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More on one-loop massless amplitudes of superstring theories

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A systematic way of explicitly evaluating one-loop superstring amplitudes with $M>4$ external massless bosons is studied. In particular, we give an algorithm to derive various new theta-function identities which are useful for directly summing over spin structures of massless amplitudes. These identities make the validity of the cancellation mechanism of the dilaton tadpole divergence in higher-point amplitudes of type-I theory very transparent.

The evaluation of one-loop massless M-point amplitudes in superstring theories is an old problem. There has been a great deal of argument on massless amplitudes for various models, on various formalism for several years after Green and Schwarz discovered the divergence and the anomaly cancellation mechanism of the four-point amplitude of type-I superstring theory. It was confirmed that the divergence cancellation holds for arbitrary M, for both parity-conserving and parity-violating parts of the open sector of type-I superstring theory.[1,2] Also the ultraviolet finiteness of the amplitudes of type-II and heterotic models is assured by their modular invariance. Still, few concrete calculations of amplitudes have been done except for $M\le 4$ and some examples for $M=5$ (Refs. 3 and 4). This is partly because calculations are believed to be algebraically tedious, and somewhat opaque, even though there is no conceptual difficulty. The purpose of this paper is to diminish the cumbersomeness in calculating massless amplitudes for $M>4$, and to give the systematic way of it, especially in summing over spin structures and evaluating kinematical factors. For simplicity, we concentrate on considering the parity-conserving part of type-I theory. Some example calculations are done in the form in which it is straightforward to see that the dilaton tadpole cancellation mechanism holds. Throughout the paper we adopt the old covariant method since it is more convenient than the light-cone gauge method in many respects for evaluating loops with $M>4$ external bosons, as was already noted in Ref. 5. Schematically the whole of the M-point amplitude (corresponding to the planar diagram) has the form

$$g^M\int\prod_{I=1}^{M}[d\nu_I]\frac{dq}{q}\frac{1}{(\ln q)^5}\prod_{I<J}(\psi_{IJ})^{k_I\cdot k_J}A_M\ , \tag{1}$$

$$A_M=\sum_{i=0}^{M}(\text{kinematical factor bosonic correlation}\times\text{fermionic part } G_i)\ , \tag{2}$$

$$\psi=-2\pi i\exp\left[\frac{2\pi(\nu_I-\nu_J)^2}{\tau}\right]\theta_1(\nu_I-\nu_J|\tau)/\dot\theta_1(0|\tau)\ . \tag{3}$$

In (2) the "fermionic part" G_i means a spin structure sum over products of partition functions and fermion correlation functions, which have the form

$$G_i(x_0,x_1,\dots,x_{i-1})=\sum_{\nu=1,2,3}(-1)^\nu\left[\frac{\theta_{\nu+1}(0|\tau)}{\dot\theta_1(0|\tau)}\right]^4\times\prod_{j=0}^{i-1}P_\nu(x_j)\ , \tag{4}$$

where

$$P_\nu(x_j)=\frac{\dot\theta_1(0|\tau)\theta_{\nu+1}(x_j|\tau)}{\theta_{\nu+1}(0|\tau)\theta_1(x_j|\tau)} \tag{5}$$

is the fermion one-loop correlation function and x_j is the difference of inserting points of vertices. In the path-integral formalism the factor $[\theta_{\nu+1}(0|\tau)/\dot\theta_1(0|\tau)]^4$ comes from both fermionic (chiral Dirac and superghost) and bosonic determinants.[6] In massless amplitudes we only have to consider the case

$$\sum_{j=0}^{i-1}x_j=0\ . \tag{6}$$

It is a technical problem to evaluate the right-hand side of Eq. (4), but it considerably simplifies matters to see some important properties of amplitude such as the validity of the divergence cancellation mechanism, the pole structures of amplitudes, etc., as we will see later. As is well known, for $i=0, 2, 3, 4$,

$$\begin{aligned}&G_0\equiv\sum_{\nu=1,2,3}(-1)^\nu\left[\frac{\theta_{\nu+1}(0|\tau)}{\dot\theta_1(0|\tau)}\right]^4=0\ ,\\&G_2(x_0,x_1)=0\ ,\\&G_3(x_0,x_1,x_2)=0\ ,\\&G_4(x_0,x_1,x_2,x_3)=1\ .\end{aligned} \tag{7}$$

In the above, the first three identities are part of the contents of the so-called nonrenormalization theorem. All these identities are proved by Riemann theta formula or Jacobi fundamental formula, but neither of these two formulas is convenient for evaluating G_i for $i>5$. In Appendix B, we sketch a rather different way of summing over the right-hand side of (4), which is useful to obtain G_i consecutively for large i. The results for $i=5,6,7$ are

$$G_5(x_0,x_1,x_2,x_3,x_4)=\sum_{j=0}^{4}\frac{\partial}{\partial x_j}\ln\theta_1(x_j)\ , \tag{8}$$

$$G_6(x_0,x_1,x_2,x_3,x_4,x_5)=\frac{1}{2}\left[\left[\sum_{j=0}^{5}\frac{\partial}{\partial x_j}\ln\theta_1(x_j)\right]^2+\left[\sum_{j=0}^{5}\frac{\partial^2}{\partial x_j^2}\ln\theta_1(x_j)+12\eta_1\right]\right]\ , \tag{9}$$

$$G_7(x_0,x_1,x_2,x_3,x_4,x_5,x_6)=\frac{1}{24}\left[\sum_{j=0}^{6}\frac{\partial}{\partial x_j}\ln\theta_1(x_j)\right]^3-\frac{1}{24}\left[\sum_{j=0}^{6}\frac{\partial^3}{\partial x_j^3}\ln\theta_1(x_j)\right]$$
$$+\frac{1}{8}\left[\sum_{j=0}^{6}\frac{\partial}{\partial x_j}\ln\theta_1(x_j)\right]\left[\sum_{j=0}^{6}\frac{\partial^2}{\partial x_j^2}\ln\theta_1(x_j)+14\eta_1\right]\ . \tag{10}$$

It is straightforward to calculate G_i for $i\geq 8$.

The results (8)–(10) indicate the existence of massless poles in amplitudes with more than five external bosons (see the notes at the end of Appendix B). After averaging the fermion correlation functions over various boundary conditions, G_i can be written entirely in terms of $\theta_1(\nu|\tau)$, or in terms of constants. On the other hand, in Eq. (2), the bosonic correlation function which correctly includes contributions from zero modes has the form

$$\langle X(x_i)X(x_j)\rangle=\ln\psi(x_i-x_j)\ . \tag{11}$$

Hence one-loop amplitudes can be written in terms of the odd theta function which is unique on a torus.

Let us begin evaluating M-point amplitudes with reviewing the $M=4$ case for later convenience. The amplitude does not include the bosonic correlation function and has the form [in the following we concentrate on obtaining A_M in (2)]

$$A_4=\zeta_1^{i_1}k_1^{j_1}\zeta_2^{i_2}k_2^{j_2}\zeta_3^{i_3}k_3^{j_3}\zeta_4^{i_4}k_4^{j_4}$$
$$\times\{[t_4(1,2,3,4)G_4(\nu_1-\nu_2,\nu_2-\nu_3,\nu_3-\nu_4,\nu_4-\nu_1)+\text{combinations}]_{3\ \text{terms}}$$
$$+[t_2(1,2)t_2(3,4)G_4(\nu_1-\nu_2,\nu_2-\nu_1,\nu_3-\nu_4,\nu_4-\nu_3)+\text{combinations}]_{3\ \text{terms}}\}\ , \tag{12}$$

where

$$t_2(1,2)=[\delta^{j_1i_2}\delta^{j_2i_1}+\text{antisymmetrized terms on }(i_1\leftrightarrow j_1),\ (i_2\leftrightarrow j_2)]_{2\ \text{terms}}$$
$$=\delta^{j_1i_2}\delta^{j_2i_1}-\delta^{i_1i_2}\delta^{j_1j_2}\ , \tag{13}$$

$$t_4(1,2,3,4)=[\delta^{j_1i_2}\delta^{j_2i_3}\delta^{j_3i_4}\delta^{j_4i^1}+\text{antisymmetrized terms on }(i_1\leftrightarrow j_1),\ (i_2\leftrightarrow j_2),\ (i_3\leftrightarrow j_3),\ (i_4\leftrightarrow j_4)]_{16\ \text{terms}}\ . \tag{14}$$

Here "combination" means different possible ways of symmetrizing the interchange of a pair of pairs (i_1j_1), (i_2j_2), (i_3j_3), (i_4j_4) [see Fig. 1(a)]. All G_4 in (12) equal one by (7) and the integrand of A_4 becomes the well-known form with the following kinematical factor:

$$t^{ijklmnpq}\zeta_1^ik_1^j\zeta_2^kk_2^l\zeta_3^mk_3^n\zeta_4^pk_4^q\ , \tag{15}$$

where $t^{ijklmnpq}$ is given in Ref. 7. In covariant calculation tensors t_2,t_4 represent the way of contractions of fermionic fields in vertex operators to give one-loop correlation functions. Antisymmetry under the exchange $i_a\leftrightarrow j_a$ is due to the exchange of fermion fields in a vertex operator: $k\cdot\psi(x_a)\zeta\cdot\psi(x_a)\leftrightarrow\zeta\cdot\psi(x_a)k\cdot\psi(x_a)$. Accordingly what we need for the evaluation of kinematical factors of higher point amplitudes is the straightforward extension of t_2,t_4 (and similar tensors corresponding to contractions of boson fields which we will omit):

$$t_m(1,2,\ldots,m)=[\delta^{j_1i_2}\delta^{j_2i_3}\cdots\delta^{j_{m-1}i_m}\delta^{j_mi_1}+\text{antisymmetrized terms on }(i_1\leftrightarrow j_1),\ (i_2\leftrightarrow j_2),\ldots,(i_m\leftrightarrow j_m)]_{2^m\ \text{terms}}\ . \tag{16}$$

These observations are simple, but useful for discussing the divergence cancellation mechanism. In the light-cone gauge calculation there appear terms with an ϵ tensor in kinematical factors of $M\geq 5$ point amplitudes—and, more importantly, terms with $\epsilon\cdot\delta$, $\epsilon\cdot\delta\cdot\delta$, . . . tensors which are not obvious to be canceled and give potentially divergent terms.[3] In the covariant method such terms do not appear from the beginning.

Considering the above, the five-point massless amplitude is easily found to be

$$A_5=\xi_1^{i_1}k_1^{j_1}\xi_2^{i_2}k_2^{j_2}\xi_3^{i_3}k_3^{j_3}\xi_4^{i_4}k_4^{j_4}\xi_5^{i_5}k_5^{j_5}$$
$$\times\{[t_5(1,2,3,4,5)G_5(\nu_1-\nu_2,\ \nu_2-\nu_3,\ \nu_3-\nu_4,\ \nu_4-\nu_5,\ \nu_5-\nu_1)+\text{combinations}]$$
$$+[t_3(1,2,3)t_2(4,5)G_5(\nu_1-\nu_2,\ \nu_2-\nu_3,\ \nu_3-\nu_1,\ \nu_4-\nu_5,\ \nu_5-\nu_4)+\text{combinations}]\}$$
$$+\left[A_4(1,2,3,4)\left[\sum_{i=1}^{4}\xi_5\cdot k_i\frac{\partial}{\partial\nu_{5i}}\ln\psi(\nu_5-\nu_i)\right]+\text{combinations}\right]. \tag{17}$$

In Fig. 1(b) ⊙ represents the contraction of the bosonic field, that is $\sum_{i(\neq j)}\xi_i\langle\partial X(i)e^{ik_j\cdot X(j)}\rangle$ in this case corresponding to the last line of (17).

Similarly, six- and seven-point massless amplitudes of the parity-conserving part of type-I superstring theory are

$$A_6=\xi_1^{i_1}k_1^{j_1}\xi_2^{i_2}k_2^{j_2}\xi_3^{i_3}k_3^{j_3}\xi_4^{i_4}k_4^{j_4}\xi_5^{i_5}k_5^{j_5}\xi_6^{i_6}k_6^{j_6}$$
$$\times\{[t_6(1,2,3,4,5,6)G_6(\nu_1-\nu_2,\ \nu_2-\nu_3,\ \nu_3-\nu_4,\ \nu_4-\nu_5,\nu_5-\nu_6,\nu_6-\nu_1)+\text{combinations}]$$
$$+[t_3(1,2,3)t_3(4,5,6)G_6(\nu_1-\nu_2,\ \nu_2-\nu_3,\ \nu_3-\nu_1,\ \nu_4-\nu_5,\ \nu_5-\nu_6,\ \nu_6-\nu_4)+\text{combinations}]$$
$$+[t_4(1,2,3,4)t_2(5,6)G_6(\nu_1-\nu_2,\ \nu_2-\nu_3,\ \nu_3-\nu_4,\ \nu_4-\nu_1,\ \nu_5-\nu_6,\ \nu_6-\nu_5)+\text{combinations}]$$
$$+[t_2(1,2)t_2(3,4)t_2(5,6)G_6(\nu_1-\nu_2,\ \nu_2-\nu_1,\ \nu_3-\nu_4,\ \nu_4-\nu_3,\ \nu_5-\nu_6,\ \nu_6-\nu_5)+\text{combinations}]\}$$
$$+\left[A_5(1,2,3,4,5)\left[\sum_{i=1}^{5}\xi_6\cdot k_i\frac{\partial}{\partial\nu_{6i}}\ln\psi(\nu_6-\nu_i)\right]+\text{combinations}\right]$$
$$+\left[A_4(1,2,3,4)\left[\xi_5\cdot\xi_6\frac{\partial^2}{\partial\nu_{56}^2}\ln\psi(\nu_5-\nu_6)\right]+\text{combinations}\right], \tag{18}$$

$$A_7=\xi_1^{i_1}k_1^{j_1}\xi_2^{i_2}k_2^{j_2}\xi_3^{i_3}k_3^{j_3}\xi_4^{i_4}k_4^{j_4}\xi_5^{i_5}k_5^{j_5}\xi_6^{i_6}k_6^{j_6}\xi_7^{i_7}k_7^{j_7}$$
$$\times\{[t_7(1,2,3,4,5,6,7)G_7(\nu_1-\nu_2,\ \nu_2-\nu_3,\ \nu_3-\nu_4,\ \nu_4-\nu_5,\ \nu_5-\nu_6,\ \nu_6-\nu_7,\ \nu_7-\nu_1)+\text{combinations}]$$
$$+[t_5(1,2,3,4,5)t_2(6,7)G_7(\nu_1-\nu_2,\ \nu_2-\nu_3,\ \nu_3-\nu_4,\ \nu_4-\nu_5,\ \nu_5-\nu_1,\ \nu_6-\nu_7,\ \nu_7-\nu_6)+\text{combinations}]$$
$$+[t_3(1,2,3)t_4(4,5,6,7)G_7(\nu_1-\nu_2,\ \nu_2-\nu_3,\ \nu_3-\nu_1,\ \nu_4-\nu_5,\ \nu_5-\nu_6,\ \nu_6-\nu_7,\ \nu_7-\nu_4)+\text{combinations}]$$
$$+[t_2(1,2,3)t_2(4,5)t_2(6,7)G_7(\nu_1-\nu_2,\ \nu_2-\nu_3,\ \nu_3-\nu_1,\ \nu_4-\nu_5,\ \nu_5-\nu_4,\ \nu_6-\nu_7,\ \nu_7-\nu_6)+\text{combinations}]\}$$
$$+\left[A_6(1,2,3,4,5,6)\left[\sum_{i=1}^{6}\xi_7\cdot k_1\frac{\partial}{\partial\nu_{7i}}\ln\psi(\nu_7-\nu_i)\right]+\text{combinations}\right]$$
$$+\left[A_5(1,2,3,4,5)\left[\xi_6\cdot\xi_7\frac{\partial^2}{\partial\nu_{67}^2}\ln\psi(\nu_6-\nu_7)\right]+\text{combinations}\right]. \tag{19}$$

Equation (18) with the result of the parity-violating part[8] completes the one-loop hexagon amplitude for type-I theory after including contributions from Möbius and nonplanar parts. Similarly we can construct higher-point amplitudes. Using t_m tensors will also facilitate the calculation of string loop corrections to the effective field theory.

Next we discuss the dilation tadpole divergence cancellation mechanism. Under Jacobi transformation, A_M transforms as

$$A_M\to(\ln q)^{-M+4}A_M\left[\nu\to\frac{\nu}{\tau},\tau\to-\frac{1}{\tau}\right].$$

Also the factor $(\ln q)^{M+1}$ comes from the Jacobian; hence, the $(\ln q)^{-5}$ factor in the integrand cancels out. Note that G_i is a "modulator form" itself, while the bosonic correlation function is not. Extending the argument given in Appendix B, we can prove that A_M can be entirely written by the modular forms g_2,g_3 and some derivatives of $\ln\theta_1(\nu|\tau)$ with respect to ν for arbitrary M. All these three have the same form structure as functions of q: constant+polynomial of q^2. Therefore the dilation tadpole cancellation mechanism is always applicable.

Up till now we have discussed type-I theory. In calculating closed-superstring amplitudes, we should take care of the mixing between right and left movers. Take, for example, type-II theory. The whole of the amplitude has the form

$$\left[\frac{\kappa}{4\pi}\right]^M\int\prod_{I=1}^{M}d^2z_Id^2w|w|^{-2}\times\left[\frac{-4\pi}{\ln|w|}\right]^5\prod_{I<j}(\chi_{IJ})^{k_I\cdot k_J/2}A_M^C. \tag{20}$$

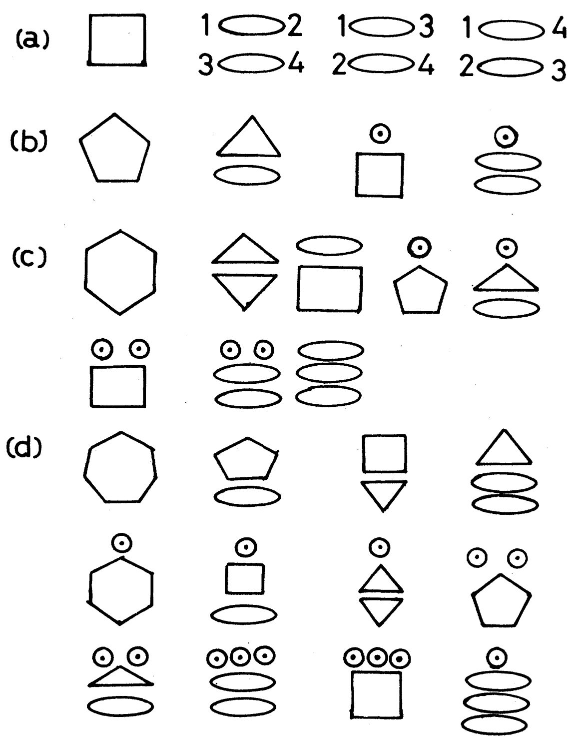

FIG. 1. Schematic explanation of contractions of vertices.

Owing to the momentum constraint $\alpha_0=\tilde{\alpha}_0$, we find after momentum integration that the bosonic fields of right movers X_R and left movers X_L have nonzero correlations on a torus:

$$\langle \partial_- X_R^\mu \partial_+ X_L^\nu \rangle = \left[\frac{1}{4\pi\,\mathrm{Im}\tau} \right] \delta^{\mu\nu} . \tag{21}$$

Therefore, contractions between right- and left-handed polarization tensors occur in evaluating $M>4$ amplitudes, and A_M^C takes a rather complicated form compared to the open-string amplitudes. However, the method of calculations is entirely similar to those of open-string amplitudes (one can consider right and left movers of the "fermionic part" separately), and we do not present the results here.

We expect that there would be an appropriate formula between higher-genus theta functions with various boundary conditions, which makes some discussions on superstring amplitudes more transparent.

Note added in proof. After submitting this work I noticed Ref. 11 in which the advantage of the covariant method has already been emphasized with concrete calculations.

APPENDIX A

We set up in this appendix the notation and some formulas of elliptic functions used in the text and Appendix B.

We first define the σ function[9] as

$$\sigma(z)=\sigma(z|2\omega_1,2\omega_3) = z \prod_{m,n}{}' \left[1-\frac{z}{\Omega_{m,n}} \right] \exp\left[\frac{z}{\Omega_{m,n}} + \frac{z^2}{2\Omega_{m,n}^2} \right] , \tag{A1}$$

where $\Omega_{m,n}=2m\omega_1+2n\omega_3$. In the text and Appendix B we choose $2\omega_1=1$, $2\omega_3=\tau$ as is often denoted in string theories. We also define the following functions:

$$\begin{aligned} &\zeta(z)=\frac{d\ln\sigma(z)}{dz}, \quad P(z)=-\frac{d\zeta(z)}{dz} , \\ &\omega_2\equiv-(\omega_1+\omega_3), \quad P(\omega_\nu)=e_\nu , \\ &\zeta(\omega_\nu)=\eta_\nu \quad (\nu=1,2,3) , \\ &g_2=60\sum_{m,n}\Omega_{m,n}^{-4}, \quad g_3=140\sum_{m,n}\Omega_{m,n}^{-6} , \\ &P_\nu(z)=[P(z)-e_\nu]^{1/2} , \end{aligned} \tag{A2}$$

$$P^{(2n)}(z)=\frac{d^{2n}P(z)}{dz^{2n}} = \text{polynomial of } P(z) \text{ of order } n+1 , \tag{A3}$$

$$P^{(2n+1)}(z)=P'(z)\times[\text{polynomial of } P(z) \text{ of order } n] , \tag{A4}$$

$$P'(\omega_\nu)=0 , \tag{A5}$$

$$\{P'(z)\}^2=4[P(z)]^3-g_2P(z)-g_3 . \tag{A6}$$

The σ function and P_ν function are related to the θ function as

$$\begin{aligned} &\sigma(2\omega_1 z)=2\omega_1\exp(2\eta_1\omega_1 z^2)\frac{\theta_1(z)}{\dot{\theta}_1(0)} , \\ &\eta_1=\zeta(\omega_1)=-\frac{1}{6}\frac{\dddot{\theta}_1(0)}{\dot{\theta}_1(0)} , \end{aligned} \tag{A7}$$

$$P_\nu(2\omega_1 z)=\frac{1}{2\omega_1}\frac{\dot{\theta}_1(0)}{\theta_{\nu+1}(0)}\frac{\theta_{\nu+1}(z)}{\theta_1(z)} . \tag{A8}$$

The following formulas are utilized:

$$\begin{aligned} &e_2-e_1=-\frac{\pi^2\theta_4^4(0)}{4\omega_1^2}, \quad e_3-e_2=-\frac{\pi^2\theta_2^4(0)}{4\omega_1^2} , \\ &e_1-e_3=\frac{\pi^2\theta_3^4(0)}{4\omega_1^2} , \end{aligned} \tag{A9}$$

$$\begin{aligned} &e_1+e_2+e_3=0, \quad e_1e_2+e_2e_3+e_3e_1=-\frac{g_2}{4} , \\ &e_1e_2e_3=\frac{g_3}{4} , \end{aligned} \tag{A10}$$

and

$$\begin{vmatrix} 1 & P(x_0) & P'(x_0) & \cdots & P^{(i-2)}(x_0) \\ 1 & P(x_1) & P'(x_1) & \cdots & P^{(i-2)}(x_1) \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & P(x_{i-1}) & P'(x_{i-1}) & \cdots & P^{(i-2)}(x_{i-1}) \end{vmatrix} = (-1)^{(i-1)(i-2)/2} 1!2! \cdots (i-1)! \frac{\sigma(x_0+\cdots+x_{i-1}) \prod_{\lambda<\mu} \sigma(x_\lambda - x_\mu)}{\prod_{k=0}^{i-1} \sigma^i(x_k)} . \tag{A11}$$

From a modern point of view, one can regard (A11) as a corollary of Fay's formula.[10]

APPENDIX B

In this appendix we give a method to sum over the right-hand side of Eq. (4) and express G_i entirely in terms of $\theta_1(\nu|\tau)$ or constants. First note that one-loop fermion correlation functions are Szegö kernels on a torus P_ν defined in in Appendix A. We can rewrite G_i as

$$G_i = \frac{1}{\Delta^{1/2}} [(e_1-e_3)R_2 + (e_3-e_2)R_1 + (e_2-e_1)R_3] , \tag{B1}$$

where

$$\Delta^{1/2} = (e_2-e_1)(e_3-e_2)(e_1-e_3) , \tag{B2}$$

$$R_\nu = \prod_{j=0}^{i-1} [P(x_j)-e_\nu]^{1/2} = \prod_{j=0}^{i-1} P_\nu(x_j) . \tag{B3}$$

When we set $x_0+x_1+\ldots+x_{i-1}=0$ as in (6), we see that the right-hand side of (A11) vanishes, so there exist $a_0, a_1, \ldots, a_{i-2}$ which satisfy a set of equations

$$1 + a_0 P(z) + a_1 P'(z) + a_2 P''(z) + \cdots + a_{i-2} P^{(i-2)}(z) = 0 \tag{B4}$$

for $z = x_0, x_1, \ldots, x_{i-1}$. By Cramer's formula, we explicitly have (omitting x_0)

$$a_n = (-1)^n \frac{\begin{vmatrix} 1 & P(x_1) & \cdots & P^{(n-1)}(x_1) & P^{(n+1)}(x_1) & \cdots & P^{(i-2)}(x_1) \\ 1 & P(x_2) & \cdots & P^{(n-1)}(x_2) & P^{(n+1)}(x_2) & \cdots & P^{(i-2)}(x_2) \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 1 & P(x_{i-1}) & \cdots & P^{(n-1)}(x_{i-1}) & P^{(n+1)}(x_{i-1}) & \cdots & P^{(i-2)}(x_{i-1}) \end{vmatrix}}{\begin{vmatrix} P(x_1) & P'(x_1) & \cdots & P^{(i-2)}(x_1) \\ \vdots & \vdots & & \vdots \\ P(x_{i-1}) & P'(x_{i-1}) & \cdots & P^{(i-2)}(x_{i-1}) \end{vmatrix}} . \tag{B5}$$

Now we consider a polynomial $f(x)$ defined by

$$f(P(z)) = [1 + a_0 P(z) + a_2 P''(z) + a_4 P^{(4)}(z) + \cdots + a_{i-2} P^{(i-2)}(z)]^2 - [a_1 P'(z) + a_3 P'''(z) + \cdots + a_{i-3} P^{(i-3)}(z)]^2 . \tag{B6}$$

We assume for awhile that i is even. Considering (A3) and (A4), $f(P(z))$ can be written as

$$f(P(z)) = [h(P(z))]^2 - [P'(z)]^2 \times [\text{polynomial of } P(z)] , \tag{B7}$$

$$h(P(z)) = 1 + a_0 P(z) + a_2 P''(z) + \cdots + a_{i-2} P^{(i-2)}(z) . \tag{B8}$$

By the (B4) equation, $f(x)=0$ has i roots at $x=P(x_0)$, $P(x_1), \ldots, P(x_{i-1})$; hence, $f(x)$ has the form

$$f(x) = a_{i-2}^2 c^2 [x - P(x_0)][x - P(x_1)] \cdots [x - P(x_{i-1})] , \tag{B9}$$

where c is a numerical constant. So, R_ν defined in (B3) can be rewritten as

$$R_\nu = \frac{[f(e_\nu)]^{1/2}}{c a_{i-2}} = \frac{[f(P(\omega_\nu))]^{1/2}}{c a_{i-2}} = \frac{h(e_\nu)}{c a_{i-2}} . \tag{B10}$$

Here we used (A5). Substituting (B10) and (A3) into (B1), G_i can be written as

$$\sum_{m,n} (\text{const})_{m,n} \frac{a_n}{a_{i-2}} \times \frac{(e_2-e_3)e_1^m + (e_3-e_1)e_2^m + (e_1-e_2)e_3^m}{(e_2-e_1)(e_3-e_2)(e_1-e_3)} . \tag{B11}$$

Only minor modifications are needed in the above argument in the case that i is odd.

In (B11), the e_j-dependent factor can be written as a symmetric polynomial of e_1, e_2, e_3 so that it can be represented by the modular form g_2, g_3 when we consider (A10). G_i depends on x_k through the ratio a_n/a_{i-2} which is a functional of $P(x)$. One can prove that

$$\frac{a_{i-3}}{a_{i-2}} = \frac{(i-1)!}{(i-2)!} \sum \zeta(x_j) \,, \tag{B12}$$

$$\frac{a_{i-4}}{a_{i-2}} = \frac{(i-1)!}{(i-3)!} \frac{1}{2} \left[\left[\sum \zeta(x_j) \right]^2 - \left[\sum P(x_j) \right] \right] , \tag{B13}$$

$$\frac{a_{i-5}}{a_{i-2}} = \frac{(i-1)!}{(i-4)!} \frac{1}{6} \left[\left[\sum \zeta(x_j) \right]^3 - \left[\sum \frac{\partial}{\partial x_j} P(x_j) \right] - 3 \left[\sum \zeta(x_j) \right] \left[\sum P(x_j) \right] \right] , \tag{B14}$$

$$\frac{a_{i-6}}{a_{i-2}} = \frac{(i-1)!}{(i-5)!} \frac{1}{4!} \left[\left[\sum \zeta(x_j) \right]^4 - \left[\sum \frac{\partial^2}{\partial x_j^2} P(x_j) \right] - 4 \left[\sum \zeta(x_j) \right] \left[\sum \frac{\partial}{\partial x_j} P(x_j) \right] \right.$$
$$\left. + 3 \left[\sum \partial x_j P(x_j) \right]^2 - 6 \left[\sum \zeta(x_j) \right]^2 \left[\sum P(x_j) \right] \right] , \tag{B15}$$

where we again defined $x_0 \equiv -(x_1 + \cdots + x_{i-1})$ and all summations are taken over from $j=0$ to $i-1$. For example, a_{i-3}/a_{i-2} has poles at $x_1=0$ and $x_1 = -(x_2 + \cdots + x_{i-1})$ as a function of x_1. Examining residues and periodicity we have

$$\frac{a_{i-3}}{a_{i-2}} = (i-1)[\zeta(x_1) + \cdots + \zeta(x_{i-1})$$
$$- \zeta(x_1 + x_2 + \cdots + x_{i-1})]$$

which coincides with the right-hand side of (B12). Note that a_{i-3}/a_{i-2} is elliptic, though each $\zeta(x)$ is not. From the form of poles and residues, a_{i-4}/a_{i-2} is found to be essentially equivalent to the elliptic function which has poles $1/x_i x_j$ $(i \neq j)$. The right-hand side of (B13) has exactly such a structure. The second term of (B13), $[\sum P(x_j)]$, is added so as to subtract poles such as $1/x_i^2$. Similarly a_{i-5}/a_{i-2} is found to be essentially the elliptic function which has poles $1/x_i x_j x_k$ $(i \neq j,\, j \neq k,\, k \neq i)$, and last two terms of the right-hand side of (B14) are added to $[\sum \zeta(x_j)]^3$ so as to subtract poles such as $1/x_i^3$, $1/x_i^2 x_j$. In this way we can easily express a_{i-s}/a_{i-2} $(s > 6)$ in terms of $\zeta(x)$ and its derivatives, though the resulting form would be long. Expressing $\zeta(x)$ and $P(x)$ in terms of $\ln\theta_1(\nu|\tau)$ using (A2) and (A7) we have (8)–(10) in the text. These structures of poles have clear physical meaning. In calculating string amplitudes x is the difference of inserting points of vertex operators, so that structures of a_n/a_{i-2} indicate the existence of massless poles which should appear in factorizing $M \geq 5$ point amplitudes.

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