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Parametrized scalar field on $\mathbb{R}\times S^1$: Dynamical pictures, spacetime diffeomorphisms, and conformal isometries

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As a preparation for a consistent Dirac constraint quantization and an anomaly-free operator representation of the spacetime diffeomorphism algebra, we develop a covariant canonical theory of a parametrized massless scalar field propagating on a cylindrical Minkowskian spacetime. We show how to pass from the Schrödinger picture to the Heisenberg picture on the extended phase space of this parametrized system, how to construct a pair of canonical representations of L Diff$M$ by using these pictures, and how to relate canonical representations of conformal isometries to those of L Diff$M$. We reconstruct the spacetime structures needed for operator ordering from the geometric data on a single embedding. We keep the formalism covariant under all relevant transformations.

I. INTRODUCTION

If a field propagates on a Minkowskian background, most people do not consider measuring it on curved hypersurfaces or studying its propagation from one such hypersurface to another. Yet this is exactly what one must do in curved spacetimes and what one should do even in flat spacetimes if one wants to use the well-understood concept of field dynamics as a testing ground for more complicated generally covariant dynamical systems such as strings or Einstein's theory of gravitation. The framework for discussing such questions was constructed by Dirac.[1] By a process known as parametrization, spacetime diffeomorphisms are adjoined to the field variables and the spacetime action is thereby made generally covariant. The phase space of the theory is correspondingly extended by addition of spacelike embeddings (parametrized hypersurfaces) and their conjugate momenta. The covariance of the spacetime action leads to constraints among the embedding data and the field data. The Hamiltonian of the system becomes a linear combination of these constraints. The Poisson brackets of the constraints have a characteristic structure known as the Dirac "algebra"[1]; this structure is common to all covariant systems (in particular, to parametrized field theories, strings, membranes, and Einstein's theory of gravitation with or without sources). It ensures that the dynamical evolution does not depend on the choice of foliation connecting the initial embedding with the final embedding.[2]

While the constraints follow from the invariance of the spacetime action under spacetime diffeomorphisms, the Dirac "algebra" does not resemble the Lie algebra L Diff$M$ of the diffeomorphism group Diff$M$. In parametrized theories, however, it is possible to replace the Dirac constraints by an equivalent set of constraints ("diffeomorphism Hamiltonians") whose Poisson algebra is homomorphic to L Diff$M$.[3] This homomorphism reflects the spacetime covariance of the canonical version of the parametrized theory.

In the Dirac constraint quantization, one attempts to turn the constraints into operators and impose them as restrictions on physical states. Our main goal is to show that this can be done without violating the spacetime covariance of the canonical theory and the associated foliation independence of the field evolution. Our task amounts to finding a factor ordering of the diffeomorphism Hamiltonians which makes their operator commutator algebra homomorphic to L Diff$M$. The difficulties in attaining this goal are obvious. It is well known that commutators of the energy-momentum-tensor operators acquire Schwinger terms[4] which, because the constraints contain projections of the energy-momentum tensor, can be expected to creep into the Dirac "algebra" or the L Diff$M$ algebra of diffeomorphism Hamiltonians as anomalies. [This seems to happen in string theory where it led people to side step the question of whether there exists a consistent Dirac constraint quantization by resorting to alternative, not manifestly equivalent, techniques, such as gauge (parametrization) fixing,[5] introduction of the ghost fields,[6] and/or the use of the Becchi-Rouet-Stora-Tyutin (BRST) formalism.[7]] We shall show that in a parametrized (free) field theory these difficulties can be surmounted by constructing a covariant (but embedding-dependent) factor ordering which cancels all anomalies in the operator representation of L Diff$M$.

Our method is applicable to an arbitrary free field theory, but we shall develop it here for a massless scalar field propagating on a flat cylindrical ($\mathbb{R}\times S^1$) Minkowskian background. There are many reasons for choosing this particular system: (i) The solution of both the classical and the quantum problems is simple and explicit. (ii) The system has an infinitely dimensional symmetry group, namely, the group C of conformal isometries. There is an interesting interplay between this group and the group of spacetime diffeomorphisms. (iii) The formalism is closely connected with that of the bosonic string model. In fact, there exists a nonlocal canonical transformation[8] in the extended phase space which takes the constraints of the parametrized theory of D-2 independent scalar fields propagating on a flat background into those of a string in a D-dimensional target space. (iv) The Schwinger terms are finite. (v) Products of the field operators at the same point can be well defined.[9] (vi) The closing of space into a circle S^1 makes the system analogous to a closed string and also gets rid of the infrared problem.

We shall reach our goal, a consistent Dirac constraint quantization of the parametrized massless field theory, in the following paper.[10] Here we shall develop and emphasize several features of the classical formalism without which, we believe, one cannot properly understand how the constraint quantization proceeds. (1) The possibility of reconstructing privileged spacetime structures from the geometric data on a single embedding; (2) the relation between the Schrödinger and Heisenberg pictures for parametrized systems; (3) the way in which representations of L Diff$M$ are constructed in these pictures; (4) the interplay between representation of L Diff$M$ and representations of a possible symmetry group; and (5) the manifest covariance of the formalism. Let us comment on these topics one by one.

(1) The normal factor ordering of a free field theory depends on such geometric spacetime structures as the Killing vector fields. In parametrized field theories one is asked to write the factor-ordering prescription entirely in terms of the data on a given embedding. One must thus be able to reconstruct the appropriate spacetime structures from the hypersurface data. This problem is solved in Sec. II.

(2) Finding a consistent factor ordering of the constraints in the Schrödinger picture is not easy: it amounts to an infinite number of ordering decisions, one on each embedding, which must be made so that L Diff$M$ is represented by the diffeomorphism Hamiltonian operators without any anomaly. It is much easier to perform the factor ordering once and for all in the Heisenberg picture, and then find the embedding-dependent reordering term which compensates for the anomaly. This requires, however, a clarification of what is meant by the Heisenberg picture in a parametrized theory.

For ordinary physical systems the Heinsenberg picture variables are the initial canonical data. After parametrization, time itself (represented by embeddings for a field system) becomes one of the canonical variables. It thus seems that in the Heisenberg picture on the extended phase space of a parametrized system, time, together with the rest of the canonical variables, should be replaced by its initial value; if so, dynamics comes to a standstill. This accounts for a widespread feeling that in parametrized theories one should work in the Schrödinger picture and that the transition to the Heisenberg picture is not feasible.

We have argued elsewhere[11] that time plays a different role in parametrized formalism than the dynamical variables and we gave the canonical transformation from the Schrödinger canonical variables to the Heisenberg canonical variables on the extended phase space. This transformation leaves the Heisenberg time equal to the Schrödinger time (the Heisenberg embedding equal to the Schrödinger embedding), but it turns the suitably chosen constraints into the Heisenberg embedding momenta. This is the main advantage of the Heisenberg picture over the Schrödinger picture; it follows that in the Dirac constraint quantization the algebra of the constraints is trivially taken over to the quantum theory without any anomaly. The anomaly appears in the algebra of the Heisenberg operators which evolve the fields, but because these operators are not subject to constraints, the anomaly still leads to consistent evolution. Moreover, one can redefine the Heisenberg evolutions operators so that one cancels the anomaly, and then pass to the Schrödinger picture.

Section III is an explicit and detailed account of how the transition from the Schrödinger picture to the Heisenberg picture and back works in classical dynamics of our system of interest (the massless scalar field on $\mathbb{R}\times S^1$). We are using a statistical description of an ensemble of such fields by means of the distribution function on the extended phase space. This sets the stage for the Dirac constraint quantization of the field in the following paper.

(3) The Lie algebra L Diff$M$ allows two representations by the Poisson algebra of suitable dynamical variables on the extended phase space: one based on the Heisenberg embedding momenta and the other on the Schrödinger embedding momenta. Depending on what picture is used, one of these representations is trivially realized and the other ensures the independence of the dynamical evolution of the choice of foliation. The generators of the two representations evolve, respectively, the states (distribution functions) or the Schrödinger field variables. The diffeomorphism group thus plays a dynamical role in the theory. This topic is developed in Secs. III and IV. The main theme of the following paper is to turn these two representations into operator representations while at the same time avoiding the anomalies.

(4) The group of conformal isometries C may be considered as a subgroup of the dynamical group Diff$M$, but it is more important as the symmetry group of the system. In this way of looking at things, the representation of L Diff$M$ is based on the Heinsenberg embedding momenta and the representation of LC on the Schrödinger embedding momenta. One can show that the mixed Poisson brackets between two such generators, one of L Diff$M$ and another of LC, (weakly) vanishes. As a consequence, the generators of LC are constants of motion and they simultaneously generate canonical transformations which leave the diffemorphism Hamiltonians (weakly) invariant. This relation between the two groups and their representations is described in Sec. IV. In the following paper we show that this relationship is disturbed by the anomaly. This does not affect, however, the consistency of the Dirac quantization.

(5) So far we have talked about the active actions of Diff$M$ and C in the framework of the parametrized canonical formalism. The formalism also uses coordinates in M and on Σ. One should check that all formulas are covariant under passive transformations of these coordinates and thus express genuine relationships between geometric objects. We take special care both in this paper and in the following paper always to exhibit this covariance manifestly.

II. SPACELIKE EMBEDDINGS IN $M=\mathbb{R}\times S^1$

A. A cylindrical Minkowskian spacetime

A cylindrical Minkowskian spacetime ($M=\mathbb{R}\times S^1,\mathbf{G}$) can be foliated by maximal hypersurfaces (circles); we choose the unit of length such that their circumference equals 2π. We orient the circles "counterclockwise" and denote the unit vector field tangent to them by $\mathbf{s}$. The congruence of geodesics orthogonal to the foliation of maximal circles is timelike; we denote the timelike future-oriented unit vector field tangent to it by $\mathbf{t}$. We thus obtain an oriented Minkowskian basis $\mathbf{t},\mathbf{s}$:

$$t^\alpha t_\alpha=-1,\quad s^\alpha s_\alpha=1,\quad \text{and } t^\alpha s_\alpha=0 . \tag{2.1}$$

We cover M by a single patch of Minkowskian coordinates $T(X)$, $Z(X)$,

$$X^\alpha=(T,Z),\quad T\in(-\infty,\infty),\quad Z\in[-\pi,\pi] , \tag{2.2}$$

such that

$$t_\alpha=-T_{,\alpha}\quad\text{and } s_\alpha=Z_{,\alpha} \tag{2.3}$$

and identify the points $(T,Z=-\pi)$ with the points $(T,Z=\pi)$.

The null vectors

$$\mathbf{e}_{(\pm)}=\mathbf{t}\pm\mathbf{s} \tag{2.4}$$

are tangent to the counterclockwise (clockwise) propagating future-oriented null rays. They are normalized by

$$e_{(\pm)\alpha}e_{(\pm)}{}^\alpha=0,\quad e_{(+)}{}^\alpha e_{(-)\alpha}=-2 , \tag{2.5}$$

and related to the affine parameters

$$T^\pm=T\pm Z \tag{2.6}$$

by the equations

$$e_{(\pm)\alpha}=-T^{\mp}{}_{,\alpha} ,\quad e_{(\pm)}=2\partial_\pm:=2\partial/\partial T^\pm . \tag{2.7}$$

The advanced and retarded time variables $T^\pm$ have the range $T^+\in[T^--2\pi,T^-+2\pi]$, $T^-\in(-\infty,\infty)$; the point $T^+-T^-=-2\pi$ must be identified with the point $T^+-T^-=2\pi$. Both of them increase in the future direction, but each time the null ray crosses the "dateline" $Z=\pm\pi$, T^+, or T^- jumps back by 2π. In the null coordinates $X^\alpha=(T^-,T^+)$,

$$\begin{aligned} &e_{(-)}{}^\alpha=(2,0),\quad e_{(+)}{}^\alpha=(0,2) ,\\ &e_{(-)\alpha}=(0,-1),\quad e_{(+)\alpha}=(-1,0) ,\\ &G_{+-}=-\tfrac{1}{2},\quad G^{+-}=-2 ,\\ &G_{++}=G_{--}=0=G^{++}=G^{--} . \end{aligned} \tag{2.8}$$

Most calculations involving a massless scalar field are simplest when done in the null coordinates. The results can then be cast into a form which is valid in arbitrary coordinates by using the geometrically defined vector fields $\mathbf{e}_{(\pm)}$.

The local frame $\mathbf{t},\mathbf{s}$ of the privileged Minkowskian observer can be boosted into the local frame

$$\begin{aligned} \bar{\mathbf{t}}&=\mathbf{t}\cosh\eta+\mathbf{s}\sinh\eta ,\\ \bar{\mathbf{s}}&=\mathbf{t}\sinh\eta+\mathbf{s}\cosh\eta , \end{aligned} \tag{2.9}$$

of an observer moving with the speed $v=\tanh\eta$ with respect to the privileged observer. The boosted null frame

$$\bar{\mathbf{e}}_{(\pm)}=\bar{\mathbf{t}}\pm\bar{\mathbf{s}} \tag{2.10}$$

is related to the privileged null frame (2.4) by scaling factors:

$$\bar{\mathbf{e}}_{(\pm)}=\lambda^{\pm1}e_{(\pm)},\quad\text{with } \lambda=\cosh\eta+\sinh\eta=\left[\frac{1+v}{1-v}\right]^{1/2} . \tag{2.11}$$

B. Spacelike embeddings in $M=\mathbb{R}\times S^1$

In hypersurface dynamics one addresses the question of how does a field evolve from one Cauchy hypersurface to another. Good Cauchy hypersurfaces are only those globally spacelike hypersurfaces in M which are homeomorphic to the $T=0$ circle. We represent them by (orientation-preserving) embeddings

$$X\in \mathrm{Emb}(\Sigma,M):\Sigma\to M=\mathbb{R}\times S^1 \tag{2.12}$$

of an oriented circle Σ into the spacetime M.

We label the points x of Σ by a single coordinate $x^1\in[-\pi,\pi]$ and identify the points $x^1=-\pi$ and $x^1=\pi$. The embedding is then described by the equation

$$X^\alpha=X^\alpha(x^1) . \tag{2.13}$$

(It is not necessary that the join $x^1=\pm\pi$ be mapped into the dateline $Z=\pm\pi$, but in practical calculations it is convenient to assume this.) The spatial metric induced on Σ,

$$\begin{aligned} &g_{11}(x;X]:=G_{\alpha\beta}(X(x))X^\alpha_1(x)X^\beta_1(x) ,\\ &X^\alpha_1:=\partial X^\alpha(x^1)/\partial x^1 , \end{aligned} \tag{2.14}$$

must be spacelike, $g_{11}>0$. The notation $(x;X]$ which we shall use extensively in the following emphasizes that a quantity is a function of $x\in\Sigma$ and simultaneously a functional of $X\in\mathrm{Emb}(\Sigma,M)$. In the null coordinates

$$g_{11}(x^1)=-T^-{}_{,1}(x^1)T^+{}_{,1}(x^1) . \tag{2.15}$$

The metrics $G^{\alpha\beta},G_{\alpha\beta}$ and g^{11},g_{11} are used for the raising and lowering of the corresponding indices.

The projector X^α_1 transforms as a covector and g_{11} as a covariant tensor under DiffΣ. In a one-dimensional space, densities of weight $w>0$ transform as covariant tensors of rank w and densities of weight $w<0$ as contravariant tensors of rank w. This fact provides a simple bookkeeping for the density character of a quantity by the number and the position of the attached index 1.

Besides g_{11}, we shall introduce two other metrics $g^{(\pm)}{}_{11}$ associated with the "distances" $ds^{(\pm)}\equiv\pm dT^\pm$ on Σ given by the differences of the null coordinates $T^\pm$ which the embedding intersects:

$$g^{(\pm)}{}_{11}=g^{(\pm)}{}_1g^{(\pm)}{}_1\quad\text{with } g^{(\pm)}{}_1:=\pm T^\pm{}_{,1} . \tag{2.16}$$

Similarly,

$$g_{11}=g_1 g_1 \quad \text{with } g_1 := (\det g_{11})^{1/2} \; . \tag{2.17}$$

Each metric induces an appropriate covariant derivative, D_1 and $D^{(\pm)}{}_1$. The Christoffel affine connection $\Gamma^1{}_{11}$ is the arithmetic mean of the affine connections $\Gamma^{(\pm)1}{}_{11}$:

$$\Gamma^{(\pm)1}{}_{11}=(\ln(\pm T^{\pm}{}_{,1}))_{,1}=\frac{T^{\pm}{}_{,11}}{T^{\pm}{}_{,1}} \; , \tag{2.18}$$

$$\Gamma^1{}_{11}=(\ln g_1)_{,1}=\tfrac{1}{2}(\Gamma^{(+)1}{}_{11}+\Gamma^{(-)1}{}_{11}) \; . \tag{2.19}$$

The Levi-Civita pseudotensor $\epsilon_{\alpha\beta}$ on M enables us to introduce the normal

$$n_{1\alpha}:=\epsilon_{\alpha\beta}X^{\beta}_{1}, \quad n_{1\alpha}n_1{}^{\alpha}=-g_{11} \; , \tag{2.20}$$

which is a covector, and the unit normal $n_\alpha := g^1 n_{1\alpha}$. The natural basis (n^α_1, X^α_1) and cobasis (n^1_α, X^1_α) associated with the hypersurface have the standard orthonormality and completeness properties. In the null coordinates

$$\begin{aligned} &X^{\pm}_1=T^{\pm}{}_{,1}, \quad X^1_{\pm}=\tfrac{1}{2}(T^{\pm}{}_{,1})^{-1} \; , \\ &n^{\pm}_1=\pm T^{\pm}{}_{,1}, \quad n^1_{\pm}=\mp\tfrac{1}{2}(T^{\pm}{}_{,1})^{-1} \; . \end{aligned} \tag{2.21}$$

The change of the hypersurface basis along the embedding is related to the extrinsic curvature:

$$K_{11}:=X^{\beta}_1(\nabla_\beta X^\alpha_1)n_\alpha \quad \text{and } K:=K^1_1 \; . \tag{2.22}$$

From definition (2.22) we get

$$g_1 K=\frac{1}{2}\left[\frac{T^{-}{}_{,11}}{T^{-}{}_{,1}}-\frac{T^{+}{}_{,11}}{T^{+}{}_{,1}}\right]=\tfrac{1}{2}(\Gamma^{(-)1}{}_{11}-\Gamma^{(+)1}{}_{11}) \; . \tag{2.23}$$

The difference of two affine connections is a (mixed) tensor which, in a one-dimensional space, can be identified with a covector. From Eq. (2.23) we see that the difference of $\Gamma^{(\pm)1}{}_{11}$ yields the mean extrinsic curvature. Equations (2.19) and (2.23) then enable us to write[12]

$$\Gamma^{(\pm)1}{}_{11}=\Gamma^1{}_{11}\mp g_1 K \; . \tag{2.24}$$

In a given spacetime, the extrinsic curvature of a hypersurface is limited by the Gauss-Codazzi equations and its change from one hypersurface to another by the Mainardi equation (Appendix A). In two-dimensional spacetime, the Gauss-Codazzi equations are identically satisfied and $K(x)$ on Σ can thus have an arbitrary distribution. The Mainardi equation implies that the normal change of $g_1 K$ on a flat background depends only on the intrinsic metric of Σ,

$$(\delta g_1(x)K(x)/\delta X^\alpha(x'))n^\alpha(x')=-g_1(x)\Delta\delta_{1'}(x,x') \; . \tag{2.25}$$

Here $\Delta\equiv g^{11}D_1 D_1$ is the covariant Laplacian on Σ. The δ function, as indicated by the attached index $1'$, is a scalar in the first argument and a scalar density in the second argument.

C. The spacetime basis and the hypersurface basis

From the hypersurface basis vectors $(n^\alpha, g^1 X^\alpha_1)$ we can form the null vectors

$$n_{(\pm)}{}^{\alpha}:=n^\alpha \pm g^1 X^\alpha_1 \; . \tag{2.26}$$

In the null coordinates

$$\begin{aligned} &n_{(-)}{}^{\alpha}_{1}=(-2T^{-}{}_{,1},0), \quad n_{(+)}{}^{\alpha}_{1}=(0,2T^{+}{}_{,1}) \; , \\ &n_{(-)}{}^{1}_{\alpha}=(0,-(T^{+}{}_{,1})^{-1}), \quad n_{(+)}{}^{1}_{\alpha}=((T^{-}{}_{,1})^{-1},0) \; . \end{aligned} \tag{2.27}$$

Because the scalar product $n_{(+)}{}^{\alpha}n_{(-)\alpha}=-2$ is the same as the product (2.5), $n_{(\pm)}{}^{\alpha}$ can differ from $e_{(\pm)}{}^{\alpha}$ at most by a position-dependent boost (2.11):

$$\begin{aligned} &n_{(\pm)}{}^{\alpha}(x;X]=\Lambda^{\pm 1}(x;X]e_{(\pm)}{}^{\alpha}(X(x)) \; , \\ &\Lambda=-\tfrac{1}{2}n_{(+)}{}^{\alpha}e_{(-)\alpha} \; . \end{aligned} \tag{2.28}$$

Here, $\Lambda(x)$ is related to the speed $v(x)$ of the hypersurface observer $\mathbf{n}(x)$ with respect to the privileged inertial observer $\mathbf{t}(x)$ as in Eq. (2.11). We shall call Λ the slope factor because it determines the slope of the hypersurface $X(x)$ with respect to the foliation of maximal hypersurfaces $T=\text{const}$. The invariant scalar product in (2.28) can be evaluated in the null coordinates:

$$\Lambda=T^{+}{}_{,1}/-T^{-}{}_{,1} \; . \tag{2.29}$$

It is obvious that Λ is a scalar on Σ, i.e., that it does not depend on the parametrization of Σ.

An observer who is confined to a small piece of an embedding cannot determine $\Lambda(x)$ because the privileged inertial frame $e_{(\pm)}{}^{\alpha}$ is fixed only by global considerations. In particular, he cannot recognize $e_{(\pm)}{}^{\alpha}$ from the frame field $\bar{e}_{(\pm)}{}^{\alpha}$ boosted by a constant λ, Eq. (2.11). However, from the geometric data on the whole embedding $X:\Sigma\rightarrow M$ (from the intrinsic metric g_{11} and the extrinsic curvature K_{11}) he can reconstruct $\Lambda(x)$ without going directly to the privileged $T=\text{const}$ foliation. This is a fundamental point in our discussion of potentials designed to cancel the anomalies in the Dirac constraint algebra.

The reconstruction of $\Lambda(x)$ starts from an observation that its logarithmic derivative $(\ln\Lambda(x))_{,1}$, unlike $\Lambda(x)$ itself, is a local function of the geometric data:

$$(\ln\Lambda(x))_{,1}=-g_1(x)K(x) \; . \tag{2.30}$$

By integrating this equation we get

$$\Lambda(x)=\Lambda(0)\exp\left[-\int_0^{x^1}dx^{1'}g_{1'}(x')K(x')\right] \; . \tag{2.31}$$

Note that $\Lambda(-\pi)=\Lambda(\pi)$ because for any smooth embedding

$$\int_\Sigma dx^1 g_1(x)K(x)=0 \; . \tag{2.32}$$

To determine the integration constant $\Lambda(0)$, we use the fact that the privileged Killing vector field **t** has no circulation around Σ:

$$\int_\Sigma dx^1 t_1(x)=0, \quad \text{where } t_1:=t_\alpha X_1^\alpha \ . \tag{2.33}$$

[Indeed, $t_1(x)=-T_{,1}(X(x))$ is a gradient of a globally defined time scalar T.] We find t_1 as a function of Λ from Eqs. (2.4) and (2.28):

$$t_1(x)=\tfrac{1}{2}g_1(x)(\Lambda^{-1}(x)-\Lambda(x)) \ . \tag{2.34}$$

Equation (2.33) then determines the integration constant $\Lambda(0)$ in expression (2.31):

$$\Lambda(0)^2=\int_\Sigma dx^1 g_1(x)\exp\left[\int_0^{x^1} dx^{1'}g_{1'}(x')K(x')\right]\Big/\int_\Sigma dx^1 g_1(x)\exp\left[-\int_0^{x^1} dx^{1'}g_{1'}(x')K(x')\right] \ . \tag{2.35}$$

Note that in spite of the fact that $\Lambda(x)$ is obtained from $g_{11}(x)$ and $K_{11}(x)$ by nonlocal operations, namely, through the integrals (2.31) and (2.35) along Σ, it is still a local functional of the embedding $X(x)$. Indeed, any deformation of $X(x)$ outside a small interval about x does not change $\Lambda(x;X]$. This is seen directly from Eq. (2.29) and, in the integral expressions for $\Lambda(x;X]$, it is ensured by Eq. (2.32). In its turn, Eq. (2.32) is related to the Mainardi equation (2.25) which implies that

$$\delta\int_\Sigma dx^1 g_1(x)K(x)/\delta X^\alpha(x')=0 \ . \tag{2.36}$$

Once we know the slope factor (2.29) as a functional (2.31) and (2.35) of the intrinsic and extrinsic geometry of an embedding, we also know the metric covectors

$$g^{(\pm)}{}_1=\pm T^{\pm}{}_{,1}=g_1\Lambda^{\pm1} \tag{2.37}$$

as functionals of g_1 and K. By integrating Eq. (2.37) along an arc of the embedding which does not cross the dateline we reconstruct the finite differences

$$T^\pm(x)-T^\pm(x')=\pm\int_{x'}^{x} dx^{1''}g_{1''}(x'')\Lambda^{\pm1}(x'') \tag{2.38}$$

of the null coordinates from the geometry of the hypersurface. These differences enter as arguments of various singular functions which characterize the solution of the wave equation on $M=\mathbb{R}\times S^1$. In particular, we shall use Eq. (2.38) for writing the normal-ordering kernel of the Hamiltonian flux on a spatial hypersurface.

III. PARAMETRIZED SCALAR FIELD ON $M=\mathbf{R}\times S^1$

A. Spacetime approach

After understanding the geometry of the spacetime background $M=\mathbb{R}\times S^1$, we introduce a massless scalar field ϕ propagating on that background. The dynamics of $\phi(X)$ follows from the spacetime action

$$S[\phi]=-\tfrac{1}{2}\int_M dV G^{\alpha\beta}(X)\phi_{,\alpha}(X)\phi_{,\beta}(X) \ ; \tag{3.1}$$

here

$$dV=\epsilon_{\alpha\beta\,1}dX^\alpha{}_2dX^\beta \tag{3.2}$$

is an invariant volume element on M. The field equations $\Box\phi:=G^{\alpha\beta}\nabla_\alpha\nabla_\beta\phi=0$ take the form

$$-\phi_{,TT}+\phi_{,ZZ}=0 \quad \text{and} \quad \phi_{,+-}=0 \tag{3.3}$$

in the privileged systems of coordinates (T,Z) and (T^-,T^+). The two forms of the wave equation (3.3) lead naturally to the two forms of its solution traditionally associated with the names of Bernoulli and of D'Alembert. The Bernoulli solution is a superposition of a homogeneous mode and the harmonic components in the coordinate Z:

$$\phi(T,Z)=\frac{1}{\sqrt{2\pi}}\left[\mathbf{q}+\mathbf{p}T+\frac{1}{\sqrt{2}}\sum_{k=-\infty}^{\infty}{}'\frac{1}{|k|}(\mathbf{a}_k e^{i(kZ-|k|T)}+\text{c.c.})\right] \ . \tag{3.4}$$

The coefficients $\mathbf{q},\mathbf{p}$ are arbitrary real numbers, the coefficients $\mathbf{a}_k$ arbitrary complex numbers; the summation $\sum'$ is over all whole numbers k except zero. The D'Alembert solution is a superposition of two arbitrary functions $\phi_\odot$ and $\phi_\otimes$:

$$\phi(T^-,T^+)=\phi_\odot(T^-)+\phi_\otimes(T^+) \ . \tag{3.5}$$

The $\phi_\odot(T^-)$ wave travels counterclockwise and the $\phi_\otimes(T^+)$ wave travels clockwise; neither of them changes its profile. The Bernoulli solution is related to the D'Alembert solution by

$$\phi_{\otimes,\odot}(T^\pm)=\frac{1}{\sqrt{2\pi}}\left[\frac{1}{2}(\mathbf{q}+\mathbf{p}T^\pm)+\frac{1}{\sqrt{2}}\sum_{k=1}^{\infty}\frac{1}{k}(\mathbf{a}_{\mp k}e^{-ikT^\pm}+\text{c.c.})\right] \ . \tag{3.6}$$

In Eq. (3.6) we have divided the constant level $\mathbf{q}$ equally between the $\phi_\odot$ and $\phi_\otimes$ solutions. We see that the coefficients $\mathbf{a}_k$ with a positive k describe waves propagating counterclockwise and those with a negative k describe waves propagating clockwise. Note that

$$\phi_{,+}=\phi_{\otimes,+} \quad \text{and} \quad \phi_{,-}=\phi_{\odot,-} \ . \tag{3.7}$$

The energy-momentum tensor $T_{\alpha\beta}(X)$ of the scalar field is symmetric ($T_{\alpha\beta}=T_{\beta\alpha}$), conserved ($\nabla_\beta T^\beta_\alpha=0$), and trace-free ($T^\alpha_\alpha=0$). In the null coordinates these equations imply

$$T_{\pm\pm}=T_{\pm\pm}(T^\pm)=(\phi_{,\pm}(T^\pm))^2 \ . \tag{3.8}$$

B. Parametrized field: The Schrödinger picture

In hypersurface dynamics we follow the evolution of the scalar field $\phi(t,x):=\phi(X(t,x))$ along a foliation

$$X^\alpha = X^\alpha(t,x) \tag{3.9}$$

of spacelike embeddings.[13] We express the spacetime action (3.1) as a functional of $\phi(t,x)$ and of the foliation (3.9). We differentiate Eq. (3.9) and substitute the differentials into the expression (3.2), thus recasting the invariant volume element into the form

$$dV = (-n_\alpha(t,x;X]\dot{X}^\alpha(t,x)dt)(g_1(x;X]dx^1)\ . \tag{3.10}$$

We write $\phi_{,\alpha}$ in terms of $\dot\phi$ and $\phi_{,1}$,

$$\phi_{,\alpha} = (\dot\phi - \phi_{,1}X^1_\beta \dot{X}^\beta)(n_\gamma \dot{X}^\gamma)^{-1} n_\alpha + \phi_{,1}X^1_\alpha\ , \tag{3.11}$$

and substitute expressions (3.10) and (3.11) into the spacetime action (3.1). We obtain the hypersurface action

$$S[X^\alpha(t,x),\phi(t,x)] = \int_{\mathbb{R}} dt \int_\Sigma dx^1 l_1(x;X,\phi,\dot{X},\dot\phi) \tag{3.12}$$

whose Lagrangian density

$$l_1 = -\tfrac{1}{2} g_1(-n_\gamma \dot{X}^\gamma) G^{\alpha\beta}(X(t,x))\phi_{,\alpha}\phi_{,\beta} \tag{3.13}$$

is a homogeneous function of the velocities $\dot{X}^\alpha$, $\dot\phi$. The hypersurface action leads to valid equations under independent variations of $X^\alpha(t,x)$ and $\phi(t,x)$ (Ref. 14).

We define the momenta $\pi_1(x)$ conjugate to $\phi(x)$,

$$\pi_1(x) := \partial l_1/\partial\dot\phi = n^\alpha_1 \phi_{,\alpha}\ , \tag{3.14}$$

and cast the hypersurface action into a canonical form in the field variables $\phi(x),\pi_1(x)$:

$$S[X^\alpha,\phi,\pi] = \int_{\mathbb{R}} dt \int_\Sigma dx^1(\pi_1\dot\phi - h_1)\ . \tag{3.15}$$

The Hamiltonian density

$$h_1 := \pi_1\dot\phi - l_1 = h_{1\alpha}\dot{X}^\alpha \tag{3.16}$$

is linear in the embedding velocity $\dot{X}^\alpha$. The coefficient

$$h_{1\alpha} = T_{\alpha\beta} n^\beta_1 \tag{3.17}$$

is the energy-momentum flux in the direction normal to the hypersurface. To express it as a functional of the canonical variables ϕ and π_1 we invert Eqs. (3.11) and (3.14),

$$\phi_{,\alpha} = -\pi_1 n^1_{\ \alpha} + \phi_{,1}X^1_\alpha\ , \tag{3.18}$$

and substitute expression (3.18) into Eq. (3.17):

$$h_{1\alpha} = -h_{11\perp} n^1_{\ \alpha} + h_{11}X^1_\alpha\ . \tag{3.19}$$

The coefficients

$$h_{11\perp} := h_{1\alpha} n^\alpha_1 = \tfrac{1}{2}((\pi_1)^2 + (\phi_{,1})^2)$$

and (3.20)

$$h_{11} := h_{1\alpha} X^\alpha_1 = \pi_1\phi_{,1}$$

[the energy and momentum two-densities measured by the hypersurface observer (n^α_1, X^α_1)] do not depend explicitly on the embedding.

The action (3.15) is still not written in a canonical form with respect to the embedding. Because h_1 is linear in the embedding velocity, the embedding momentum

$$P_{1\alpha}(x) := \partial(\pi_1\dot\phi - h_{1\beta}\dot{X}^\beta)/\partial\dot{X}^\alpha = -h_{1\alpha} \tag{3.21}$$

is subject to the constraints

$$\mathbf{P}_{1\alpha}(x) = 0\ , \tag{3.22}$$

with

$$\mathbf{P}_{1\alpha}(x) := P_{1\alpha}(x) + h_{1\alpha}(x;X,\phi,\pi]\ . \tag{3.23}$$

The canonical action

$$S[X,\phi,P,\pi] := \int_{\mathbb{R}} dt \int_\Sigma dx^1(P_{1\alpha}\dot{X}^\alpha + \pi_1\dot\phi) \tag{3.24}$$

in the extended phase space must be varied subject to these constraints to yield the correct field equations.

The action (3.24) and the constraints (3.23) are written in terms of the Schrödinger fundamental variables $X^\alpha(x),P_{1\alpha}(x),\phi(x),\pi_1(x)$ which provide a canonical chart in the extended phase space. The only nonvanishing fundamental Poisson brackets are those between canonically conjugate pairs of variables:

$$\begin{aligned}
&\{X^\alpha(x),\ P_{1'\beta}(x')\} = \delta^\alpha_\beta \delta_{1'}(x,x')\ ,\\
&\{\phi(x),\ \pi_{1'}(x')\} = \delta_{1'}(x,x')\ ,\\
&\{X^\alpha(x),\ X^\beta(x')\} = 0 = \{P_{1\alpha}(x),\ P_{1'\beta}(x')\}\ ,\\
&\{\phi(x),\ \phi(x')\} = 0 = \{\pi_1(x),\ \pi_{1'}(x')\}\ ,\\
&\{X^\alpha(x),\ \phi(x')\} = 0 = \{X^\alpha(x),\ \pi_{1'}(x')\}\ ,\\
&\{P_{1\alpha}(x),\ \phi(x')\} = 0 = \{P_{1\alpha}(x),\ \pi_{1'}(x')\}\ .
\end{aligned} \tag{3.25}$$

Note that these Poisson brackets remain invariant under a spacetime transformation $X^\alpha \to X^{\alpha'} = X^{\alpha'}(X^\beta)$ which induces the transformation

$$\begin{aligned}
&X^\alpha \to X^{\alpha'}(x) := X^{\alpha'}(X^\beta(x))\ ,\\
&P_{1\beta}(x) \to P_{1\beta'}(x) := X^\alpha_{\beta'}(X^\gamma(x))P_{1\alpha}(x)\ ,\\
&X^\alpha_{\beta'} := \partial X^\alpha/\partial X^{\beta'}
\end{aligned} \tag{3.26}$$

of the embeddings and their conjugate momenta.

C. The algebra of constraints: The Schrödinger picture

The algebra of the constraints (3.23) can be determined by direct calculation. In the projected form

$$\mathbf{P}_{11\perp} := P_{11\perp} + h_{11\perp},\quad \mathbf{P}_{11} = P_{11} + h_{11}\ , \tag{3.27}$$

the embedding variables $X^\alpha(x),P_{1\alpha}(x)$ decouple from the field variables $\phi(x)$, $\pi_1(x)$: $P_{11\perp}$ and P_{11} depend only on the embedding variables and the energy-momentum two-densities (3.20) depend only on the field variables.

Evaluation of the Poisson brackets is most straightforward in the hypersurface null basis $n_{(\pm)}{}^\alpha_1 = n^\alpha_1 \pm X^\alpha_1$. The projections $h_{11(\pm)} := h_{1\alpha} n^\alpha_1$ of the energy-momentum flux into the null basis are related to the normal and tangential projections (3.20) by $h_{11(\pm)} = h_{11\perp} \pm h_{11}$ and

$$h_{11(\pm)}=h_{11\perp}\pm h_{11} \quad \text{and}$$
$$h_{11\perp}=\tfrac{1}{2}(h_{11(+)}+h_{11(-)}) \,, \tag{3.28}$$
$$h_{11}=\tfrac{1}{2}(h_{11(+)}-h_{11(-)}) \,.$$

One can express $h_{11(\pm)}$ in terms of the null momenta

$$\pi_{1(\pm)}:=n_{(\pm)}{}^{\alpha}_{1}\phi_{,\alpha}=\pi_1\pm\phi_{,1} \,. \tag{3.29}$$

This leads to

$$h_{11(\pm)}=\tfrac{1}{2}(\pi_{1(\pm)})^2 \,. \tag{3.30}$$

From the fundamental field brackets (3.25) we get the Poisson brackets of the null momenta:

$$\{\pi_{1(\pm)}(x),\, \pi_{1'(\pm)}(x')\}=\pm 2\delta_{1',1}(x,x') \,,$$
$$\{\pi_{1(+)}(x),\, \pi_{1'(-)}(x')\}=0 \,. \tag{3.31}$$

This yields the algebra of the null projections:

$$\{h_{11(\pm)}(x), h_{1'1'(\pm)}(x')\}$$
$$=\pm 2(h_{11(\pm)}(x)\delta_{1',1}(x,x')-(x\leftrightarrow x'))$$
$$=\mp 2(h_{11(\pm)}(x)D_{1'}\delta_{1'}(x,x')-(x\leftrightarrow x')) \tag{3.32}$$

and

$$\{h_{11(+)}(x),\, h_{1'1'(-)}(x')\}=0 \,. \tag{3.33}$$

The first form of Eq. (3.32) makes manifest the fact that the Poisson brackets do not depend on the embedding variables, the second form makes manifest the two-density character of the brackets in both the x and x' arguments. It is easy to prove that the projections $P_{11(\pm)}(x)$ obey the same algebra (3.32) and (3.33) as the flux projections $h_{11(\pm)}(x)$, and hence the projected constraints $\mathbf{P}_{11(\pm)}(x)$ also obey the same algebra.

Similarly, the normal and tangential flux projections $h_{11\perp}(x)$, $h_{11}(x)$ obey the same algebra as the embedding momentum projections $P_{11\perp}(x)$, $P_{11}(x)$ and as the projected constraints $\mathbf{P}_{11\perp}(x)$, $\mathbf{P}_{11}(x)$. We shall write this algebra for the last set of quantities:

$$\{\mathbf{P}_{11}(x),\, \mathbf{P}_{1'1'}(x')\}=\mathbf{P}_{11}(x)\delta_{1',1}(x,x')-(x\leftrightarrow x') \,,$$
$$\{\mathbf{P}_{11\perp}(x),\, \mathbf{P}_{1'1'}(x')\}=\mathbf{P}_{11\perp}(x)\delta_{1',1}(x,x')-(x\leftrightarrow x') \,, \tag{3.34}$$
$$\{\mathbf{P}_{11\perp}(x),\, \mathbf{P}_{1'1'\perp}(x'\}=\mathbf{P}_{11}(x)\delta_{1',1}(x,x')-(x\leftrightarrow x') \,.$$

As in Eq. (3.32) one can also use the form which makes manifest the density character of the Poisson brackets. The algebra (3.34) of the projected constraints is the Dirac algebra.[1] (In two dimensions, when using the two-density constraints, the Dirac algebra is a true algebra.[15])

In hypersurface dynamics, however, it is far more convenient to work with the unprojected constraints (3.23). It holds that

$$\mathbf{P}_{1\alpha}=P_{1\alpha}-\tfrac{1}{2}h_{11(+)}n_{(-)}{}^{1}_{\alpha}-\tfrac{1}{2}h_{11(-)}n_{(+)}{}^{1}_{\alpha} \,. \tag{3.35}$$

In the null coordinates, in view of Eq. (2.27),

$$\mathbf{P}_{1\pm}(x)=P_{1\pm}(x)\pm\tfrac{1}{2}h_{11(\pm)}(x)(T^{\pm}{}_{,1}(x))^{-1} \,. \tag{3.36}$$

Evaluation of the Poisson brackets is straightforward and leads to the conclusion that in these coordinates the brackets of the unprojected constraints have to vanish:

$$\{\mathbf{P}_{1\alpha}(x),\, \mathbf{P}_{1'\beta}(x')\}=0 \,. \tag{3.37}$$

However, the bracket (3.37) transforms as a bicovector at $X(x)$ and $X(x')$ under spacetime transformations (3.26) and hence Eq. (3.37) must hold in arbitary spacetime coordinates. The algebra of the unprojected constraints is Abelian. One can trace this back to the independence of the spacetime action of the choice of foliation.[14]

D. Parametrized field: From the Schrödinger picture to the Heisenberg picture

Each one of the Schrödinger fundamental variables $X^{\alpha}(x)$, $P_{1\alpha}(x)$, $\phi(x)$, $\pi_1(x)$, and hence also any dynamical variable $F[X,P,\phi,\pi]$ constructed out of them, evolves from one embedding $\mathbf{X}^{\alpha}(x)$ to another according to the Hamilton equations of motion

$$\delta F[X,P,\phi,\pi]/\delta\mathbf{X}^{\alpha}(x)$$
$$=\{F[X,P,\phi,\pi],\mathbf{P}_{1\alpha}(x;X,P,\phi,\pi]\} \,. \tag{3.38}$$

We do not *a priori* identify the embedding $\mathbf{X}^{\alpha}(x)$ which is an independent variable in the variational differential equation (3.38) with the embedding $X^{\alpha}(x)$ which is one of the canonical variables describing the parametrized system. Such an identification, Eq. (3.42), follows naturally from the Hamilton equations (3.38) themselves. At the end of the day, the embedding $\mathbf{X}^{\alpha}(x)$ becomes one of the Heisenberg variables in the extended phase space.

We assume that the Schrödinger variables take the initial values

$$X^{\alpha}(x)={}_0X^{\alpha}(x), \quad P_{1\alpha}(x)={}_0P_{1\alpha}(x) \,,$$
$$\phi(x)=\boldsymbol{\phi}(x), \quad \pi_1(x)=\boldsymbol{\pi}_1(x) \,, \tag{3.39}$$

on an initial embedding $\mathbf{X}^{\alpha}(x)={}_0X^{\alpha}(x)$, say,

$${}_0X^{\alpha}(x):\ {}_0T(x)=0, \quad {}_0Z(x)=x \,. \tag{3.40}$$

Of course, ${}_0P_{1\alpha}(x)$ is determined from ${}_0X^{\alpha}(x)$ and $\boldsymbol{\phi}(x),\boldsymbol{\pi}_1(x)$ by the initial constraint (3.22).

In principle, we can solve the Hamilton equations of motion for these initial conditions. First of all, due to the structure (3.23) of $\mathbf{P}_{1\alpha}(x)$, Eq. (3.38) yields

$$\delta X^{\beta}(x')/\delta\mathbf{X}^{\alpha}(x)=\delta^{\beta}_{\alpha}\delta_1(x',x) \,. \tag{3.41}$$

Under the initial conditions (3.39) this has the solution

$$\mathbf{X}^{\alpha}(x)=X^{\alpha}(x) \,. \tag{3.42}$$

Second, by solving the Hamilton equations (3.38) for the field variables we get

$$\phi(x)=\phi(x;\mathbf{X},\boldsymbol{\phi},\boldsymbol{\pi}]$$
and
$$\pi_1(x)=\pi_1(x;\mathbf{X},\boldsymbol{\phi},\boldsymbol{\pi}] \,. \tag{3.43}$$

These solutions can be inverted to yield

$$\boldsymbol{\phi}(x)=\boldsymbol{\phi}(x;X,\phi,\pi]$$
and (3.44)
$$\boldsymbol{\pi}_1(x)=\boldsymbol{\pi}_1(x;X,\phi,\pi]\ .$$

Any dynamical variable F which has vanishing Poisson brackets with the constraints,

$$\{F[X,P,\phi,\pi],\mathbf{P}_{1\alpha}(x;X,P,\phi,\pi]\}=0\ , \tag{3.45}$$

has the same value on any embedding $\mathbf{X}^{\alpha}(x)$ along the dynamical trajectory (3.38) and is thus a constant of motion. By the method of their construction, the initial fields (3.44) are constants of motion:

$$\{\boldsymbol{\phi}(x),\mathbf{P}_{1'\alpha}(x')\}=0=\{\boldsymbol{\pi}_1(x),\mathbf{P}_{1'\alpha}(x')\}\ . \tag{3.46}$$

Also, the constraint functions (3.23) are constants of motion; this is the content of Eq. (3.37). The constraints tell us that the value of these constants of motion must be necessarily put equal to zero. It we wish, we can now complete the solution of the Hamilton equations of motion by substituting the field solutions (3.43) into the constraint functions (3.23) and writing

$$P_{1\alpha}(x;\mathbf{X},\boldsymbol{\phi},\mathbf{P},\boldsymbol{\pi}]=\mathbf{P}_{1\alpha}(x)-h_{1\alpha}(x;\mathbf{X},\boldsymbol{\phi},\boldsymbol{\pi}]\ . \tag{3.47}$$

By imposing the constraints $\mathbf{P}_{1\alpha}=0$, we get the solution of the Hamilton equation for $P_{1\alpha}(x)$.

The constraint functions (3.23) and the embedding variables (3.42) have the Poisson brackets

$$\{\mathbf{X}^{\alpha}(x),\ \mathbf{P}_{1'\beta}(x')\}=\delta^{\alpha}_{\beta}\delta_{1'}(x,x')\ ,$$
$$\{\mathbf{X}^{\alpha}(x),\ \mathbf{X}^{\beta}(x')\}=0\ . \tag{3.48}$$

The initial field variables, as the field variables on any hypersurface, have the fundamental brackets

$$\{\boldsymbol{\phi}(x),\boldsymbol{\pi}_{1'}(x')\}=\delta_{1'}(x,x') \tag{3.49}$$

(the other brackets being zero). We have thus discovered that the dynamical variables $\mathbf{X}^{\alpha}(x),\mathbf{P}_{1\alpha}(x),\boldsymbol{\phi}(x),\boldsymbol{\pi}_1(x)$ have Poisson brackets which are appropriate for a canonical chart on the extended phase space: Eqs. (3.48), (3.37) (3.46), (3.49). These variables are nothing other than the Heisenberg fundamental variables on the extended phase space of the parametrized scalar field theory. Equations (3.23) and (3.44) give the canonical transformation from the Schrödinger variables to the Heisenberg variables; Eqs. (3.43) and (3.47) give the inverse canonical transformation from the Heisenberg variables to the Schrödinger variables. The argument we have just sketched illustrates for a simple example of a massless scalar field in two dimensions the general connection between Schrödinger and Heisenberg pictures for parametrized systems.[11]

While the Heisenberg field variables are the initial values of the fields and hence constants of motion, the Heisenberg embedding variable is *not* the initial embedding, but it is identical with the Schrödinger embedding variable. Also, while the Heisenberg embedding momentum *is* a constant of motion, the Schrödinger embedding momentum is *not,* and the canonical transformation between them, Eqs. (3.23) and (3.47), is *not* an identity transformation. The Heisenberg time $\mathbf{X}^{\alpha}(x)$ still runs; the transition from the Schrödinger picture to the Heisenberg picture does not yield the "frozen time formalism."[16]

A classical state of the field is described by the statistical distribution function ρ on the extended phase space. One can always eliminate the embedding momenta by using the constraints. When this is done, the statistical distribution is a functional $\rho[X,\phi,\pi]$ in the Schrödinger picture and a functional $\rho[\mathbf{X},\boldsymbol{\phi},\boldsymbol{\pi}]$ in the Heisenberg picture. We postulate that the statistical distribution is a constant of motion on the extended phase space:

$$\{\rho,\mathbf{P}_{1\alpha}(x)\}=0\ . \tag{3.50}$$

In the Heisenberg picture, $\mathbf{P}_{1\alpha}(x)$ is a fundamental variable and Eq. (3.50) merely tells us that ρ does not depend on the Heisenberg embedding:

$$\rho=\rho[\boldsymbol{\phi},\boldsymbol{\pi}]\ . \tag{3.51}$$

The distribution (3.51) is interpreted as the probability distribution of the initial field data $\boldsymbol{\phi}(x),\boldsymbol{\pi}_1(x)$ on the initial embedding. In the Schrödinger picture, $\mathbf{P}_{1\alpha}$ is not a fundamental variable, but it is constructed from the fundamental Schrödinger variables according to Eq. (3.23). Equation (3.50) in the Schrödinger picture has the meaning of the Liouville equation:

$$\delta\rho[X,\phi,\pi]/\delta X^{\alpha}(x)=-\{\rho[X,\phi,\pi],h_{1\alpha}(x;X,\phi,\pi]\}\ . \tag{3.52}$$

It determines how the probability distribution of the Schrödinger variables $\phi(x),\pi_1(x)$ on an embedding $X^{\alpha}(x)$ changes if we change the embedding.

Equation (3.50) is thus trivial in the Heisenberg picture, but nontrivial in the Schrödinger picture. We encounter an opposite situation when we study the equations

$$\{\phi(x),\ P_{1'\alpha}(x')\}=0=\{\pi_1(x),\ P_{1'\alpha}(x')\}\ . \tag{3.53}$$

In the Schrödinger picture, these equations are trivial, being merely a part of the fundamental Poisson-brackets relations. In the Heisenberg picture, $P_{1'\alpha}(x')$ is given by Eq. (3.47); Eq. (3.53) then acquires the meaning of the evolution equation for the field variables:

$$\delta\phi(x;\mathbf{X},\boldsymbol{\phi},\boldsymbol{\pi}]/\delta\mathbf{X}^{\alpha}(x')=\{\phi(x;\mathbf{X},\boldsymbol{\phi},\boldsymbol{\pi}],h_{1'\alpha}(x';\mathbf{X},\boldsymbol{\phi},\boldsymbol{\pi}]\}\ , \tag{3.54}$$

and similarly for $\pi_1(x;\mathbf{X},\boldsymbol{\phi},\boldsymbol{\pi}]$. The functionals (3.43) are supposed to solve this equation. Equations (3.53) in the Heisenberg picture are nontrivial: ultimately they are equivalent to the wave equation (3.3).

The consistency of Eq. (3.50) is ensured by the vanishing of the Poisson brackets (3.37). In the Heisenberg picture, Eqs. (3.37) serve as a starting point of the formalism and require no proof. In the Schrödinger picture, their proof is nontrivial; we have given it in Sec. III C. Their meaning is also nontrivial; they ensure that the evolution of the Schrödinger state $\rho[X,\phi,\pi]$ from an initial embedding ${}_0X^{\alpha}(x)$ to a final embedding $X^{\alpha}(x)$ does not depend on the foliation $X^{\alpha}(t,x)$ along which we evolve the state.

Similarly, the consistency of Eqs. (3.53) is ensured by equation

$$\{P_{1\alpha}(x), P_{1'\beta}(x')\}=0 \ . \tag{3.55}$$

Equation (3.55) is a starting point of the Scrödinger picture formalism; in the Heisenberg picture the $P_{1\alpha}(x)$ are given by the expressions (3.47) and Eq. (3.55) need to be proved (see Sec. III E). Their role in the Heisenberg picture is equally nontrivial: they ensure that the evolution of the field variables $\phi(x;\mathbf{X}],\pi_1(x;\mathbf{X}]$ by Eqs. (3.54) from the initial data $\boldsymbol{\phi}(x),\boldsymbol{\pi}_1(x)$ on an initial embedding ${}_0X^\alpha(x)$ does not depend on the foliation $\mathbf{X}^\alpha(t,x)$ along which we reach the final embedding $\mathbf{X}^\alpha(x)$.

To summarize, both $P_{1\alpha}(x)$ and $\mathbf{P}_{1\alpha}(x)$ form an Abelian Poisson-brackets system, Eqs. (3.55) or (3.37). Whether we are working in the Schrödinger picture or in the Heisenberg picture, one set of these equations is always trivial and another set requires proof.

In the Dirac quantization of a constrained system, one usually thinks in terms of the Schrödinger picture. Unfortunately, a consistent factor ordering is difficult to find and the commutators of the constraint functions typically acquire an anomaly. Even the form of this anomaly on curved embeddings is not properly given in the existing literature. Moreover, when one imposes the constraints as limitations on the Schrödinger picture states, the anomaly leads to inconsistencies. It is thus better to quantize the system first in the Heisenberg picture. The constraints in this picture preserve their trivial algebra and one can impose them on the Heisenberg picture states. It is the algebra of the $P_{1\alpha}(x)$ operators which acquires an anomaly. Fortunately, it is quite straightforward to find the correct form of this anomaly and to show that the consistency of the quantum evolution equations analogous to Eqs. (3.54) is still maintained. The anomaly in the algebra (3.55) does not matter because the $P_{1\alpha}$, unlike the $\mathbf{P}_{1\alpha}$, is not constrained to vanish. In the Heisenberg picture it is also easier to see how the anomaly in the algebra of the $P_{1\alpha}(x)$ variables can be removed by their redefinition. It is then possible to pass to the quantum Schrödinger picture in which the algebra of the constraints is also free of the anomaly.

To implement this program one must first learn how to do the classical calculations directly in the Heinsenberg picture.

E. Parametrized field: From the Heisenberg picture to the Schrödinger picture

A major advantage of the Heisenberg picture is that the Poisson brackets can be evaluated directly between spacetime fields and then restricted and projected to embeddings. As usual we assign the Poisson brackets

$$\begin{aligned}&\{\mathbf{q},\mathbf{p}\}=1,\quad \{\mathbf{a}_k,\mathbf{a}^*{}_l\}=-i|k|\delta_{kl}\ ,\\ &\{\mathbf{q},\mathbf{q}\}=0=\{\mathbf{p},\mathbf{p}\},\quad \{\mathbf{a}_k,\mathbf{a}_l\}=0=\{\mathbf{a}^*{}_k,\mathbf{a}^*{}_l\}\end{aligned} \tag{3.56}$$

to the coefficients $\mathbf{q},\mathbf{p},\mathbf{a}_k$ characterizing the Bernoulli solution (3.4). [We shall soon show that this amounts to assigning the correct Poisson brackets (3.49) to the Heisenberg fields $\boldsymbol{\phi}(x),\boldsymbol{\pi}_1(x)$.]

Equations (3.56) imply that the D'Alembert fields (3.6) have the Poisson brackets

$$\{\phi_{\otimes,\odot}(T^\pm),\phi_{\otimes,\odot}(T^{\pm\prime})\}=-\tfrac{1}{2}\theta(T^\pm-T^{\pm\prime})+\frac{1}{4}\frac{1}{2\pi}(T^\pm-T^{\pm\prime}) \tag{3.57}$$

and

$$\{\phi_\otimes(T^+),\phi_\odot(T^-)\}=-\frac{1}{4}\frac{1}{2\pi}(T^+-T^{-\prime})\ , \tag{3.58}$$

with

$$\theta(T^\pm):=\frac{1}{2\pi}\left[T^\pm+2\sum_{k=1}^{\infty}\frac{1}{k}\sin kT^\pm\right]\ . \tag{3.59}$$

The linear terms in (3.57)–(3.59) are due to the equipartition of the homogeneous mode between $\phi_\odot$ and $\phi_\otimes$. The expression (3.59) is representation of the step function on a circle:

$$\theta(T^\pm)=\begin{cases}-\frac{1}{2}\ \text{ for } T^\pm\in(-2\pi,0)\ ,\\ 0\ \text{ for } T^\pm=0\ ,\\ \frac{1}{2}\ \text{ for } T^\pm\in(0,2\pi)\ ,\end{cases} \tag{3.60}$$

$$\theta(T^\pm+2\pi n)=\theta(T^\pm)+n\quad\text{for } n=0,\pm1,\pm2,\ldots\ .$$

Its derivative yields the δ function

$$\begin{aligned}\theta_{,\pm}(T^\pm)&=\frac{1}{2\pi}\left[1+2\sum_{k=1}^{\infty}\cos kT^\pm\right]\\ &=\frac{1}{2\pi}\sum_{k=-\infty}^{\infty}e^{ikT^\pm}=\delta(T^\pm)\ .\end{aligned} \tag{3.61}$$

As with any function which depends either on T^+ or on T^-, $\theta(T^\pm)$ and $\delta(T^\pm)$ are solutions of the wave equation; $\theta(T^\pm)$ is an odd solution and $\delta(T^\pm)$ an even one.

From Eqs. (3.57) and (3.58) we get the Poisson bracket of the total D'Alembert solution (3.5) with itself:

$$\{\phi(X),\phi(X')\}=-\tfrac{1}{2}\theta(T^+-T^{+\prime})-\tfrac{1}{2}\theta(T^--T^{-\prime})\ . \tag{3.62}$$

Its successive differentiations yield

$$\{\phi(X),\phi_{,\pm}(X')\}=\tfrac{1}{2}\delta(T^\pm-T^{\pm\prime}) \tag{3.63}$$

and

$$\{\phi_{,\pm}(T^\pm),\phi_{,\pm}(T^{\pm\prime})\}=\tfrac{1}{2}\delta_{,\pm}(T^\pm-T^{\pm\prime})\ . \tag{3.64}$$

From the last equation we obtain the Poisson brackets between the components (3.8) of the energy-momentum tensor:

$$\begin{aligned}&\{T_{\pm\pm}(T^\pm),T_{\pm\pm}(T^{\pm\prime})\}=T_{\pm\pm}(T^\pm)\delta_{,\pm}(T^\pm-T^{\pm\prime})-(T^\pm\leftrightarrow T^{\pm\prime})\ ,\\ &\{T_{++}(T^+),T_{--}(T^{-\prime})\}=0\ .\end{aligned} \tag{3.65}$$

Finally, Eq. (3.63) yields the Poisson bracket of the field with the energy-momentum tensor,

$$\{\phi(X),T_{\pm\pm}(X')\}=\phi_{,\pm}(T^{\pm})\delta(T^{\pm}-T^{\pm'}) \ , \qquad (3.66)$$

and Eq. (3.64) its differentiated form

$$(\phi_{,\pm}(T^{\pm}),T_{\pm\pm}(T^{\pm'})\}=\phi_{,\pm}(T^{\pm'})\delta_{,\pm}(T^{\pm}-T^{\pm'}) \ . \qquad (3.67)$$

We start now restricitng these relations to a spacelike embedding $T^{\pm}=\mathbf{T}^{\pm}(x)$. Two points on such an embedding have the null coordinates $T^{\pm}=\mathbf{T}^{\pm}(x)$ and $T^{\pm'}=\mathbf{T}^{\pm}(x')$ whose differences have an opposite sign, $(T^{+}-T^{+'})(T^{-}-T^{-'})<0$, and range $|T^{\pm}-T^{\pm'}|\leq 2\pi$. The step functions in Eqs. (3.62) then cancel each other and the Schrödinger field variables

$$\phi(x):=\phi(\mathbf{X}(x)) \qquad (3.68)$$

have the vanishing Poisson brackets on the hypersurface:

$$\{\phi(x),\phi(x')\}=0 \ . \qquad (3.69)$$

Define the field momentum by Eq. (3.14), i.e.,

$$\pi_1(x):=\phi_{,+}(\mathbf{T}^{+}(x))\mathbf{T}^{+}{}_{,1}(x)-\phi_{,-}(\mathbf{T}^{-}(x))\mathbf{T}^{-}{}_{,1}(x) \ . \qquad (3.70)$$

From Eqs. (3.63),

$$\{\phi(x),\pi_{1'}(x')\}=\tfrac{1}{2}[\delta(\mathbf{T}^{+}(x)-\mathbf{T}^{+}(x'))\mathbf{T}^{+}{}_{,1'}(x') -\delta(\mathbf{T}^{-}(x)-\mathbf{T}^{-}(x'))\mathbf{T}^{-}{}_{,1'}(x')] \ . \qquad (3.71)$$

The spatial delta functions $\delta_{1'}(x,x')$ and their derivatives ∂_1 and $\partial_{1'}$, are related to the spacetime delta functions $\delta(T^{\pm}-T^{\pm'})$ and their derivatives $\partial_{\pm}$ by the relations which are summarized in Appendix B. By Eq. (B1),

$$\{\phi(x),\pi_{1'}(x')\}=\delta_{1'}(x,x') \ . \qquad (3.72)$$

Finally, from Eqs. (3.64) and (3.70),

$$\{\pi_1(x),\pi_{1'}(x')\}=0 \ . \qquad (3.73)$$

Equations (3.69), (3.72), and (3.73) show that the transformation (3.68), (3.70), (3.5), and (3.6) from the mode variables $\mathbf{q},\mathbf{p};\mathbf{q}_k,\mathbf{p}_k$,

$$\mathbf{q}_k=\frac{1}{\sqrt{2}}\frac{1}{|k|}(\mathbf{a}^*{}_k+\mathbf{a}_k), \quad \mathbf{p}_k=\frac{1}{\sqrt{2}}i(\mathbf{a}^*{}_k-\mathbf{a}_k) \ , \qquad (3.74)$$

to the Schrödinger variables $\phi(x),\pi_1(x)$ is (an embedding-dependent) canonical transformation.

The mode coefficients $\mathbf{q},\mathbf{p};\mathbf{q}_k,\mathbf{p}_k$ have a simple relation to the field variables $\boldsymbol{\phi}(x)$ and $\boldsymbol{\pi}_1(x)$ on the initial hypersurface $T^{+}=x, T^{-}=-x$. When we decompose $\boldsymbol{\phi}(x)$ and $\boldsymbol{\pi}_1(x)$ into the real Fourier components

$$\boldsymbol{\phi}(x)=\frac{1}{\sqrt{2\pi}}\left[\boldsymbol{\phi}_0+\sqrt{2}\sum_{k=1}^{\infty}(\boldsymbol{\phi}_k^{(c)}\cos kx+\boldsymbol{\phi}_k^{(s)}\sin kx)\right] \ , \qquad (3.75)$$

$$\boldsymbol{\pi}_1(x)=\frac{1}{\sqrt{2\pi}}\left[\boldsymbol{\pi}_0+\sqrt{2}\sum_{k=1}^{\infty}(\boldsymbol{\phi}_k^{(c)}\cos kx+\boldsymbol{\pi}_k^{(s)}\sin kx)\right] \ ,$$

we get

$$\boldsymbol{\phi}_0=\mathbf{q}, \quad \boldsymbol{\pi}_0=\mathbf{p} \ ,$$

$$\boldsymbol{\phi}_k^{(c)}=\frac{1}{\sqrt{2}}(\mathbf{q}_k+\mathbf{q}_{-k}), \quad \boldsymbol{\phi}_k^{(s)}=\frac{1}{\sqrt{2}}\frac{1}{k}(\mathbf{p}_{-k}-\mathbf{p}_k) \ , \qquad (3.76)$$

$$\boldsymbol{\pi}_k^{(c)}=\frac{1}{\sqrt{2}}(\mathbf{p}_k+\mathbf{p}_{-k}), \quad \boldsymbol{\pi}_k^{(s)}=\frac{1}{\sqrt{2}}k(\mathbf{q}_k-\mathbf{q}_{-k}) \ ,$$

$$k=1,2,\ldots \ .$$

The Fourier coefficients $\boldsymbol{\phi}_0,\boldsymbol{\phi}_k^{(c)},\boldsymbol{\phi}_k^{(s)}$ and $\boldsymbol{\pi}_0,\boldsymbol{\pi}_k^{(c)},\boldsymbol{\pi}_k^{(s)}$ have the Poisson brackets appropriate to canonical coordinates and their conjugate momenta. We can think of them, or the related coefficients $\mathbf{q},\mathbf{p},\mathbf{a}_k,\mathbf{a}^*{}_k$, as about two alternative sets of Heisenberg data. The transformation (3.68), (3.70), (3.5), and (3.6) together with the identifications (3.74)–(3.76) spells thus explicitly what Eq. (3.43) implied as a program.

Canonical transformation of the field variables must be complemented by the corresponding transformation of the embeddings and their conjugate momenta. While the embeddings are the same in the two pictures, Eq. (3.42), the embedding momenta change. In Sec. III E we proved that the transition (3.36) from the Schrödinger variables to the Heisenberg embedding momentum is a canonical transformation, Eqs. (3.37), (3.46), and (3.48). Here we are going to prove that the inverse relation (3.47) is a canonical transformation by performing all calculations in the Heinsenberg picture.

We define the Schrödinger embedding momentum by Eq. (3.36) and show that it has the correct Poisson brackets

$$\{\phi(x),P_{1'\pm}(x')\}=0=\{\pi_1(x),P_{1'\pm}(x')\} \qquad (3.77)$$

with the Schrödinger field variables (3.68) and their conjugate momenta (3.70). By projecting the spacetime brackets (3.66) we learn that

$$\{\phi(x),h_{1'\pm}(x')\}=\phi_{,\pm}(x)\delta_{1'}(x,x') \ . \qquad (3.78)$$

Similarly, from the definition (3.70) of $\pi_1(x)$ and by projecting the spacetime bracket (3.67) we learn that

$$\{\pi_1(x),h_{1'\pm}(x')\}=\pm\phi_{,\pm}(\mathbf{T}^{\pm}(x'))\delta_{1',1}(x,x') \ . \qquad (3.79)$$

The same expressions, (3.78) and (3.79), however, also follow from the action of the Heisenberg embedding momentum $\mathbf{P}_{1'\pm}(x')$ on the Schrödinger field variables (3.68) and (3.70):

$$\{\phi(x),\mathbf{P}_{1'\pm}(x')\}=\phi_{,\pm}(x)\delta_{1'}(x,x') \ ,$$

$$\{\pi_1(x),\mathbf{P}_{1'\pm}(x')\}=\pm[\phi_{,\pm\pm}(T^{\pm}(x))T^{\pm}{}_{,1}(x)\delta_{1'}(x,x') +\phi_{,\pm}(T^{\pm}(x))\delta_{1',1}(x,x')]$$

$$=\pm\phi_{,\pm}(T^{\pm}(x'))\delta_{1',1}(x,x') \ . \qquad (3.80)$$

This leads to Eq. (3.77).

We now turn to the brackets between the variables $X^\alpha(x)$ and $P_{1\alpha}(x)$ themselves. Among these, only the verification of

$$\{P_{1\pm}(x),P_{1'\pm}(x')\}=0 \tag{3.81}$$

requires some work. We start from the brackets

$$\begin{aligned}\{P_{1\pm}(x),P_{1'\pm}(x')\}&=\{\mathbf{P}_{1\pm}(x)-h_{1\pm}(x),\mathbf{P}_{1'\pm}(x')-h_{1'\pm}(x')\}\\&=\{h_{1\pm}(x),h_{1'\pm}(x')\}\\&\quad-(\{h_{1\pm}(x),\mathbf{P}_{1'\pm}(x')\}-(x\leftrightarrow x'))\ .\end{aligned} \tag{3.82}$$

The definition (3.17) relates the Hamiltonian flux to the components of the energy-momentum tensor. From the Poisson brackets (3.65) and the identity (B2) for the differentiated δ functions we immediately get the algebra of the flux components:

$$\{h_{1\pm}(x),h_{1'\pm}(x')\}=(\mathbf{T}^{\pm}{}_{,1}(x))^{-1}h_{1\pm}(x)\delta_{1',1}(x,x')-(x\leftrightarrow x')\ , \tag{3.83}$$

$$\{h_{1+}(x),h_{1'-}(x')\}=0\ .$$

On the other hand, we evaluate

$$\{h_{1\pm}(x),\mathbf{P}_{1'\pm}(x')\}=\pm\mathbf{T}^{\pm}{}_{,1}(x)T_{\pm\pm,\pm}\delta_{1',1}(x,x') \tag{3.84}$$

and interchange the points:

$$\{h_{1\pm}(x),\mathbf{P}_{1'\pm}(x')\}-(x\leftrightarrow x')=(\mathbf{T}^{\pm}{}_{,1}(x))^{-1}h_{1\pm}(x)\delta_{1',1}(x,x')-(x\leftrightarrow x')\ . \tag{3.85}$$

We see that expressions (3.85) and (3.83) exactly cancel each other in Eq. (3.82). This proves Eq. (3.81).

We carried out all calculations in the null coordinates, but the transformation properties of the Poisson brackets under spacetime transformations (3.26) enable us to write our results in a form which is valid in an arbitrary coordinate system. In particular, the fundamental Poisson brackets (3.77) and (3.81) take the covariant form (3.25). The algebra (3.83) of the Hamiltonian fluxes also has a covariant transcription,

$$\{h_{1\alpha}(x),h_{1'\beta}(x')\}=C^{1\gamma}{}_{\alpha\beta'}(x,x';\mathbf{X}]h_{1\gamma}(x)\delta_{1',1}(x,x')-(\alpha x\leftrightarrow\beta x')\ , \tag{3.86}$$

whose coefficients $C^{1\gamma}{}_{\alpha\beta'}$ are constructed from the null hypersurface basis (2.26) and (2.27):

$$\begin{aligned}C^{1\gamma}{}_{\alpha\beta'}(x,x';\mathbf{X}]:=\tfrac{1}{2}(&n_{(+)}{}^{1\gamma}(x)\,n_{(-)\alpha}(x)n_{(-)\beta}(x')\\&-n_{(-)}{}^{1\gamma}(x)\,n_{(+)\alpha}(x)n_{(+)\beta}(x'))\ .\end{aligned} \tag{3.87}$$

Both the spatial and the spacetime indices at the two points involved are correctly matched to produce a manifestly covariant result.

This completes our proof that the transitions (3.68), (3.70), (3.5), (3.6), (3.42), and (3.35) are a canonical transformation from the Heisenberg picture to the Schrödinger picture.

IV. SPACETIME DIFFEOMORPHISMS AND CONFORMAL ISOMETRIES IN THE HAMILTONIAN FORMALISM

The hypersurface action (3.12) of the parametrized theory is invariant under spacetime diffeomorphisms and also under conformal scalings of the metric G. This leads us to the question of what role the diffeomorphism group Diff$M$ and the group of conformal isometries C play in the parametrized canonical formalism. We shall see that Diff$M$ is a dynamical group of the theory: the generators of L Diff$M$ can be represented by Hamiltonians, i.e., by generators of canonical transformations which yield the actual motion of the system in the extended phase space. On the other hand, C is a symmetry group of the theory: the generators of LC can be represented by dynamical variables which generate canonical transformations that leave the Hamiltonians (weakly) invariant and which thus are constants of motion.

A. Spacetime diffeomorphisms and conformal isometries

Modulo certain technical niceties, the Lie algebra L Diff$M$ can be identified with the set of all complete vector fields $\mathbf{U}$ on M whose Lie bracket is (up to sign) their commutator

$$[\mathbf{U},\mathbf{V}]=(U^\beta V^\alpha{}_{,\beta}-V^\beta U^\alpha{}_{,\beta})\frac{\partial}{\partial X^\alpha}\ . \tag{4.1}$$

Conformal Killing vector fields $\mathbf{u}$ form a subalgebra LC of L Diff$M$. When acting on the (flat) metric $\mathbf{G}$, a conformal Killing vector field $\mathbf{u}$ scales it by a factor $W[\mathbf{u}]$:

$$(\pounds_{\mathbf{u}}\mathbf{G})_{\alpha\beta}=\nabla_\alpha u_\beta+\nabla_\beta u_\alpha=W[\mathbf{u}]G_{\alpha\beta}\ . \tag{4.2}$$

From the conformal Killing equation (4.2) it follows that

$$W[\mathbf{u}]=\operatorname{div}\mathbf{u}=\nabla_\alpha u^\alpha\ . \tag{4.3}$$

When written in the null coordinates, the conformal Killing equation yields

$$u^-{}_{,+}=0=u^+{}_{,-} \tag{4.4}$$

and

$$W[\mathbf{u}]=u^+{}_{,+}+u^-{}_{,-}\ . \tag{4.5}$$

From Eq. (4.4) we see that the general conformal Killing vector has the form

$$u^\alpha=(u^-(T^-),u^+(T^+)) \tag{4.6}$$

and that $W[\mathbf{u}]$ given by Eq. (4.5) has the form of a D'Alembert solution to the wave equation $\Box W=0$.

The null conformal Killing vectors $\mathbf{u}_{(\pm)}$ parallel to the counterclockwise (clockwise) propagating null rays $e_{(\pm)}$,

$$e_{(\pm)\alpha}u_{(\pm)}{}^{\alpha}=0\ , \quad \text{or}$$
$$u_{(+)}{}^{\alpha}=(0,u^{+}(T^{+}))\ ,$$
$$u_{(-)}{}^{\alpha}=(u^{-}(T^{-}),0) \tag{4.7}$$

form two subalgebras, $L_{(\pm)}C$, of LC. The elements of $L_{(+)}C$ commute with those of $L_{(-)}C$, $[\mathbf{u}_{(+)},\mathbf{u}_{(-)}]=0$, and LC is a direct product of the $L_{(\pm)}C$ subalgebras, $LC=L_{(+)}C\otimes L_{(-)}C$. The vector fields $\mathbf{u}_{(\pm)}(T^{-},T^{+})$ must satisfy the matching conditions $\mathbf{u}_{(\pm)}(T+\pi,T-\pi)=\mathbf{u}_{(\pm)}(T-\pi,T+\pi)$ which imply that the $u^{\pm}(T^{\pm})$ are periodic functions of $T^{\pm}$. Therefore, if we know these function in the interval $T^{\pm}\in[-\pi,\pi]$, we know the fields $\mathbf{u}_{(\pm)}$ everywhere. The composition law of the elements of $L_{(\pm)}C$,

$$[\mathbf{u}_{(\pm)},\mathbf{v}_{(\pm)}]=[u^{\pm}(T^{(\pm)})\partial_{\pm},v^{\pm}(T^{\pm})\partial_{\pm}]\ , \tag{4.8}$$

is the same as the composition law of two elements of $L\ \mathrm{Diff}S^1$. Each of the two algebras $L_{(\pm)}C$ is thus isomorphic with $L\ \mathrm{Diff}S^1$.

An arbitrary vector field $\mathbf{N}(x)$ on a spacelike hypersurface can be written in a unique way as a sum of two conformal Killing vector fields $\mathbf{u}_{(\pm)}$ restricted to that hypersurface. This follows from the geometric fact that each null line intersects a spacelike hypersurface once and only once; we can thus put $u^{\pm}(T^{\pm})=N^{\pm}(x(T^{\pm}))$.

The periodic functions $u^{\pm}(T^{\pm})$ which characterize the fields $\mathbf{u}_{(\pm)}$ can be decomposed into the Fourier components $u^{\pm}{}_{n}$:

$$\mathbf{u}_{(\pm)}=\sum_{n=-\infty}^{\infty}u^{\pm}{}_{n}\mathbf{L}^{(\pm)}{}_{n}\ , \tag{4.9}$$

where

$$\mathbf{L}^{(\pm)}{}_{n}:=e^{-inT^{\pm}}\partial_{\pm} \tag{4.10}$$

and

$$u^{\pm}{}_{n}:=\frac{1}{2\pi}\int_{-\pi}^{\pi}dT^{\pm}\,u^{\pm}(T^{\pm})e^{inT^{\pm}}\ . \tag{4.11}$$

One can think about $\mathbf{u}_{(\pm)}$ as an element of $L_{(\pm)}C\approx L\ \mathrm{Diff}S^1$ and about $u^{\pm}{}_{n}$ as its components in the basis $\mathbf{L}^{(\pm)}{}_{n}$ in $L_{(\pm)}C$. The commutators of the basis vectors $\mathbf{L}^{(\pm)}{}_{n}$ close according to the Virasoro algebra

$$[\mathbf{L}^{(\pm)}{}_{m},\mathbf{L}^{(\pm)}{}_{n}]=i(m-n)\mathbf{L}^{(\pm)}{}_{m+n} \tag{4.12}$$

and they commute:

$$[\mathbf{L}^{(+)}{}_{m},\mathbf{L}^{(-)}{}_{n}]=0\ . \tag{4.13}$$

B. Canonical relatizations of L Diff$M$

We shall now let the spacetime diffeomorphisms act in the extended phase space of our parametrized system by canonical transformations. We have previously introduced two such actions and discussed their geometric meaning.[3] Let us briefly summarize these results and connect them with the Schrödinger and Heisenberg pictures of the parametrized field dynamics.

The basis idea behind canonical realizations of the Lie algebra $L\ \mathrm{Diff}M$ is to map each generator $\mathbf{U}\in L\ \mathrm{Diff}M$ into a dynamical variable on the extended phase space by using the generator as a smearing function of the embedding momenta. This can be done either with the Schrödinger or with the Heisenberg momenta. Also, the resulting dynamical variables can be alternatively considered as functionals either of the Schrödinger or of the Heisenberg fundamental variables. We are thus led to consider the dynamical variables

$$P(\mathbf{U})[X,P]:=\int_{\Sigma}dx^{1}U^{\alpha}(X(x))P_{1\alpha}(x) \tag{4.14}$$

and

$$\mathbf{P}(\mathbf{U})[X,P,\phi,\pi]:=\int_{\Sigma}dx^{1}U^{\alpha}(X(x))\mathbf{P}_{1\alpha}(x;X,P,\phi,\pi] \tag{4.15}$$

in the Schrödinger picture and their counterparts

$$P(\mathbf{U})[\mathbf{X},\mathbf{P},\boldsymbol{\phi},\boldsymbol{\pi}]:=\int_{\Sigma}dx^{1}u^{\alpha}(\mathbf{X}(x))P_{1\alpha}(x;\mathbf{X},\mathbf{P},\boldsymbol{\phi},\boldsymbol{\pi}] \tag{4.16}$$

and

$$\mathbf{P}(\mathbf{U})[\mathbf{X},\mathbf{P}]:=\int_{\Sigma}dx^{1}U^{\alpha}(\mathbf{X}(x))\mathbf{P}_{1\alpha}(x) \tag{4.17}$$

in the Heisenberg picture. The dynamical variable $P(\mathbf{U})$ acts in the Schrödinger kinematical sector X,P of the extended phase space and $\mathbf{P}(\mathbf{U})$ acts in the Heisenberg kinematical sector $\mathbf{X},\mathbf{P}$. From the fundamental Poisson brackets (3.55) and (3.37) it follows that

$$\{P(\mathbf{U}),P(\mathbf{V})\}=P(-[\mathbf{U},\mathbf{V}]) \tag{4.18}$$

and

$$\{\mathbf{P}(\mathbf{U}),\mathbf{P}(\mathbf{V})\}=\mathbf{P}(-[\mathbf{U},\mathbf{V}])\ . \tag{4.19}$$

The mappings $\mathbf{U}\to P(\mathbf{U})$ and $\mathbf{U}\to\mathbf{P}(\mathbf{U})$ are thus homomorphisms from $L\ \mathrm{Diff}M$ into the Poisson-brackets algebra of the dynamical variables.

According to Eqs. (3.50) and (3.53) the statistical distribution function ρ and the Schrödinger field variables $\phi(x),\pi_1(x)$ satisfy the requirements

$$\{\rho,\mathbf{P}(\mathbf{U})\}=0 \tag{4.20}$$

and

$$\{\phi(x),P(\mathbf{U})\}=0=\{\pi_1(x),P(\mathbf{U})\}\ . \tag{4.21}$$

The dynamical variable $\mathbf{P}(\mathbf{U})$ expressed in terms of the Schrödinger data, Eq. (4.15), generates by Eq. (4.20) the evolution of the state $\rho[X,\phi,\pi]$ under an infinitesimal diffeomorphism $\mathbf{U}$ which displaces the embedding $X(x)$ into $X(x)+\mathbf{U}(X(x))$. Similarly, the dynamical variable $P(\mathbf{U})$ expressed in terms of the Heisenberg data, Eq. (4.16), generates by Eqs. (4.21) the evolution of the field variables $\phi(x)$ and $\pi_1(x)$ under an infinitesimal diffeomorphism which displaces the embedding $\mathbf{X}(x)$ into $\mathbf{X}(x)+\mathbf{U}(\mathbf{X}(x))$. The representation equations (4.18) and (4.19) guarantee the consistency of these two alternative evolutions.

One can restrict the canonical realizations (4.18) and (4.19) of the spacetime diffeomorphism algebra L Diff$M$ to one of its subalgebras LC or $L_{(\pm)}C$. We have already noticed that the action $\mathbf{U}\in L$ Diff$M$ at any given embedding can be composed from the actions $\mathbf{u}_{(\pm)}\in L_{(\pm)}C$. These remarks, however, do not capture the full importance of the subalgebras $L_{(\pm)}C$ for the dynamics of the scalar field in a two-dimensional spacetime M. The group of conformal isometries C and the related algebras $L_{(\pm)}C$ is not important as the *dynamical* group according to which the system is evolved, but rather as a *symmetry* group of diffeomorphism Hamiltonians. We shall explore this aspect of C in the following section.

C. Conformal isometries and constants of motion

We have seen that any dynamical variable F which has vanishing Poisson brackets with the constraints, Eq. (3.45), is a constant of motion, i.e., has the same value on any embedding. It is well known that any covariantly conserved symmetric trace-free tensor $T_{\alpha\beta}$ yields a conserved current when multiplied by a conformal Killing vector field u^α, i.e., that

$$\nabla_\beta(u^\alpha T^\beta_\alpha)=0 . \tag{4.22}$$

As a consequence of Eq. (4.22), the integral

$$\int_\Sigma dx^1 u^\alpha T_{\alpha\beta} n^\beta_1 \tag{4.23}$$

has the same value on any embedding $\mathbf{X}:\Sigma\to M$. When expressed in terms of the canonical variables, this integral is nothing other than the smeared Hamiltonian flux:

$$h(\mathbf{u}):=\int_\Sigma dx^1 u^\alpha(\mathbf{X}(x))h_{1\alpha}(x) . \tag{4.24}$$

One thus expects that the Poisson brackets of $h(\mathbf{u})$ with the constraints $\mathbf{P}_{1\alpha}(x)$ will vanish

$$\{h(\mathbf{u}),\mathbf{P}_{1\alpha}(x)\}=0 . \tag{4.25}$$

This can be corroborated in many different ways, e.g., from Eqs. (3.84) and (4.4). The most explicit method, however, is to express $h(\mathbf{u})$ directly in terms of the initial data. By Eqs. (3.8), (3.17), and (4.6),

$$\begin{aligned} h(\mathbf{u})= &\int_\Sigma dx^1[\mathbf{T}^+{}_{,1}(x)T_{++}(\mathbf{T}^+(x))u^+(\mathbf{T}^+(x)) \\ &\qquad -\mathbf{T}^-{}_{,1}(x)T_{--}(\mathbf{T}^-(x))u^-(\mathbf{T}^-(x))] \\ = &\int_{-\pi}^{\pi} dT^+T_{++}(T^+)u^+(T^+) \\ &+\int_{-\pi}^{\pi} dT^-T_{--}(T^-)u^-(T^-) . \end{aligned} \tag{4.26}$$

The dependence of $T_{\pm\pm}(T^\pm)$ on $T^\pm$ is fixed by Eqs. (3.8), (3.5), and (3.6); the last version of Eq. (4.26) then makes it clear that $h(\mathbf{u})$ depends only on $\mathbf{q},\mathbf{p}$ and $\mathbf{a}_k,\mathbf{a}^*{}_k$, but not on the embedding.

We thus see that each conformal Killing vector field $\mathbf{u}\in LC$ yields a dynamical variable $h(\mathbf{u})$ which is a constant of motion. The Poisson bracket of two such dynamical variables, $h(\mathbf{u})$ and $h(\mathbf{v})$, must again be a constant of motion. Indeed, by smearing Eq. (3.83), we learn that it is a constant of motion generated by the conformal Killing vector field $[\mathbf{u},\mathbf{v}]$:

$$\{h(\mathbf{u}),h(\mathbf{v})\}=h([\mathbf{u},\mathbf{v}]) . \tag{4.27}$$

The constants of motion $h(\mathbf{u})$ thus form a Poisson algebra which is antihomomorphic to the Lie algebra LC.

Let us take those constants of motion $h^{(\pm)}{}_m$ which correspond to the basis elements (4.10) of the Lie algebras $L_{(\pm)}C$. By Eq. (4.26),

$$\begin{aligned} h^{(\pm)}{}_m:&=h(\mathbf{L}^{(\pm)}{}_m) \\ &=\int_{-\pi}^{\pi} dT^\pm e^{-imT^\pm}T_{\pm\pm}(T^\pm;\mathbf{q},\mathbf{p},\mathbf{a}_k,\mathbf{a}^*{}_k) . \end{aligned} \tag{4.28}$$

By using Eqs. (3.8), (3.5) and (3.6), we express $h^{(\pm)}{}_m$ explicitly in terms of the Heisenberg data:

$$\begin{aligned} h^{(\pm)}{}_0&=\tfrac{1}{2}\mathbf{p}^2+\sum_{k=1}^{\infty}\mathbf{a}^*{}_{\mp k}\mathbf{a}_{\mp k} , \\ h^{(\pm)}{}_m&=i\frac{1}{\sqrt{2}}\mathbf{p}\mathbf{a}^*{}_{\mp m}-\tfrac{1}{2}\sum_{k=1}^{m-1}\mathbf{a}^*{}_{\mp k}\mathbf{a}^*{}_{\mp(m-k)} \\ &\quad+\sum_{k=1}^{\infty}\mathbf{a}^*{}_{\mp(k+m)}\mathbf{a}^*{}_{\mp k}, \quad m>0 \\ h^{(\pm)}{}_{-m}&=(h^{(\pm)}{}_m)^* . \end{aligned} \tag{4.29}$$

From Eqs. (4.12) and (4.27) it follows that these constants of motion have the Poisson brackets which close according to the Virasoro algebra:

$$\{h^{(\pm)}{}_m,h^{(\pm)}{}_n\}=i(m-n)h^{(\pm)}{}_{m+n} . \tag{4.30}$$

There is a fundamental difference between Eqs. (4.18) and (4.19), and Eq. (4.27). Equations (4.18) and (4.19) are valid for *arbitrary* vector fields $\mathbf{U}$, $\mathbf{V}$, and the dynamical variables $P(\mathbf{U})$ or $\mathbf{P}(\mathbf{V})$ thus represent the full diffeomorphism algebra L Diff$M$. On the other hand, Eq. (4.27) is valid only for *conformal* Killing vector fields $\mathbf{u},\mathbf{v}$ and the dynamical variables $h(\mathbf{u})$ thus represent merely the conformal algebra LC (which is a subalgebra of L Diff$M$).

We can connect the representation equations (4.18), (4.19) and (4.27) with the conservation equation (4.25) by the following chain of equations which are valid for arbitrary vector fields $\mathbf{U}$ and $\mathbf{V}$:

$$\mathbf{P}(-[\mathbf{U},\mathbf{V}])-h(-[\mathbf{U},\mathbf{V}])=P(-[\mathbf{U},\mathbf{V}])=\{\mathbf{P}(\mathbf{U})-h(\mathbf{U}),\mathbf{P}(\mathbf{V})-h(\mathbf{V})\}$$
$$=\{\mathbf{P}(\mathbf{U}),\mathbf{P}(\mathbf{V})\}+\{h(\mathbf{U}),h(\mathbf{V})\}+(\{h(\mathbf{U}),\mathbf{P}(\mathbf{V})\}-(\mathbf{U}\leftrightarrow\mathbf{V}))$$
$$=\mathbf{P}(-[\mathbf{U},\mathbf{V}])+\{h(\mathbf{U}),h(\mathbf{V}\}+(\{h(\mathbf{U}),\mathbf{P}(\mathbf{V})\}-(\mathbf{U}\leftrightarrow\mathbf{V}))\ . \tag{4.31}$$

As a result,

$$h[\mathbf{U},\mathbf{V}])=\{h(\mathbf{U}),h(\mathbf{V})\}+(\{h(\mathbf{U}),\mathbf{P}(\mathbf{V})\}-(\mathbf{U}\leftrightarrow\mathbf{V})) \tag{4.32}$$

for arbitrary $\mathbf{U}$, $\mathbf{V}\in L\,\mathrm{Diff}M$. For conformal Killing vector fields $\mathbf{u}$ and $\mathbf{v}$, $h(\mathbf{u})$ and $h(\mathbf{v})$ are conserved, Eq. (4.25), and the representation equation (4.27) follows from Eq. (4.32). On the other hand, for arbitrary vector fields $\mathbf{U}$ and $\mathbf{V}$, $h(\mathbf{U})$ and $h(\mathbf{V})$ are not conserved, and the relation (4.32) implies that the representation equation (4.27) is in general violated.

Two dynamical variables F_1 and F_2 on the extended phase space are considered to be equivalent (weakly equal) to each other, $F_1\approx F_2$, if they coincide on the constraint surface $\mathbf{P}_{1\alpha}(x)=0$. The constraints themselves are equivalent to the zero dynamical variable, $\mathbf{P}_{1\alpha}(x)\approx 0$. By Eq. (3.47), the smeared Hamiltonian flux $-h(\mathbf{U})$, $\mathbf{U}\in L\,\mathrm{Diff}M$, is equivalent to the smeared Schrödinger momentum $P(\mathbf{U})$: $P(\mathbf{U})\approx -h(\mathbf{U})$. For $\mathbf{u}\in LC$, smeared Schrödinger momentum is weakly conserved,

$$\{P(\mathbf{u}),\mathbf{P}(\mathbf{V})\}=\mathbf{P}(-[\mathbf{u},\mathbf{V}])\approx 0 \quad \forall\mathbf{u}\in LC \text{ and } \mathbf{V}\in L\,\mathrm{Diff}M\ . \tag{4.33}$$

In the Schrödinger picture, this equation,

$$\{\mathbf{P}(\mathbf{V})[X,P,\phi,\pi],P(\mathbf{u})\}\approx 0\ , \tag{4.34}$$

has an alternative interpretation. The dynamical variable $P(\mathbf{u})$, $\mathbf{u}\in LC$, generates a conformal motion in the Schrödinger kinematical sector $X(x)$, $P_{1\alpha}(x)$ of the phase space while leaving the Schrödinger field variables $\phi(x),\pi_1(x)$ unchanged. By Eq. (4.33), the smeared constraint $\mathbf{P}(\mathbf{V})[X,P,\phi,\pi]$ (which generates the evolution of the state under an infinitesimal diffeomorphism $\mathbf{V}\in L\,\mathrm{Diff}M$ and thus plays the role of the Hamiltonian) is left weakly invariant under this conformal motion. We say that this Hamiltonian is *conditionally symmetric*[17] under a conformal motion generated by $P(\mathbf{u}),\mathbf{u}\in LC$. One can also see that the parametrized canonical action functional (3.22)–(3.24) is left weakly invariant under the conformal motion generated by $P(\mathbf{u})$ (Ref. 17). It is thus the (conditional) symmetry of the dynamical Hamiltonian or of the corresponding canonical action which leads to the conservation laws for the generators $P(\mathbf{u})\approx -h(\mathbf{u})$ of this symmetry.

To summarize, the group of conformal motions plays the role of the symmetry group, the group of all spacetime diffeomorphisms the role of the dynamical group. The elements of LC are homomorphically mapped into the generators $P(\mathbf{u})$ of conformal motions in the phase space, and the elements of $L\,\mathrm{Diff}M$ are mapped into the diffeomorphism Hamiltonians $\mathbf{P}(\mathbf{V})$. By Eq. (4.34), the diffeomorphism Hamiltonians are left (conditionally) invariant by conformal motions and hence the generators of such motions are (conditionally) conserved in the dynamical evolution of the system.

We shall see that the Dirac constraint quantization leaves the operator version of Eqs. (4.18) and (4.19) untouched, but the generators $P(\mathbf{u}),\mathbf{u}\in LC$ of conformal motions no longer keep the diffeomorphism Hamiltonians $\mathbf{P}(\mathbf{V})$, $\mathbf{V}\in L\,\mathrm{Diff}M$ conditionally invariant: Eq. (4.34) breaks down. Quantum mechanically, the diffeomorphism Hamiltonians are left conditionally invariant by a different group (which is a central extension of C). The symmetry group which leaves the diffeomorphism Hamiltonians invariant thus develops an anomaly upon quantization, but the dynamical group of the quantum diffeomorphism Hamiltonians themselves does not. Fortunately, it is the commutator algebra of quantum diffeomorphism Hamiltonians, not the algebra of the symmetry generators, which ensures the consistency of the Dirac constraint quantization. We shall pursue this theme in the following paper.

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APPENDIX A: THE MAINARDI EQUATION

The Mainardi equation for a hypersurface $X^\alpha=X^\alpha(x^a)$, $\alpha=1,\dots,n$, $a=1,\dots,n-1$, embedded in an n-dimensional spacetime connects the normal derivative of the extrinsic curvature $K_{ab}(x;X]$ to the $R_{\perp a\perp b}:=R_{\mu\alpha\nu\beta}n^\mu X^\alpha_a n^\nu X^\beta_b$ projection of the spacetime curvature tensor $R_{\mu\alpha\nu\beta}$:

$$n^\gamma\partial_\gamma K_{ab}=(R_{\perp a\perp b}(x)-K^c_a(x)K_{cb}(x))\ . \tag{A1}$$

In terms of the variational derivatives,[8]

$$(\delta K_{ab}(x;X]/\delta X^\gamma(x'))n^\gamma(x';X] =(R_{\perp a\perp b}(x)-K^c_a(x)K_{cb}(x))\delta(x,x')-D_aD_b\delta(x,x')\ . \tag{A2}$$

The extrinsic curvature is in its turn given by the normal variational derivative of the metric:

$$(\delta g_{ab}(x;X]/\delta X^{\gamma}(x'))n^{\gamma}(x';X]=-2K_{ab}(x)\delta(x,x') \ . \tag{3}$$

In a flat two-dimensional spacetime, Eqs. (A2) and (A3) reduce to

$$(\delta K_{11}(x)/\delta X^{\alpha}(x'))n^{\alpha}(x') = -K(x)K_{11}(x)\delta_{1'}(x,x')-D_1D_1\delta_{1'}(x,x') \tag{A4}$$

and

$$(\delta g_{11}(x)/\delta X^{\alpha}(x'))n^{\alpha}(x')=-2K_{11}(x)\delta_{1'}(x,x') \ . \tag{A5}$$

From them we obtain the form of the Mainardi equation which is most useful for our purposes: namely,

$$(\delta g_1(x)K(x)/\delta X^{\alpha}(x'))n^{\alpha}(x')=-g_1(x)\Delta\delta_{1'}(x,x') \ . \tag{A6}$$

APPENDIX B: SPACETIME AND SPATIAL δ FUNCTIONS

The Poisson brackets of the null components of various spacetime tensors in the Heisenberg picture yield the δ functions $\delta(T^{\pm}-T^{\pm'})$ of the null coordinates or their $\partial_{\pm}$ derivatives. When projecting these on a spatial embedding we need to express the results in terms of the spatial δ functions $\delta_{1'}(x,x')$ and their spatial derivatives. We give a sequence of identities which help us to deal with this problem.

First of all,

$$\delta_{1'}(x,x')=\delta(T^{\pm}(x)-T^{\pm}(x'))(\pm T^{\pm}{}_{,1'}(x')) \ . \tag{B1}$$

We can verify this identity by smearing it by a scalar function $N(T^{\pm}(x'))$ and changing the $x^{1'}$ integration into a $T^{\pm'}$ integration [the $\pm$ sign comes in because $T^{+}(x'=\pm\pi)=T\pm\pi$, but $T^{-}(x'=\pm\pi)=T\mp\pi$].

By differentiating Eq. (B1) we obtain

$$\delta_{1',1}(x,x')=\delta_{,\pm}(T^{\pm}(x)-T^{\pm}(x'))(\pm T^{\pm}{}_{,1}(x)T^{\pm}{}_{,1'}(x')) \ . \tag{B2}$$

To differentiate further and preserve the spatial covariance of the formulas we must use covariant derivatives. In case of the $\delta(T^{\pm}(x)-T^{\pm}(x'))$ functions it is advantageous to use the $D^{(\pm)}{}_1$ derivatives. When applied to a density (a covector),

$$(T^{\pm}{}_{,1}(x))^{-1}D^{(\pm)}{}_1D^{(\pm)}{}_1\delta_{1'}(x,x') =\delta_{,\pm\pm}(T^{\pm}(x)-T^{\pm}(x'))(\pm T^{\pm}{}_{,1}(x)T^{\pm}{}_{,1'}(x')) \ . \tag{B3}$$

The process can easily be continued. Thus, for the third derivative,

$$(T^{\pm}{}_{,1}(x))^{-2}(D^{(\pm)}{}_1)^3\delta_{1'}(x,x') =\delta_{,\pm\pm\pm}(T^{\pm}(x)-T^{\pm}(x'))(\pm T^{\pm}{}_{,1}(x)T^{\pm}{}_{,1'}(x')) \ . \tag{B4}$$

This is the Schwinger term which appears in the commutator of the null components of the energy-momentum tensor.

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