}_n\ ,\\
&[L_0,l^m]=-\omega^m{}_pl^p\ ,\\
&[L_0,l_m]=\omega^p{}_ml_p\ ,
\end{aligned} \tag{3.6}$$

where, numerically

$$\omega^p{}_{mn}=\omega_p{}^{mn}=\frac{(p+m)n}{\sqrt{2pmn}}\delta_{p,m+n}\ , \tag{3.7}$$

$$\mathcal{T}_p{}^{mn}=\omega_p{}^{mn}-\omega_p{}^{nm}=\mathcal{T}^p{}_{mn}\ , \tag{3.8}$$

$$\Delta^m{}_n = \delta^m_n \ , \tag{3.9}$$

$$\Omega^m{}_n = \tfrac{1}{24} D(\Delta^m{}_n - \omega^m{}_p \omega^p{}_n) = \tfrac{1}{24} D(1-m^2)\delta^m_n \ , \tag{3.10}$$

$$\omega^m{}_n = m\delta^m_n \ . \tag{3.11}$$

From (3.3) and (3.5) we extract the general rule that complex (or Hermitian) conjugation converts the position of all indices carried by some object. This agrees with the fact that the Virasoro structure constants are real, and we may write

$$(\omega_p{}^{mn})^* = \omega^p{}_{mn} \ , \tag{3.12}$$

$$(\omega^m{}_n)^* = \omega_m{}^n \equiv \omega^n{}_m \tag{3.13}$$

as an illustration.

One may ask now, why it was necessary to modify the basis of the Virasoro Lie algebra according to (3.4) and (3.5). The reason for these definitions is the numerical identity (3.9). One may, however, extend the formalism to more general bases. This is presented in the Appendix. Applied correctly, this general formalism would not change any of the relevant expressions occurring in this paper, just as a formula in general relativity may be established in a special coordinate system but written down in a generally covariant form. Hence, the possibility of more general bases is automatically ensured.

There are some relations between the structure constants, some of them just reflecting the Jacobi identity, others characterizing (3.6) as the Virasoro algebra. Two of them, which we will need in the following, are

$$\omega^m{}_{rs}\omega_n{}^{sr} = \tfrac{13}{12}\omega^m{}_p\omega^p{}_n - \omega^m{}_n - \tfrac{1}{12}\Delta^m{}_n \tag{3.14}$$

and

$$\omega^r{}_{mq}\omega^q{}_{np} - \omega^r{}_{nq}\omega^q{}_{mp} - \mathcal{T}^q{}_{mn}\omega^r{}_{qp} = 0 \ . \tag{3.15}$$

Dealing with identities such as these, it may be necessary to evaluate the whole expression *before* taking the sum, in order not to run into things like the difference of two infinite terms. In this sense, all expressions occurring in the following are well defined.

The essential step is now to introduce a concept which formally looks like covariant derivatives, but with the partial derivatives replaced by the Virasoro operators (3.4), (3.5), and L_0. For any covariant string field with an arbitrary index structure (here we show only one index of each position), we define

$$\mathcal{D}_p|\chi^m{}_n\rangle = l_p|\chi^m{}_n\rangle + \omega^m{}_{pq}|\chi^q{}_n\rangle - \omega^q{}_{pn}|\chi^m{}_q\rangle \ , \tag{3.16}$$

$$\mathcal{D}^p|\chi^m{}_n\rangle = l^p|\chi^m{}_n\rangle + \omega_q{}^{pm}|\chi^q{}_n\rangle - \omega_n{}^{pq}|\chi^m{}_q\rangle \ . \tag{3.17}$$

Moreover, with L_0 we associate the "derivative"

$$\mathcal{D}|\chi^m{}_n\rangle = L_0|\chi^m{}_n\rangle + \omega^m{}_q|\chi^q{}_n\rangle - \omega^q{}_n|\chi^m{}_q\rangle \ . \tag{3.18}$$

The operations $\mathcal{D}_m$, $\mathcal{D}^m$, and $\mathcal{D}$ shall be called "covariant Virasoro operators" in order to indicate their formal structure as well as their connection to the ordinary Virasoro operators. According to their definition, they commute with the index contraction, e.g.,

$$\delta^n_m \mathcal{D}_p|\chi^m{}_n\rangle = l_p|\chi^m{}_m\rangle \ . \tag{3.19}$$

Moreover, they are compatible with complex conjugation in the sense that for expressions with total index saturation we have identities such as

$$\langle \chi_m{}^r|\mathcal{D}_p|\lambda^{pm}{}_r\rangle^* = \langle \lambda_{pm}{}^r|\mathcal{D}^p|\chi^m{}_r\rangle \tag{3.20}$$

and analogous formulas for $\mathcal{D}$. Note also that the contractions $\omega^m{}_{mn}$ and $\omega^m{}_{nm}$ vanish, which implies

$$\mathcal{D}_p|\xi^p\rangle = l_p|\xi^p\rangle \ . \tag{3.21}$$

The formal advantage of the covariant Virasoro operators in contrast with the ordinary ones lies in their surprisingly simple algebra. Applied to any covariant string field, we have

$$[\mathcal{D}_m,\mathcal{D}_n] = [\mathcal{D}^m,\mathcal{D}^n] = 0 \ , \tag{3.22}$$

$$[\mathcal{D}_m,\mathcal{D}] = [\mathcal{D}^m,\mathcal{D}] = 0 \ . \tag{3.23}$$

To prove these relations, one must take into account the identity (3.15). The remaining commutator needs a sort of "curvature tensor":

$$\mathcal{R}^{mr}{}_{ns} = \omega_q{}^{mr}\omega^q{}_{ns} - \omega^m{}_{nq}\omega_s{}^{qr} - \omega^r{}_{nq}\omega_s{}^{mq} - \omega^r{}_{qs}\omega_n{}^{mq} - \Delta^m{}_n\omega^r{}_s \ , \tag{3.24}$$

which satisfies

$$\mathcal{R}^{mr}{}_{ns} = \mathcal{R}^{rm}{}_{ns} = \mathcal{R}^{mr}{}_{sn} = (\mathcal{R}^{ns}{}_{mr})^* \tag{3.25}$$

and, using (3.14),

$$\mathcal{R}^{mr}{}_{rn} = -\Delta^m{}_n + \frac{26}{D}\Omega^m{}_n \ . \tag{3.26}$$

The only nonvanishing commutator is then given by

$$[\mathcal{D}^m,\mathcal{D}_n]|\chi^p{}_q\rangle = \mathcal{R}^{mp}{}_{nr}|\chi^r{}_q\rangle - \mathcal{R}^{mr}{}_{nq}|\chi^p{}_r\rangle + \left[\Delta^m{}_n(\mathcal{D}-1) - \mathcal{R}^{mr}{}_{rn} - \frac{D-26}{D}\Omega^m{}_n\right] \times|\chi^p{}_q\rangle \tag{3.27}$$

(and of course a further contraction with $\mathcal{R}$ for any additional index). Note that in $D=26$, the right-hand side involves only $\Delta^m{}_n(\mathcal{D}-1)$ and certain combinations and contractions of $\mathcal{R}^{mr}{}_{ns}$. This simple structure will be responsible for the gauge invariance of our action.

Having written down all essential features of the framework, we turn now to the reformulation of the free string field theory as given in the previous section. According to the definitions (3.2), (3.5), and the identity (3.21), the gauge transformation law (2.14a) reads

$$\delta|\psi\rangle = \mathcal{D}_p|\xi^p\rangle \ , \tag{3.28a}$$

its bra version, in a somewhat symbolic fashion, being

$$\delta\langle\psi| = \langle\xi_p|\mathcal{D}^p \ . \tag{3.28b}$$

Closer inspection of the operators C^p_{mn} defined in (2.19) shows that they contain the combination $\omega_p{}^{mn} + \omega_p{}^{nm}$. The correct gauge transformation law (2.23) turns out to be in the new language

$$\delta|\phi^{mn}\rangle = \mathcal{D}^m|\xi^n\rangle + \mathcal{D}^n|\xi^m\rangle \tag{3.29a}$$

or

$$\delta\langle \phi_{mn}| = \langle \xi_n|\mathcal{D}_m + \langle \xi_m|\mathcal{D}_n \ . \tag{3.29b}$$

This remarkable formula encourages us to translate the action functional as well. The corresponding contributions are now denoted by

$$S_{\text{open}} = S_{\psi\psi} + S_{\psi\phi} + S_{\phi\psi} + S_{\phi\phi} \ . \tag{3.30}$$

Since $|\psi\rangle$ does not carry any index, it is trivial to check that

$$S_{\psi\psi} = -\langle \psi|\mathcal{D} - 1 - \mathcal{D}_p\mathcal{D}^p|\psi\rangle \ . \tag{3.31}$$

Applying the definitions of the covariant Virasoro operators, one readily verifies that

$$S_{\psi\phi} = -\tfrac{1}{2}\langle \psi|\mathcal{D}_m\mathcal{D}_n|\phi^{mn}\rangle \ , \tag{3.32}$$

$$S_{\phi\psi} = -\tfrac{1}{2}\langle \phi_{mn}|\mathcal{D}^m\mathcal{D}^n|\psi\rangle = S_{\psi\phi}{}^* \ , \tag{3.33}$$

according to the structure of the operators $\mathcal{A}_{mn}$ which were used to define $S_{\psi\lambda}$ and $S_{\lambda\psi}$ in Sec. II. If appropriate reality conditions are imposed, these two contributions are equal.

In order to obtain the part containing only the supplementary fields, a lot of long computations is necessary. One may begin with the numerical values for c_{mnrs}^{pq} as given in (2.35) to read off the terms quadratic in the $\mathcal{D}$'s. With some skill, one may guess the correct result and then check that the bulk of expressions (2.36)–(2.40) just comes from explicitly inserting the definitions of the covariant Virasoro operators. Thereby, one makes use of numerical relations such as those given in (3.14) and (3.15). In this way, one shows that, apart from a term proportional to $D-26$, the ϕ^2 part of the action is given by

$$S_{\phi\phi} = \tfrac{1}{4}\langle \phi_{mn}|\mathcal{D} - 1 - \mathcal{D}_p\mathcal{D}^p|\phi^{mn}\rangle + \tfrac{1}{4}\langle \phi_{mn}|\mathcal{D}^n\mathcal{D}_p + \mathcal{D}_p\mathcal{D}^n|\phi^{mp}\rangle \ . \tag{3.34}$$

The operator contained in the first term formally agrees with the one appearing in (3.31) and constitutes a "covariant" extension of $\mathcal{K}$.

One may now like to check the gauge invariance of the action in this new formalism. It turns out that one just has to use the abstract form of the algebra between the $\mathcal{D}$'s, especially (3.27), without ever inserting numerical values or additional properties of $\mathcal{R}^{mr}{}_{ns}$. As an example, we compute the $\psi\xi$ contribution to δS_{open}: One has to show that

$$-(\mathcal{D} - 1 - \mathcal{D}_p\mathcal{D}^p)|\psi\rangle - \tfrac{1}{2}\mathcal{D}_m\mathcal{D}_n|\phi^{mn}\rangle \tag{3.35}$$

is gauge invariant [this is in fact nothing but $-\frac{1}{2}$ times (2.22)]. Inserting (3.28) and (3.29) and using the algebra for $D=26$, one immediately arrives at zero. One encounters the interesting relation

$$[\mathcal{D}^m, \mathcal{D}_n]|\xi^n\rangle = (\mathcal{D} - 1)|\xi^m\rangle \ , \tag{3.36}$$

where two $\mathcal{R}$ terms of the algebra (3.27) have canceled due to the contraction over n.

It should be emphasized that in all computations concerning the gauge invariance, one may avoid to encounter the structure constants $\omega^m{}_n$, $\Omega^m{}_n$, and $\omega_p{}^{mn}$ as well as the "naked" operators l_n. On a purely aesthetic level, one would call such a formalism "covariant." However, it is not clear to what extent the $\mathcal{D}$'s may in fact be interpreted as "derivative" operators and with respect to what "covariance" is meant. The idea of treating the Virasoro operators as the partial derivatives contained in a covariant derivative, has also been pursued by Gervais,[22–24] but what he does is not exactly the same thing, because the structure constants enter his formalism in a different way. Moreover, his index structure does not agree with ours. This provides also a difference to the formalism of Banks and Peskin.[12,13]

One could, of course, reformulate the definitions (3.16)–(3.18) in a more abstract language, but the essential point seems to me to be that so far the $\mathcal{D}$'s act only on the Fock space as a vector space and not upon objects quadratic or of higher order in the fields. However, derivatives need a kind of Leibniz rule. On the other hand, it is just the interacting string field theory which requires objects of higher order. Thus, a more geometric view of the formalism presented here might be provided by an extension to the interacting case. There, one encounters vertex operators (or overlap integrals) defining noncommutative products of string fields (cf. Ref. 8) which might qualify the $\mathcal{D}$'s as derivatives in a more precise way than it was done here.

The next section is devoted to the field theory of closed-bosonic strings. It turns out that the formalism may be carried over to this case without any additional computation.

IV. CLOSED-STRING FIELD ACTION IN THE COVARIANT FORMALISM

The fields of the closed strings live in the symmetric tensor product of two open-string Fock spaces[1,2] sharing the same space-time variable $x \equiv (x^\mu)$. The quantities belonging to the one copy (e.g., the one containing the left-moving modes) are underlined, the others are characterized by an overbar. The oscillator variables $\underline{\alpha}_m{}^\mu$ and $\bar{\alpha}_m{}^\mu$ build up two independent sets of Virasoro operators $\underline{L}_m$ and $\bar{L}_m$ according to (2.1) and (2.6). Two operators of different types commute with each other. The only difference to the open-string formulas is

$$\underline{\alpha}_0{}^\mu = \bar{\alpha}_0{}^\mu = \tfrac{1}{2}p^\mu \equiv -\frac{i}{2}\partial^\mu \tag{4.1}$$

instead of (2.3). The fields belonging to the closed-string Fock space are denoted by capital greek letters.

A string field $|\Psi\rangle$ describes the physical closed-string states if

$$(\underline{L}_0 - \bar{L}_0)|\Psi\rangle = 0 \ , \tag{4.2}$$

$$(\underline{L}_0 + \bar{L}_0 - 2)|\Psi\rangle = \underline{L}_n|\Psi\rangle = \bar{L}_n|\Psi\rangle = 0 \tag{4.3}$$

for all $n \geq 1$. The condition (4.2) is only an algebraic constraint which excludes some of the space-time fields contained in $|\Psi\rangle$. The general expansion of such a string field is given by

$$|\Psi\rangle=[h_{\mu\nu}(x)\underline{\alpha}_{-1}{}^{\mu}\overline{\alpha}_{-1}{}^{\nu}+\cdots]|\underline{0}\rangle|\overline{0}\rangle \quad (4.4)$$

and again the scalar product is accompanied by an integration over D-dimensional Minkowski space.

In order to find an action functional which reproduces (4.3) in a special gauge one may proceed as in the open-string case. However, a shorter way to obtain the same result is provided by the following arguments. Consider "covariant string fields" which may now carry indices of the types $\underline{m}$ and $\overline{m}$, e.g., $|\Lambda^{\underline{m}\overline{n}}{}_{\underline{p}\overline{q}}\rangle$. Then define the covariant Virasoro operators $\mathcal{D}_{\underline{m}},\mathcal{D}^{\underline{m}},\underline{\mathcal{D}}$ and $\mathcal{D}_{\overline{m}},\mathcal{D}^{\overline{m}},\overline{\mathcal{D}}$ as in the previous section, the contractions with the ω's occurring only for indices of the appropriate type, e.g.,

$$\mathcal{D}_{\underline{m}}|\Lambda^{\underline{n}\,\overline{p}}\rangle=l_{\underline{m}}|\Lambda^{\underline{n}\,\overline{p}}\rangle+\omega^{\underline{n}}{}_{\underline{mq}}|\Lambda^{\underline{q}\overline{p}}\rangle \ , \quad (4.5)$$

where

$$l_{\underline{m}}=\frac{1}{\sqrt{2m}}\underline{L}_{-m} \quad (4.6)$$

and numerically

$$\omega^{\underline{n}}{}_{\underline{mq}}=\omega^{n}{}_{mq} \ . \quad (4.7)$$

As a consequence, two $\mathcal{D}$'s of different type commute with each other, and two $\mathcal{D}$'s of the same type obey exactly the same algebra as in the open-string case (written down in terms of the appropriate index type).

Now we restrict ourselves to covariant string fields satisfying the condition

$$(\underline{\mathcal{D}}-\overline{\mathcal{D}})|\Lambda^{\underline{m}\overline{n}\cdots}{}_{\underline{p}\overline{q}\cdots}\rangle=0 \ , \quad (4.8)$$

which is purely algebraic and reduces to (4.2) for a field without indices. Since $\underline{\mathcal{D}}$ and $\overline{\mathcal{D}}$ commute with all other covariant Virasoro operators of both types, successive application of $\mathcal{D}$'s will always maintain (4.8). Thus, in effect, we may simply write

$$\mathcal{D}=\underline{\mathcal{D}}=\overline{\mathcal{D}} \quad (4.9)$$

and omit the bars. In order to unify the remaining $\mathcal{D}$'s, define a new index type M which takes the values $(\underline{m},\overline{m})$, i.e.,

$$\mathcal{D}_M=(\mathcal{D}_{\underline{m}},\mathcal{D}_{\overline{m}}) \quad (4.10)$$

and

$$\mathcal{D}^M=(\mathcal{D}^{\underline{m}},\mathcal{D}^{\overline{m}}) \ . \quad (4.11)$$

Moreover, we consider covariant string fields carrying indices of this type, e.g.,

$$|\Lambda_M\rangle=(|\Lambda_{\underline{m}}\rangle,|\Lambda_{\overline{m}}\rangle) \ ,$$

the two "components" being independent of each other and subject to (4.8). Applied to such objects, the algebra of the covariant Virasoro operators $\mathcal{D}$, $\mathcal{D}^M$, and $\mathcal{D}_M$ takes the form

$$[\mathcal{D}_M,\mathcal{D}_N]=[\mathcal{D}^M,\mathcal{D}^N]=[\mathcal{D}_M,\mathcal{D}]=[\mathcal{D}^M,\mathcal{D}]=0 \quad (4.12)$$

and, for $D=26$,

$$[\mathcal{D}^M,\mathcal{D}_N]|\Lambda^P{}_Q\rangle=\mathcal{R}^{MP}{}_{NR}|\Lambda^R{}_Q\rangle-\mathcal{R}^{MR}{}_{NQ}|\Lambda^P{}_R\rangle +[\Delta^M{}_N(\mathcal{D}-1)-\mathcal{R}^{MR}{}_{RN}]|\Lambda^P{}_Q\rangle \ , \quad (4.13)$$

where

$$\Delta^M{}_N=\delta^M_N \quad (4.14)$$

and numerically

$$\mathcal{R}^{\underline{m}\,\underline{n}}{}_{\underline{pq}}=\mathcal{R}^{\overline{m}\overline{n}}{}_{\overline{pq}}=\mathcal{R}^{mn}{}_{pq} \ , \quad (4.15)$$

all other components of $\mathcal{R}^{MN}{}_{PQ}$ vanish. (4.13) has exactly the same structure as (3.27) which was responsible for the gauge invariance of the open-string action (3.30).

Thus we may apply the whole formalism, converting all indices into capital ones. We introduce supplementary fields

$$|\Phi^{MN}\rangle=|\Phi^{NM}\rangle \quad (4.16)$$

and write down the gauge transformation laws

$$\delta|\Psi\rangle=\mathcal{D}_M|\Xi^M\rangle \ , \quad (4.17)$$

$$\delta|\Phi^{MN}\rangle=\mathcal{D}^M|\Xi^N\rangle+\mathcal{D}^N|\Xi^M\rangle \ . \quad (4.18)$$

Apart from a conventional factor 2, the action is taken over from (3.30):

$$\begin{aligned} S_{\text{closed}}=&-2\langle\Psi|\mathcal{D}-1-\mathcal{D}_P\mathcal{D}^P|\Psi\rangle-\langle\Psi|\mathcal{D}_M\mathcal{D}_N|\Phi^{MN}\rangle\\ &-\langle\Phi_{MN}|\mathcal{D}^M\mathcal{D}^N|\Psi\rangle\\ &+\tfrac{1}{2}\langle\Phi_{MN}|\mathcal{D}-1-\mathcal{D}_P\mathcal{D}^P|\Phi^{MN}\rangle\\ &+\tfrac{1}{2}\langle\Phi_{MN}|\mathcal{D}^N\mathcal{D}_P+\mathcal{D}_P\mathcal{D}^N|\Phi^{MP}\rangle \ . \end{aligned} \quad (4.19)$$

The external condition (4.8) which has to be satisfied by $|\Psi\rangle$ and the respective parts of $|\Phi^{MN}\rangle$ and $|\Xi^M\rangle$ relate the right- and left-moving modes according to the index structure carried by the fields. As an example, one has numerically [cf. (3.11)]

$$(\underline{\mathcal{D}}-\overline{\mathcal{D}})|\Xi^{\underline{m}}\rangle=(\underline{L}_0-\overline{L}_0+m)|\Xi^{\underline{m}}\rangle=0 \ , \quad (4.20)$$

which automatically implies that $\mathcal{D}_{\underline{m}}|\Xi^{\underline{m}}\rangle$ (which is a part of $\delta|\Psi\rangle$) is subject to (4.2).

The Ψ^2 part of the action agrees with the one given in Ref. 21 whereas the structure and the couplings of the supplementary fields at the lowest levels reproduce the results of Ref. 14. The field equations in the Virasoro gauge take the form

$$(\mathcal{D}-1)|\Psi\rangle=\mathcal{D}^M|\Psi\rangle=0 \ , \quad (4.21)$$

which is identical to (4.3), all $|\Phi^{MN}\rangle$ being zero.

Thus, all essential results are formally identical to the corresponding open-string expressions. This confirms the expectation that a deeper geometric understanding of the formalism presented here might be fruitful for the interacting string field theory as well.

V. SUMMARY

As we have shown, the free bosonic string field theory may be formulated in terms of the covariant string fields $|\psi\rangle, |\phi^{mn}\rangle, |\Psi\rangle, |\Phi^{MN}\rangle$, the action functionals, and the gauge transformation laws involving the covariant Virasoro operators $\mathcal{D}, \mathcal{D}_m, \mathcal{D}^m, \mathcal{D}_M, \mathcal{D}^M$. By using $|0\rangle|\underline{0}\rangle|\overline{0}\rangle$ as the vacuum, one may place all these fields into the same vector space. The supplementary fields are symmetric in their indices.

The gauge transformation laws (3.28),(3.29) and (4.17),(4.18) resemble well-known differential expressions from tensor analysis. However, this resemblance should not lead to confusion: $\delta|\psi\rangle$ looks like a divergence, although it is rather a "gradient," e.g., at the first mass level it reads

$$\delta A_\mu(x) = \partial_\mu \xi(x) \ . \tag{5.1}$$

Note that $\mathcal{D}_p|\xi^p\rangle$ is just the expression for a general spurious state. On the other hand, $\delta|\phi^{mn}\rangle$ has the form of a metric (or graviton) variation under a coordinate transformation, although it is rather a "divergence," e.g., at the zeroth mass level the $|\phi^{11}\rangle$ variation becomes

$$\delta\phi^{11}(x) = \partial_\mu \xi^\mu(x) \ . \tag{5.2}$$

One essential feature is the absence of a metric. Index positions may only be converted collectively by complex conjugation. This contrasts many other approaches with string field theory (see, e.g., Refs. 12, 13, and 22–24) where one usually has a metric of the numerical form $m\delta_{mn}$.

The main question concerning the formalism is whether or not its remarkable simplicity carries over to the interacting case. It is intended that we study this question in a forthcoming paper.

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APPENDIX

Here it is shown how the formalism presented in Sec. III may be extended to the use of more general bases of the Virasoro Lie algebra. Let

$$l^a = \sum_{p=1}^{\infty} c_{ap} L_p \quad (a \ge 1) \ , \tag{A1}$$

$$l_n = \sum_{p=1}^{\infty} d_{np} L_{-p} \quad (n \ge 1) \tag{A2}$$

with nonsingular matrices c and d. The distinction between the space spanned by all L_p $(p \ge 1)$, which annihilate physical states, and the space spanned by all L_{-p} $(p \ge 1)$, which gives rise to the spurious states, is maintained. Hence the set of operators $\{l^a, l_n, L_0, 1\}$ is the most general basis we consider. The commutation relations define the structure constants by

$$\begin{aligned} &[l^a, l^b] = \mathcal{T}_c{}^{ba} l^c \ , \\ &[l_m, l_n] = \mathcal{T}^p{}_{mn} l_p \ , \\ &[l^a, l_n] = \omega^a{}_{nb} l^b + \omega_n{}^{am} l_m + \Delta^a{}_n L_0 - \Omega^a{}_n \ , \\ &[L_0, l^a] = -\omega^a{}_b l^b \ , \\ &[L_0, l_m] = \omega^n{}_m l_n \ . \end{aligned} \tag{A3}$$

Under an arbitrary change of the basis, i.e.,

$$\tilde{l}^a = A^a{}_b l^b \ , \tag{A4}$$

$$\tilde{l}_n = B^m{}_n l_m \ , \tag{A5}$$

the structure constants transform according to their index structure. This is the reason why we have to use two different types of indices. In particular, we may summarize the transformation laws by the scheme

$$\begin{aligned} &\tilde{f}^a = A^a{}_b f^b, \quad \tilde{f}_a = (A^{-1})^b{}_a f_b \ , \\ &\tilde{f}^m = (B^{-1})^m{}_n f^n, \quad \tilde{f}_m = B^n{}_m f_n \ , \end{aligned} \tag{A6}$$

which implies, for example,

$$\tilde{\Delta}^a{}_m = A^a{}_b B^n{}_m \Delta^b{}_n \ . \tag{A7}$$

Thus, relations which are written down by means of "covariant" rules such as index contraction are "tensorial" in the sense that they are valid in *all* bases or in *none*. Choosing a special basis, e.g., $l^a = L_a$, $l_m = L_{-m}$, one may establish the following relations which are valid generally:

$$\Omega^a{}_m = \tfrac{1}{24} D(\Delta^a{}_m - \omega^a{}_b \omega^b{}_c \Delta^c{}_m) \ , \tag{A8}$$

$$\omega^m{}_n = \Delta^m{}_a \Delta^b{}_n \omega^a{}_b \ , \tag{A9}$$

$$\mathcal{T}_c{}^{ab} = \Delta^m{}_c \Delta^b{}_n \omega_m{}^{an} - (a \leftrightarrow b) \ , \tag{A10}$$

$$\mathcal{T}_a{}^{ab} = \omega^a{}_{ma} = \omega_m{}^{am} = 0 \ , \tag{A11}$$

where $\Delta^n{}_a$ is the inverse of $\Delta^a{}_n$:

$$\Delta^m{}_a \Delta^a{}_n = \delta^m_n, \quad \Delta^a{}_m \Delta^m{}_b = \delta^a_b \ . \tag{A12}$$

Relation (A9) suggests to use $\Delta^a{}_m$ and $\Delta^m{}_a$ for index conversion, according to the rules

$$f^a = \Delta^a{}_m f^m \ , \tag{A13}$$

$$f_a = \Delta^m{}_a f_m \ . \tag{A14}$$

Then, for example, (A10) takes the form

$$\mathcal{T}_c{}^{ab} = \omega_c{}^{ab} - \omega_c^{ba} \ . \tag{A15}$$

In order to convert also the index positions, we define the matrices γ and β by

$$(l^a)^\dagger = \gamma^{an} l_n \ , \tag{A16}$$

$$l_n{}^\dagger = \beta_{an} l^a \ , \tag{A17}$$

which implies

$$\beta^\dagger \gamma = 1 \ . \tag{A18}$$

Taking the Hermitian adjoint of the algebra (A3), we obtain relations for all quantities obtained so far, summarized by the rule

$$f^{ma}{}_{nb}=\gamma^{cm}(\beta^{-1})^{pa}(\gamma^{-1})_{nd}\beta_{bq}(f_{cp}{}^{dq})^* \ , \tag{A19}$$

where one may identify

$$\omega_m{}^n=\omega^n{}_m \ , \tag{A20}$$

$$\omega_a{}^b=\omega^b{}_a \tag{A21}$$

and similarly for Ω and Δ. This provides the possibility to convert *all* upper and lower indices of a certain object into each other. Note that it is not possible to raise and lower indices *individually* (e.g., to obtain ω_{amb}). One may easily verify from (A13), (A14), and (A19) that the laws for index type conversion and index position conversion are compatible. Thus one may express all quantities with respect to the index type, say, m.

The covariant fields may now carry arbitrary indices, e.g., $|\chi^{am}\rangle$. The rule (A19) describes also the transition from a bra to a ket. Consider, for example, a field $|\chi^m\rangle$. Its naive bra is denoted by ${}_{\text{naive}}\langle\chi^m|$. The correct "covariant" bra is defined according to (A19) as

$$\langle\chi_a|={}_{\text{naive}}\langle\chi^m|(\gamma^{-1})_{ma} \ . \tag{A22}$$

A possible form for a "covariant" scalar product is

$$\langle\chi_a|\Delta^a{}_m|\chi^m\rangle\equiv\langle\chi_m|\chi^m\rangle \ . \tag{A23}$$

It is clear now how to define the covariant Virasor operators $\mathcal{D}^a,\mathcal{D}^m$, etc.

The formalism of Sec. III is recovered by choosing the basis

$$l^a=\frac{1}{\sqrt{2a}}L_a \ , \tag{A24}$$

$$l_m=\frac{1}{\sqrt{2m}}L_{-m} \ . \tag{A25}$$

In this case,

$$\Delta^a{}_m=\delta^a{}_m \tag{A26}$$

and

$$\gamma=\beta=1 \ . \tag{A27}$$

Hence, it is not necessary to distinguish between the two index types, and the rule (A19) becomes the law that complex conjugation converts the index position, as in (3.5), (3.12), and (3.13). Note that in order to obtain (A26), the square roots in the definitions above are necessary. The most general basis satisfying (A26) is obtained from the above one by applying (A4) and (A5) with $AB=1$. In order to maintain also (A27), one has to choose in addition $A^\dagger A=1$. The basis (A24) and (A25) we used in the text is just the most convenient one.

All "covariant" formulas of Secs. III and IV may as well be interpreted with respect to a general basis without any further modification.

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