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Evolution of superconducting cosmic loops

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The evolution of superconducting cosmic loops is studied numerically. The back reaction of the electromagnetic radiation on the string and the current is considered. It is shown that the infinitely thin string does not lead to a "renormalizable" back reaction, contrary to the case of a pointlike charge. However, a tentative formula for the electromagnetic back reaction is derived and several arguments are given to justify it. The full set of equations of motion for the string and the current is solved, both with and without the back reaction of the radiation. I discuss the stability of the current, the electromagnetic radiation, and the rate of particle production near the string. The main result is that the high-frequency modes of the current are very rapidly excited and the further evolution seems to be chaotic. This conclusion is based on the study of the Fourier spectrum of the current and an approximate computation of the largest Liapunov exponent, which turned out to be positive. For the current $j \approx 0.005 j_{\max}$ particle production is the main mechanism of the dissipation of energy for small loops of size $L \le 10^7$ cm.

I. INTRODUCTION

Superconducting cosmic strings have attracted the attention of many physicists and astronomers since their very conception.[1] This is not surprising since their peculiar properties and very rich phenomenology can be used to explain, or to try to explain, many phenomena. However, despite the growing popularity of superconducting strings very little is known about their motion. This is a rather strange state of affairs—since the time of Newton it has been commonly accepted that knowledge of any physical system starts from the study of its evolution. On the other hand, it is not very surprising because the dynamics of superconducting strings is rather complicated. However, this fact should not be used to explain away the call for action. This paper is conceived as a preliminary report of the investigation of the dynamics of superconducting loops.

It is of course worthwhile to maintain good tradition in physics but there are also other good reasons to study the behavior of the superconducting loops.

(1) Solutions obtained for ordinary strings display some pathological features such as cusps and kinks. These pathologies should be cured by the inertia of the charge carriers, reduction of the string tension caused by the current, and, last but not least, the effects of the back reaction of the radiation.

(2) Superconducting loops may exist as stable rings with the string tension balanced by the current. If such a state is the generic end point of the evolution of a generic loop then it seems that their existence is ruled out by the experimental evidence.

(3) Superconducting loops may be unstable against the exponential growth of the current.

(4) It is necessary to know reasonably well the electromagnetic field around the loop in order to compute the rate of particle production due to the vacuum instability.

(5) Astrophysical applications of superconducting loops usually require a good knowledge of the angular and spectral distribution of their electromagnetic radiation.

The most fundamental approach to the problem of motion is to solve the full set of field equations describing the "thick" superconducting vortex. Several authors have used this approach to study the internal structure of static superconducting strings;[2–5] also Matzner and Laguna[6] used this method to study the intercommutation of the strings. Unfortunately, this approach is too difficult in general. The more modest approach is to replace the fundamental action of the superconducting vortex by the effective Nambu action of the infinitely thin string with a current defined on its world sheet. This method significantly simplifies the dynamics but one should stress that the problem of deriving the Nambu action from the underlying fundamental action expressed in terms of the microphysical fields has not been completely solved yet.[7,8]

The Nambu action for an ordinary cosmic string leads to the simple wave equation for the world-sheet vector $Z^{\mu}(\tau,\sigma)$. However, since the solution must satisfy two constraints it is not all that easy to find some analytic solutions. Despite these difficulties, several families of solutions[9,10] are known. The similar problem for a superconducting loop is much harder since $Z^{\mu}(\tau,\sigma)$ is coupled to the scalar field ϕ describing the current and to the electromagnetic field. It is clear that the analytic approach is either hopeless or reserved for a mathematical genius; here I shall use purely numerical means.

As I have already said, this is a preliminary report. In Sec. II I derive the equations of motion for $Z^{\mu}(\tau,\sigma)$ and $\phi(\tau,\sigma)$. These equations are known from the work of other authors[11] but I include their derivation for the completeness of the paper and to fix the notation and the normalization. In Sec. III I give the derivation of the formula describing the back reaction of the electromagnetic radiation on the loop. Section IV is devoted to the numerics: I describe the numerical methods and the results obtained. In this section I also discuss the validity of the mechanical approach and compare the results of this work with the recent work of Spergel, Press, and Scherrer.[12] Finally, in Sec. V I compute the rate of particle creation from the vacuum in the vicinity of the loop.

II. EQUATIONS OF MOTION

The basic variables for the infinitely thin superconducting string are the string world-sheet vector $Z^\mu(\xi)$ and the scalar field $\phi(\xi)$. ξ^a are coordinates on the world sheet of the string. Usually I shall use τ for the time coordinate ξ^0 and σ for the spatial coordinate ξ^1. Therefore $-\infty \le \tau \le +\infty$ and $0 \le \sigma \le L$. The two-dimensional metric induced on the world sheet of the string is

$$h_{ab}=\partial_a Z^\mu(\xi)\partial_b Z_\mu(\xi) \ . \tag{2.1}$$

The determinant of h_{ab} will be denoted by h. The scalar field $\phi(\xi)$ is related to the two-dimensional current density j^a:

$$j^a(\xi)=q e^{ab}\partial_b\phi(\xi) \ ; \tag{2.2}$$

where q is the charge and e^{ab} is the antisymmetric tensor,

$$e^{ab}=\frac{\epsilon^{ab}}{\sqrt{-h}} \ ;$$

e^{ab} is the antisymmetric symbol ($\epsilon^{01}=1$).

The Nambu action written in terms of these variables is

$$S_N=\int d^2\xi\sqrt{-h}\,(-T+\tfrac{1}{2}h^{ab}\partial_a\phi\partial_b\phi-j^a\partial_a Z^\mu A_\mu) \ , \tag{2.3}$$

where T is the string tension and A_μ is the electromagnetic potential. The derivation of the equations of motion is a simple exercise in the variation calculus. The equation of motion for $Z^\mu(\xi)$ is

$$\frac{1}{\sqrt{-h}}\partial_a\{\sqrt{-h}\,[h^{ab}(T-\tfrac{1}{2}\partial_c\phi\partial^c\phi)+\partial^a\phi\partial^b\phi]\partial_b Z^\mu\} - q e^{ab}\partial_a\phi\partial_b Z^\nu F_\nu{}^\mu=0 \tag{2.4}$$

and for the scalar field

$$\frac{1}{\sqrt{-h}}\partial_a(\sqrt{-h}\,h^{ab}\partial_b\phi)+\tfrac{1}{2}q e^{ab}\partial_a Z^\mu\partial_b Z^\nu F_{\mu\nu}=0 \ . \tag{2.5}$$

Equations (2.4) and (2.5) are invariant under the change of coordinates on the world sheet. To solve them one has to fix the coordinates. A convenient choice is the so-called conformal gauge. In this gauge the induced metric h_{ab} is proportional everywhere to the two-dimensional flat metric with the signature $(+,-)$. The conformal gauge is specified by two conditions:

$$\dot Z^\mu Z'_\mu=0, \quad \dot Z^\mu \dot Z_\mu+Z'^\mu Z'_\mu=0 \ ,$$

where the overdot denotes derivative with respect to τ and the prime derivative with respect to σ. In the conformal gauge the equations of motion for $Z^\mu(\tau,\sigma)$ and $\phi(\tau)$ take the form

$$\frac{\partial}{\partial\tau}\left[\left(T+\frac{\dot\phi^2+\phi'^2}{\dot Z^\nu \dot Z_\nu}\right)\frac{\partial Z^\mu(\tau,\sigma)}{\partial\tau}-\frac{\dot\phi\phi'}{\dot Z^\nu\dot Z_\nu}\frac{\partial Z^\mu(\tau,\sigma)}{\partial\sigma}\right]-\frac{\partial}{\partial\sigma}\left[\left(T-\frac{\dot\phi^2+\phi'^2}{\dot Z^\nu\dot Z_\nu}\right)\frac{\partial Z^\mu(\tau,\sigma)}{\partial\sigma}+\frac{\dot\phi\phi'}{\dot Z^\nu\dot Z_\nu}\frac{\partial Z^\mu(\tau,\sigma)}{\partial\tau}\right] - q\epsilon^{ab}\partial_a\phi\partial_b Z^\nu(\tau,\sigma)F_\nu{}^\mu=0 \ , \tag{2.6}$$

and

$$\frac{\partial^2\phi}{\partial\tau^2}-\frac{\partial^2\phi}{\partial\sigma^2}+\tfrac{1}{2}q\epsilon^{ab}\partial_a Z^\mu(\tau,\sigma)\partial_b Z^\nu(\tau,\sigma)F_{\mu\nu}=0 \ . \tag{2.7}$$

It is worth stressing that in the absence of any interactions both equations reduce to the simple two-dimensional wave equation. This point will be important later.

The electromagnetic tensor $F_{\mu\nu}$ is the sum of any external field present and the electromagnetic field of the loop. The latter can be divided into two parts: at every point on the loop there are contributions to the electromagnetic field from all other points and also the field due to self-interaction at this point. The first part can be readily calculated using the retarded Green's function. The derivation follows closely the standard reasoning for a moving charge.[13] The four-dimensional current density is related to the current density on the world sheet by

$$J^\mu(x)=\int d^2\xi\sqrt{-h}\,j^a(\xi)\partial_a Z^\mu(\xi)\delta^{(4)}(x^\mu-Z^\mu(\xi)) \ . \tag{2.8}$$

It is convenient to introduce an abbreviation

$$c^\mu(\tau,\sigma)=\sqrt{-h}\,j^a(\tau,\sigma)\partial_a Z^\mu(\tau,\sigma) \ , \tag{2.9}$$

so the electromagnetic potential of the current in the string can be written as

$$A^\mu(x)=\frac{1}{2\pi}\int d\tau\, d\sigma\, c^\mu(\tau,\sigma)\theta(x^0-Z^0(\tau,\sigma)) \times\delta([x^\nu-Z^\nu(\tau,\sigma)]^2) \ . \tag{2.10}$$

Now it is straightforward to compute the derivatives with respect to x^ν and obtain the electromagnetic tensor

$$F^{\mu\nu}(x)=\frac{1}{2\pi}\int\frac{d\sigma}{n\cdot\dot Z}\left[\frac{1}{R(\tau,\sigma)}\left[-\frac{\dot c^{[\mu}n^{\nu]}}{n\cdot\dot Z}+\frac{c^{[\mu}n^{\nu]}}{(n\cdot\dot Z)^2}n\cdot\ddot Z\right]+\frac{1}{R^2(\tau,\sigma)}\left[\frac{c^{[\mu}\dot Z^{\nu]}}{n\cdot\dot Z}-\frac{c^{[\mu}n^{\nu]}}{(n\cdot\dot Z)^2}\dot Z\cdot\dot Z\right]\right]_{\rm ret} \ , \tag{2.11}$$

where n^{ν} is a null vector of the form $(1,\mathbf{n})$ and

$$x^{\nu}-Z^{\nu}(\tau,\sigma)=n^{\nu}R(\tau,\sigma) \ .$$

The subscript ret means that the integrand should be evaluated at the retarded time $\tau_{\rm ret}$ such that

$$t_x-Z^0(\tau_{\rm ret},\sigma)=|\mathbf{x}-\mathbf{Z}(\tau_{\rm ret},\sigma)| \ . \qquad (2.12)$$

The expression (2.11) is independent of the choice of coordinates on the world sheet. It also displays clearly the separation of the "velocity" and "acceleration" fields in the same manner as it is usually done for the pointlike charge. The only problem with the formula (2.11) is that the integrand becomes singular if the point x is on the world sheet of the string. This singularity has an obvious source—it is due to the self-interaction of the charges circulating around the loop. The standard solution of this problem is to impose a cutoff so that the singularity is avoided. This may be right from the practical point of view provided that one is not interested in the long time evolution of the string. Otherwise one has to consider carefully the limit as the cutoff goes to zero and to extract the finite part that describes the back reaction of the electromagnetic radiation on the loop. I shall discuss this problem in the next section.

Expression (2.11) can be used to calculate the angular distribution and the spectrum of the emitted radiation. To this end one has to make the Fourier transformation with respect to the time t_x:

$$F^{\mu\nu}(\omega,\mathbf{n})=\int dt_x e^{-i\omega t_x}F^{\mu\nu}(t_x,\mathbf{n}) \ .$$

For a distant observer $t_x=r+n\cdot Z$, where r is the fixed distance from the observer to the center of the loop and n does not depend on time. Consequently,

$$F^{\mu\nu}(\omega,\mathbf{n})=\frac{-i\omega e^{-i\omega r}}{2\pi r}\int d\tau\, d\sigma\, c^{[\mu}n^{\nu]}e^{-i\omega n\cdot Z(\tau,\sigma)} \ .$$

Now one can easily derive the formula for the angular and spectral distribution of the radiated energy

$$\frac{d^2E_{\rm rad}}{d\omega\, d\Omega}=\frac{\omega^2}{16\pi^2}\left|\int d\tau\, d\sigma\, \mathbf{n}\times\mathbf{c}e^{-i\omega n\cdot Z(\tau,\sigma)}\right|^2 . \qquad (2.13)$$

This result will be used later. Now let me discuss the back reaction of the radiation on the loop.

III. BACK REACTION OF THE RADIATION ON THE LOOP

The problem of the back reaction of the electromagnetic radiation on its source belongs to the small class of problems that are never solved definitely but only more or less so. For the pointlike particle this problem was more or less solved by Dirac[14] in the classic paper of 1938. Subsequent papers by others dealt mainly with the interpretation and the validity of the resulting equation of motion. Here I cannot review the entire history of this discussion[15] but I would like to recall very briefly the derivation offered by Dirac.

The basic idea of Dirac is to use the energy-momentum conservation and make the balance between the momentum of the electron and the momentum carried away by radiation. To this end he considers the integral of the $T^{\mu\nu}_{\ ;\nu}$ over the volume of spacetime around the world trajectory of the particle. The chosen volume is limited by a timelike tube of a very small radius ϵ that surrounds the trajectory, a second timelike tube of a very large radius R and the two "caps"—spacelike surfaces orthogonal to the trajectory at times τ and $\tau+d\tau$. Using the Gauss theorem one can replace the volume integral by the integral over the limiting surfaces. Because of the energy conservation $T^{\mu\nu}_{\ ;\nu}=0$ the integral must vanish and one gets

$$\int_{\sigma(\tau+d\tau)}T^{\mu\nu}v_\nu d^3\sigma-\int_{\sigma(\tau)}T^{\mu\nu}v_\nu d^3\sigma =-\int_{\Sigma(R)}T^{\mu\nu}n_\nu d^3\sigma-\int_{\Sigma(\epsilon)}T^{\mu\nu}n_\nu d^3\sigma \ . \qquad (3.1)$$

In the limit $d\tau\to 0$ the surface integrals over timelike surfaces are replaced by two integrals over spheres with the radii R and ϵ, and one gets

$$\frac{dP^\mu}{d\tau}=-\int_{\bar\Sigma(R)}T^{\mu\nu}n_\nu d^2\sigma-\int_{\bar\Sigma(\epsilon)}T^{\mu\nu}n_\nu d^2\sigma \ , \qquad (3.2)$$

where P^μ is the momentum of the electromagnetic field bound to the particle. Next one wants to take the limits $R\to\infty$ and $\epsilon\to 0$. It is possible to show that the integral over the large circle vanishes if the motion of the particle was uniform in the far past.[16] The integral over the small circle is harder. To compute it one should expand the retarded electromagnetic field in the neighborhood of the particle's trajectory in the powers of the separation and compute the corresponding expansion of the stress-energy tensor. After integration one gets

$$\frac{dP^\mu}{d\tau}=\left[-\frac{e^2\dot v^{\mu}}{8\pi\epsilon}+\frac{e^2}{4\pi}\left(\tfrac{2}{3}\ddot v^{\mu}-\tfrac{2}{3}v\cdot\ddot v v^\mu\right)\right] , \qquad (3.3)$$

P^μ is to be interpreted as the electromagnetic contribution to the momentum of the particle. To obtain the equation of motion for the particle one should add to P^μ the bare momentum of the particle m_0v^μ. Then the divergent term can be absorbed in the renormalized mass of the particle $m_{\rm phys}=m_0+e^2/8\pi\epsilon$. The resulting equation of motion is

$$\frac{d}{ds}(m_{\rm phys}v_\mu)=\frac{e^2}{4\pi}\left(\tfrac{2}{3}\ddot v^{\mu}-\tfrac{2}{3}v\cdot\ddot v v^\mu\right) \ . \qquad (3.4)$$

This equation is generally considered as the correct one for a classical pointlike particle though there are still differences of opinions whether it should be considered as the exact equation or merely as an approximation.[17] I should like to stress several points that will be important later.

(1) The theory of pointlike charges is "classically renormalizable," that is all infinities are absorbed into the physical parameters.

(2) The result is not invariant with respect to the change of the parameter along the trajectory. Of course, for a single particle the proper time is the "right" choice of the parameter but since for an extended object the proper time is not well defined it is a useful exercise to derive Dirac-Lorentz equation in an arbitrary gauge.

The computation is considerably more tedious since in an arbitrary gauge the velocity is no longer orthogonal to the acceleration nor is it normalized to one. In the end one gets

$$m\frac{\partial}{\partial\xi}\left[\frac{v^\mu}{|v|}\right]=\frac{e^2}{4\pi}\left[\tfrac{1}{3}(1-\tfrac{9}{4}|v|^3)v\cdot\dot{v}\frac{\partial}{\partial\xi}\left[\frac{v^\mu}{|v|}\right]+\tfrac{2}{3}\left[|v|\ddot{v}^\mu-\frac{v\cdot\ddot{v}}{|v|}v^\mu\right]\right], \quad (3.5)$$

where $v^\mu=(\partial/\partial\xi)Z^\mu(\xi)$ and $|v|=(v_\mu v^\mu)^{1/2}$. One can easily see that if ξ is set to be the proper time s then the Dirac-Lorentz equation is recovered.

(3) As Teitelboim[18] pointed out, the same equation can be derived with the help of an "averaged retarded field." The averaging is performed over a small sphere $\tilde{\Sigma}(\epsilon)$ centered on the particle at z

$$\bar{F}^{\mu\nu}_{\rm ret}(z)=\lim_{\epsilon\to 0}\frac{1}{4\pi\epsilon^2}\int_{\tilde{\Sigma}(\epsilon)}d^2\sigma\, F^{\mu\nu}_{\rm ret}(z+\epsilon u^\alpha)\ , \quad (3.6)$$

where u^α is a unit vector from z toward the surface of the sphere. One can check that this definition leads to the equation of motion written above. The same singular term appears as before and is absorbed into the physical mass of the particle.

Now let me turn to the real problem of calculating the back reaction of the radiation on the string. There are several important points to discuss, the first one being where to start. The most fundamental and obvious approach to this problem is to consider the back reaction on each of the discreet charges separately and then sum up their contribution. This approach shows that the fundamental theory of superconducting strings is certainly renormalizable in the same way as the theory of pointlike charges is. However, it is equally clear that this approach cannot be implemented until we learn more about the derivation of the Nambu action from the real one—at present it is not known how to sum up or average the contributions of all charges in the way that is consistent with the transition from the fundamental action to the effective one. Therefore the starting point must be the Nambu action. This choice has one serious drawback: the classical renormalizability is no longer assured. This is rather obvious. While deriving the effective action from the action expressed in terms of the microphysical terms one throws away many terms, which are judged to be irrelevant by some dimensional arguments. However, these terms may turn out necessary to absorb infinities, so without them the theory may become nonrenormalizable. The question now is whether one should worry. The answer seems to be *no.* The same dimensional arguments that were used to throw away these bare terms should apply to them upon renormalization. In other words, one should not care about the renormalization of the terms that are neglected anyway. If any infinities should appear that cannot be absorbed they should be simply discarded since they only contribute to the renormalization of the not essential terms.

The next point is that it is no good to compute the flow of energy and momentum through two timelike surfaces on the both sides of the string world sheet. Such a computation would give at best the change of the total energy and momentum of the string, but would not tell anything about the local forces of the back reaction acting at any particular point of the string. Therefore one has to define somehow the local sources of radiation. This can be done by dividing the string into small segments $\Delta\sigma$ and drawing the timelike tube around each of the segments separately. Next one can try to repeat Dirac's reasoning and compute the forces acting on each segment separately. The resulting equation of motion would have the form of the coarse-grained equation. More pictorially, for the computation of the back reaction the string is being replaced by a necklace of pointlike charged beads and the conducting string. The string that connects the beads must be conducting to allow for the flow of charge from one bead to the other, so the current density at a given point is due to the motion of the charge carriers and to the change of their density.

The third point is that the original reasoning of Dirac cannot be directly applied here: even if I computed the flow of energy and momentum through the tube around any given segment I would not know how to distribute the change of momentum between the string and the charge carriers. This problem can be solved with the help of the method proposed by Teitelboim.[18] Instead of calculating the energy flow I compute the averaged retarded field and add it to the field due to all other points on the string.

Therefore, the starting point of the calculation is the expression for the retarded field in the vicinity of the segment $\Delta\sigma_i$:

$$F^{\mu\nu}(x)=\frac{1}{2\pi}\frac{\Delta\sigma_i}{(x-Z)\cdot\dot{Z}}\frac{\partial}{\partial\tau}\left[\frac{c^{[\nu}(x-Z)^{\mu]}}{(x-Z)\cdot\dot{Z}}\right]. \quad (3.7)$$

The next step is to expand the integrand in powers of the separation $x-Z$. The expansion is considerably more tedious than in the case of the pointlike particle because τ is not the proper time and the time derivatives act both on the velocity and the charge density. Then one has to integrate over the sphere around the point $Z^\mu(\tau,\sigma)$ in order to compute the averaged retarded field as defined by Eq. (3.6). The finite part of the result is

$$\bar{F}^{\mu\nu}_{\rm ret}=\frac{\Delta\sigma_i}{2\pi}\left[\left[-\tfrac{4}{3}v\cdot\ddot{v}-\dot{v}^2+\frac{27(v\cdot\dot{v})^2}{4|v|^2}+\frac{9(v\cdot\dot{v})^2}{4|v|^6}\right]\frac{1}{|v|^2}c^{[\mu}v^{\nu]}+\frac{v\cdot\dot{v}}{2|v|^6}(1-9|v|^3)\dot{c}^{[\mu}v^{\nu]}-\frac{v\cdot\dot{v}}{4|v|^6}(2+9|v|^3)c^{[\mu}\dot{v}^{\nu]}+\frac{1}{|v|}(\ddot{c}^{[\mu}v^{\nu]}+\dot{c}^{[\mu}\dot{v}^{\nu]}+\tfrac{1}{2}c^{[\mu}\ddot{v}^{\nu]})\right], \quad (3.8)$$

and the divergent part is

$$\bar{F}^{\mu\nu}_{\rm div}=\frac{\Delta\sigma_i}{2\pi}\frac{1}{\epsilon|v|^3}(v^{[\mu}\dot{c}^{\,\nu]}+\tfrac{1}{2}\dot{v}^{\,[\mu}c^{\nu]}) \ . \tag{3.9}$$

The separation of the divergent and finite parts is equivalent to choosing a "subtraction scheme." Here I use the same scheme as used in the case of a pointlike particle. In fact one can replace the current c^μ by eu^μ (that means fixing the charge density) and check that the correct equation is recovered. The divergent terms cannot be absorbed into terms already present in the equations of motion derived from the Nambu action. At this point one may either conclude that the Nambu action cannot be used to compute the long time behavior of the loop or accept the arguments sketched above and discard the divergent terms altogether. In this case the question arises whether it is consistent to retain the finite terms given above while discarding other corrections to the Nambu action. The one possible argument is that the other corrections should lead to the equation of motion that conserves energy while the effects of the back reaction are dissipative and therefore should build up in time.

IV. NUMERICAL SOLUTIONS

The set of Eqs. (2.6) and (2.7) is too complicated for the analytic approach so one has to use a computer (and a big one). To prepare these equations for the machine it is convenient to introduce dimensionless variables. There are two length scales to play with, $\sqrt{T}$ and L, where L is the length of the loop. It is natural to define dimensionless coordinates $\tilde{\sigma}=\sigma/L$, $\tilde{\tau}=\tau/L$, and to rescale the fields $\tilde{Z}^\mu(\tau,\sigma)=Z^\mu(\tau,\sigma)/L$, $\tilde{\phi}=\phi/\sqrt{T}L$, and $\tilde{F}^{\mu\nu}=(L/\sqrt{T})F^{\mu\nu}$. The equations of motion become

$$\frac{\partial}{\partial\tau}\left[\left[1+\frac{\dot{\phi}^2+\phi'^2}{\dot{Z}^\nu\dot{Z}_\nu}\right]\frac{\partial Z^\mu(\tau,\sigma)}{\partial\tau}-\frac{\dot{\phi}\phi'}{\dot{Z}^\nu\dot{Z}_\nu}\frac{\partial Z^\mu(\tau,\sigma)}{\partial\sigma}\right] -\frac{\partial}{\partial\sigma}\left[\left[1-\frac{\dot{\phi}^2+\phi'^2}{\dot{Z}^\nu\dot{Z}_\nu}\right]\frac{\partial Z^\mu(\tau,\sigma)}{\partial\sigma}+\frac{\dot{\phi}\phi'}{\dot{Z}^\nu\dot{Z}_\nu}\frac{\partial Z^\mu(\tau,\sigma)}{\partial\tau}\right]-q\epsilon^{ab}\partial_a\phi\partial_b Z^\nu(\tau,\sigma)F_\nu{}^\mu=0 \ , \tag{4.1}$$

and

$$\frac{\partial^2\phi}{\partial\tau^2}-\frac{\partial^2\phi}{\partial\sigma^2}+\tfrac{1}{2}q\epsilon^{ab}\partial_a Z^\mu(\tau,\sigma)\partial_b Z^\nu(\tau,\sigma)F_{\mu\nu}=0 \ , \tag{4.2}$$

where all tildes have been dropped. These equations involve dimensionless quantities only. To get some feeling for the numbers one should realize that for the string that was formed at the grand unification energy scale $\sqrt{T}=\lambda_{\rm GUT}^{-1}\approx 10^{30}$ cm^{-1} and L, being the length of the loop, is also rather large.

The numerical integration of the equations of motion is not particularly easy. On the top of all common problems with nonlinear partial differential equations one has to deal with the cusps. At the cusps the product $\dot{Z}^\nu\dot{Z}_\nu\to 0$ so the equation for $Z^\mu(\tau,\sigma)$ becomes singular. It is true that the presence of the current should inhibit the occurrence of the true cusps, but from the numerical point of view $\dot{Z}^\nu\dot{Z}_\nu$ can become small enough to be called singularity. Moreover, as I shall show later, at the cusps the electromagnetic field becomes very strong so the source term in the wave equation for the scalar field becomes very large. To integrate successfully through the cusp one has to adjust carefully the time step of the code. The choice of the algorithm required some experimentation; in the end I used the explicit interlaced leapfrog method for the string equation and the implicit Richtmyer scheme for the scalar field. Both algorithms require the knowledge of all variables at n and $n-1$ time levels and the retarded electromagnetic field at the nth level to find the fields at the $n+1$ level. I used 256 points in the spatial direction σ, so the spatial step was $DS=L/256$. The generic time step was $DT=\frac{1}{4}DS$. Before making the next time step the largest electric and magnetic fields on the nth level are computed and the time step is adjusted according to the experimentally established rules. After every time step I check that $\dot{Z}^\nu\dot{Z}_\nu$ is positive at all points around the loop. If not, the code goes back to the previous time level and tries again with a smaller time step. The accuracy of the integration can be checked by looking at the constraints that define the conformal gauge. Generally both constraints are satisfied with the accuracy 10^{-4}; at some isolated points the errors jump to 10^{-3}. I also checked that no constant current is generated during the evolution.

The most tedious part of the computation is the evaluation of the retarded electromagnetic field. To calculate it, one has to find the intersections of the past light cones of all points around the loop with the world sheet of the string and then compute the integral (2.11). If one stores the entire computed history of the string and the current in the memory of the computer one is likely to exhaust the core memory of the machine very quickly. Moreover, it is not convenient to look for the intersections of the light cones with the world sheet if the latter is given at the grid points only. The better way is to use the fact that both $Z^\mu(\tau,\sigma)$ and ϕ satisfy simple wave equations, provided that the interactions are neglected. Consequently, one can decompose the exact solution in the modes of the free equation at an arbitrary time τ_0 and the amplitudes should vary slowly with time. Therefore, the Fourier series with the amplitudes computed at τ_0 should accurately represent the exact solution over some time interval centered at τ_0. After this time the Fourier transformation should be repeated to update the amplitudes. This way one has to store only the several sets of the Fourier amplitudes and one has a correct representation of string history between the grid points.

I shall always deal with the functions given at a discreet set of points on a circle so I should use the discreet Fourier transformation

$$f(\sigma)=\frac{1}{N}\sum_{n=-N/2}^{N/2} F_n \exp\left[-\frac{2\pi i n\sigma}{L}\right].$$

Consider the scalar field ϕ. I assume that $\phi=f(u)+g(v)$, where $u=\sigma-\tau$, $v=\sigma+\tau$, and f and g are some periodic functions. Denote by $H_k(\tau)$ the kth Fourier amplitude of the scalar field ϕ at the time τ. Obviously

$$H_k(\tau)=F_k e^{2\pi i k\tau/L}+G_k e^{-2\pi i k\tau/L}.$$

Similarly I can consider the decomposition of the τ derivative of ϕ. Let me denote the Fourier amplitudes of $\dot\phi$ by $\dot H_k(\tau)$. Now one can readily find the amplitudes F_k and G_k:

$$F_k=\frac{1}{2}\left[H_k+\frac{L}{2\pi i k}\dot H_k\right]e^{-2\pi i k\tau/L},$$
$$G_k=\frac{1}{2}\left[H_k-\frac{L}{2\pi i k}\dot H_k\right]e^{+2\pi i k\tau/L}, \tag{4.3}$$

for $k\neq 0$. The zero-frequency amplitudes have to be evaluated separately. Since the field ϕ is real one should use this fact in the Fourier decomposition. Standard manipulations lead to the expansion

$$\phi(\sigma,\tau)=H_0(\tau_0)+\sum_{k=1}^{N/2}\left[RF_k(\tau_0)\cos\frac{2\pi k u}{L}+IF_k(\tau_0)\sin\frac{2\pi k u}{L}+RG_k(\tau_0)\cos\frac{2\pi k v}{L}+IG_k(\tau_0)\sin\frac{2\pi k v}{L}\right], \tag{4.4}$$

where

$$RF_k(\tau)=\left[\mathrm{Re}H_k+\frac{L}{2\pi k}\mathrm{Im}\dot H_k\right]\cos\frac{2\pi k\tau}{L}+\left[\mathrm{Im}H_k-\frac{L}{2\pi k}\mathrm{Re}\dot H_k\right]\sin\frac{2\pi k\tau}{L},$$

$$IF_k(\tau)=\left[\mathrm{Im}H_k-\frac{L}{2\pi k}\mathrm{Re}\dot H_k\right]\cos\frac{2\pi k\tau}{L}-\left[\mathrm{Re}H_k+\frac{L}{2\pi k}\mathrm{Im}\dot H_k\right]\sin\frac{2\pi k\tau}{L},$$

$$RG_k(\tau)=\left[\mathrm{Re}H_k-\frac{L}{2\pi k}\mathrm{Im}\dot H_k\right]\cos\frac{2\pi k\tau}{L}-\left[\mathrm{Im}H_k+\frac{L}{2\pi k}\mathrm{Re}\dot H_k\right]\sin\frac{2\pi k\tau}{L},$$

$$IG_k(\tau)=\left[\mathrm{Im}H_k+\frac{L}{2\pi k}\mathrm{Re}\dot H_k\right]\cos\frac{2\pi k\tau}{L}+\left[\mathrm{Re}H_k-\frac{L}{2\pi k}\mathrm{Im}\dot H_k\right]\sin\frac{2\pi k\tau}{L}.$$

Similar expansion can be easily written for each component of the string world-sheet vector $Z^\mu(\tau,\sigma)$. In practice I used 60 Fourier modes and the Fourier decomposition was done every $\frac{1}{8}$ of the period of the free string motion, so every period of the evolution is described by the eight sets of 60 Fourier amplitudes. To check this scheme I compared the exact solution with the one recovered from the Fourier decomposition at the end of the time interval for which a given set of Fourier amplitudes was computed. The agreement was always very good. I also made one short test run with the Fourier decomposition done every $\frac{1}{16}$ of the period; the results were in a very good agreement with the results obtained with the smaller set of Fourier amplitudes.

The Fourier decomposition gives the explicit representation for all fields at arbitrary point (σ,τ). Therefore one can use Newton method to find the intersections of the past light cones with the world sheet. However, even this procedure is very slow since it requires several iterations and the summation of the Fourier series is rather time consuming. For this reason I restricted the series to the first 60 modes. Including more modes is also problematic because of the limited resolution in σ. Since the whole process is only approximate I decided to smooth the resulting retarded electromagnetic field by averaging over the five closest neighbors.

The next thing is to choose the initial conditions for the string and the scalar field ϕ. The initial configuration should be compatible with the equations of motion, but that is hard since no explicit solution is known. The possible approach is to use one of the known analytical solutions for the ordinary cosmic string to specify the initial shape of the superconducting loop with no current. The current can be generated later with the help of an external electromagnetic field. Alternatively, one can start with the initial scalar field represented by a superposition of a few free running waves with small amplitudes. If the amplitudes are sufficiently small the resulting perturbation of the string in the initial state can be safely neglected. This is the approach that I use here. In all runs the initial configuration of the scalar field was a superposition of two waves with the amplitude 0.0005 and the frequencies $\omega=2\pi/L$ and $4\pi/L$. The initial state of the string was either represented by the Burden solution with $m=n=1$ and $\theta=\pi/4$, or by the Chen solution with $\theta=\pi/10$, $\eta=\phi=\pi/3$. For each set of initial data the computations was performed twice: with and without the back reaction of the electromagnetic radiation. In the following I shall refer to the first set of initial conditions by B and to the second by C.

Now let me describe the results of the first run; one with the initial conditions B and without the back reaction of the radiation. In Fig. 1 I plotted the maximum of the Lorentz factor $\gamma_{\max}(\sigma)$ $(0 \leq \sigma \leq L)$ as a function of time. The striking feature is the presence of the very sharp peaks that correspond to cusps. (The actual values of the γ at the cusps should not be treated with confidence.) One can easily check that a peak of $\gamma(\sigma)$ is also well localized in space, that is as a function of σ at the time when the cusp occurs. Now, from the formula (2.11) one should expect a very strong electromagnetic field in the vicinity of the cusp. In Fig. 2 I plotted the maximum of the $|\mathbf{E}(\sigma)|$ as a function of time. Indeed the electric field changes by 7 orders of magnitude between the peaks and the minima and the peaks are very sharp. In Fig. 3 I plotted the $\log_{10}|\mathbf{E}(\sigma)|$ at the times $\tau=\frac{1}{8}P$, $\frac{1}{4}P$, $\frac{3}{8}P$, and $\frac{1}{2}P$. One can see that even when there is no cusp the electric field varies rapidly on short scales. The peaks of the retarded field occur at points moving with the large velocity. In short, at every cusp the string interacts with a powerful and very sharp pulse of the electromagnetic field. This traumatic event perturbes the scalar field ϕ very strongly, but this perturbation is localized in time and space. Consequently, at every near cusp very high-frequency Fourier modes are excited. One can see it clearly in Fig. 4. I plotted there $A(k)=\log_{10}(RF_k^2+IF_k^2+RG_k^2+IG_k^2)^{1/2}$ vs k, for several times. The first three curves were evaluated at the times $\tau=\frac{1}{16}P$, $\tau=\frac{3}{16}P$, and $\tau=\frac{5}{16}P$ ($P=L/2$ is the period of the string motion), that is before the occurrence of the first cusp. The high-frequency amplitudes are small and they grow slowly because even though the electric field varies rapidly on short scales its magnitude is small. The remaining curves were computed every $\frac{5}{8}P$ after the first cusp. After the first cusp essentially all high frequencies are excited and the further evolution seems chaotic. In Fig. 5 I made similar plot for the modes of the string. Although the original frequency clearly dominates one clearly sees that the loop trembles with all frequencies as it evolves. The curves representing the norm of the Fourier amplitudes tend to bunch in one well-defined band on the plots so it is useful to describe their limiting behavior by fitting the best line $\log_{10}|A(k)|=A+Bk$ $(k>5)$. First I computed the parameters A and B separately for all sets of Fourier amplitudes obtained at every $\frac{1}{8}P$, both for the scalar field and for the string. They change rapidly during the first period $(\tau \leq P)$ and are very stable later. In Table I, I listed the parameters A and B computed using the last eight sets of the Fourier modes as a single set of data points, so that the line $A+Bk$ represents them all. The errors are $\Delta A=\pm 0.03$ and $\Delta B=\pm 0.001$. The important point is that B is very small so very high-frequency modes should all contribute. This is usually taken as a sign of the chaotic behavior.

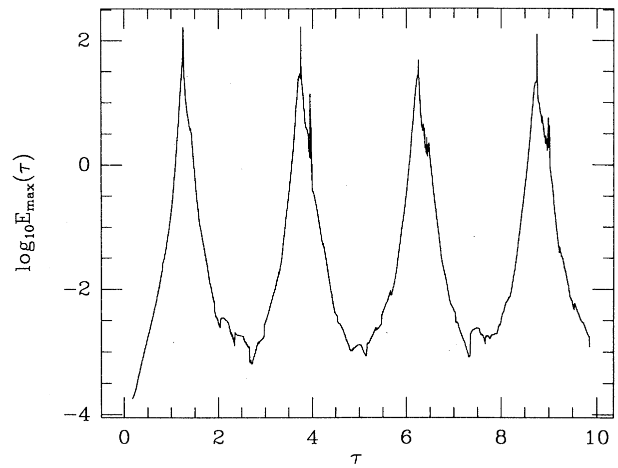

FIG. 2. The maximum of the electric field $|\mathbf{E}(\sigma)|$ $(0<\sigma<L)$ as a function of time for the initial data B.

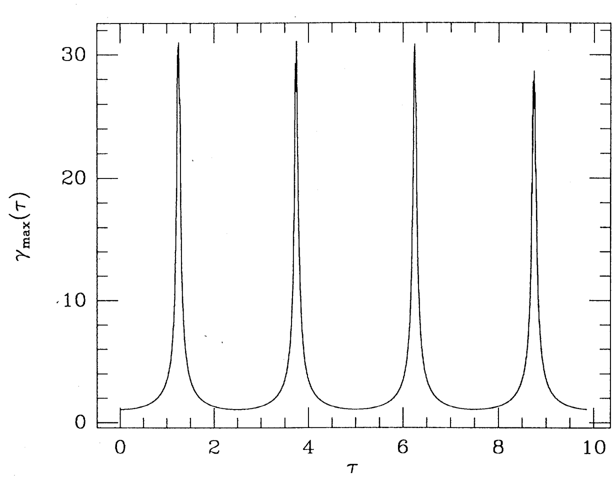

FIG. 1. The maximum of the Lorentz factor $\gamma(\sigma)$ $(0<\sigma<L)$ as a function of time for the initial data B.

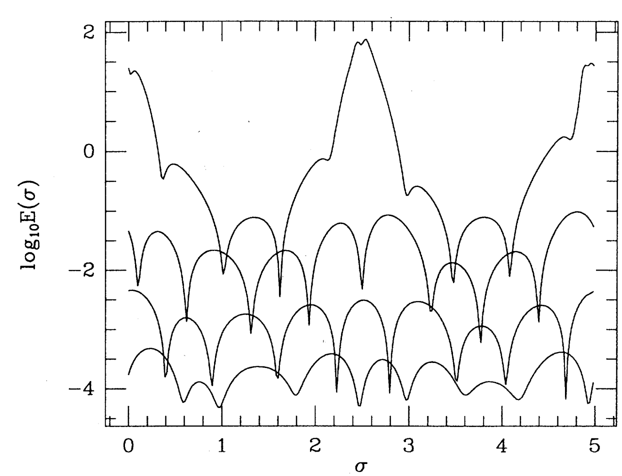

FIG. 3. The magnitude of the electric field around the loop at the times $\tau=\frac{1}{8}P$, $\frac{1}{4}P$, $\frac{3}{8}P$, and $\frac{1}{2}P$.

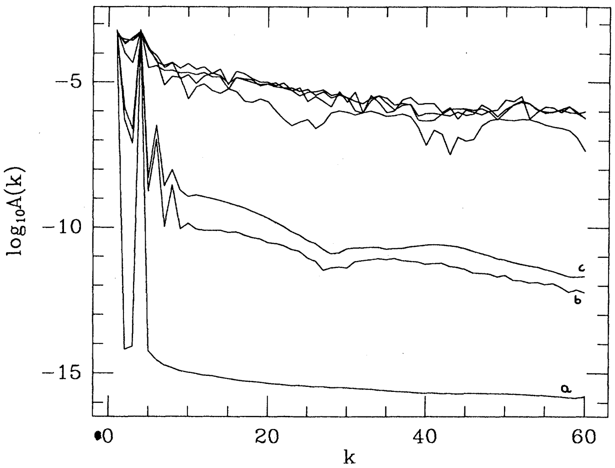

FIG. 4. The norm $A(k)$ of the Fourier modes of the scalar field for several times. The curves a, b, and c correspond to $\tau=\frac{1}{16}P$, $\frac{3}{16}P$, $\frac{5}{16}P$, and the others were taken every $\frac{5}{8}P$ later. Initial data B.

The hypothesis of the chaotic evolution can be further tested by the approximate computation of the largest Liapunov exponent. The equation of motion for the scalar field can be considered as the equation for the perturbation of ϕ around $\phi=0$. The norm of the trajectory can be defined as

$$d(\tau_n)=\left\{\sum_j^N [\phi(\sigma_j,\tau_n)^2+\dot{\phi}(\sigma_j,\tau_n)^2]\right\}^{1/2} . \tag{4.5}$$

The largest Liapunov exponent is approximately given by

$$\Lambda=\lim_{N\to\infty}\frac{1}{\tau_N}\sum_{k=1}^N \ln\frac{d(\tau_k)}{d(\tau_{k-1})} ,$$

where τ_k are times when the subsequent "measurements" of $d(\tau)$ are taken. I evaluated Λ using the interval $\tau_{k+1}-\tau_k=8DT$. The result is plotted in the Fig. 6 as the function of time τ_N. The length of the evolution was not long enough to permit the definite statement but Λ clearly approaches a positive value $\Lambda\approx 0.04$. This means that the system is either chaotic or unstable against the exponential growth of the alternating current. In the Fig. 7 I plotted the $\dot{\phi}(\sigma)$ at the times $\tau=0.12P$ and $3.12P$. One can see the correlations between the two curves but the second curve is rather messy. The peaks seem to correspond more to a little bit more constructive interference of many Fourier modes than to any smoothly growing instability. The largest peak in the final state is only 3 times larger than the amplitude of $\dot{\phi}$ in the initial time and is not appreciably bigger than the peaks of $\dot{\phi}$ at any time after the high-frequency modes were excited, that is after the first cusp.

This computation of the Liapunov exponent should be considered with a grain of salt for two reasons. First, Liapunov exponent should be evaluated studying the perturbations along many different trajectories not just one. Second, the perturbations of the string motion should be included. However, in the present context the approximation made above makes sense it corresponds to the most interesting physical situation: the stability of the

TABLE I. The parameters of the best fit $\log_{10}|A(k)|=A+Bk$ to the norm of the Fourier amplitudes of the string and the scalar field, for the initial data B or C, without the back reaction.

	A	B
B, string	−3.65	−0.045
B,ϕ	−4.49	−0.031
C, string	−4.02	−0.041
C,ϕ	−4.39	−0.028

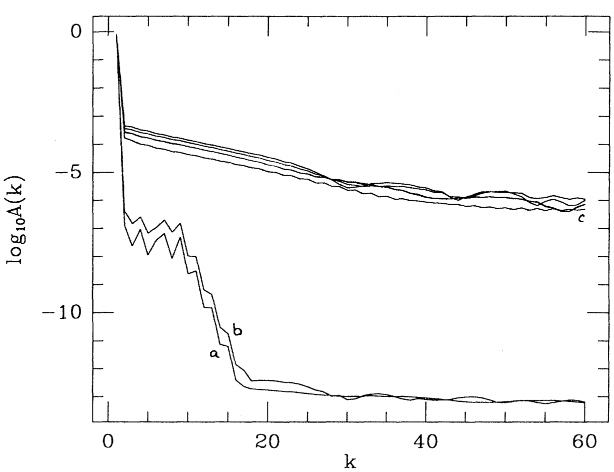

FIG. 5. The norm $A(k)$ of the Fourier modes of the string world-sheet vector for several times. The curves a, b, and c correspond to $\tau=\frac{1}{16}P$, $\frac{3}{16}P$, $\frac{5}{16}P$, and the others were taken every $\frac{5}{8}P$ later. Initial data B.

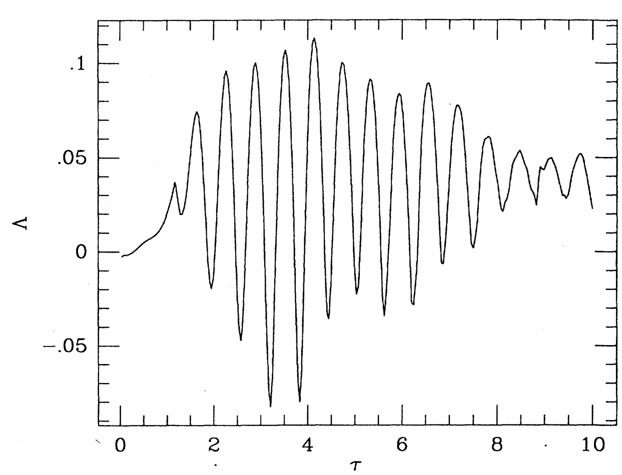

FIG. 6. The largest Liapunov exponent of the scalar field as a function of time. Initial data B.

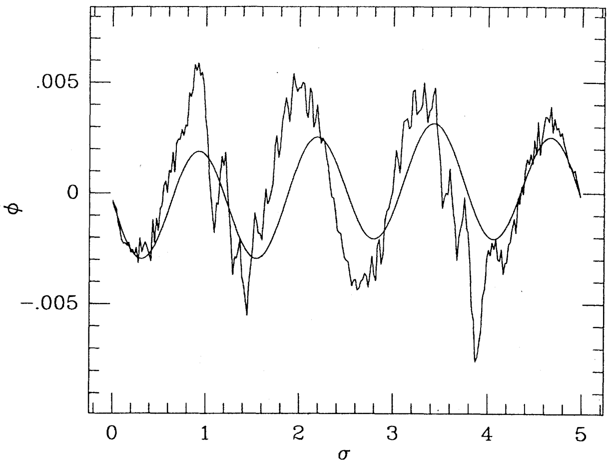

FIG. 7. The time derivative of the scalar field as a function of σ for $\tau=0.125P$ and $\tau=3.125P$. Initial data B.

superconducting string without any current.

The next question is whether this behavior is peculiar to the chosen initial conditions. To resolve this problem I evolved the system starting the initial state C. The only difference is that for this initial state the cusps occur three times during every period (see Fig. 8). Correspondingly, the high-frequency modes are excited even faster and their amplitudes decrease a little bit slower as a function of k (see Figs. 9 and 10). The approximate behavior of the Fourier spectrum is given in Table I with the help of the parameters A and B. The Liapunov exponent, evaluated as before, is also $\Lambda \approx 0.04$. However, the largest peak of $\dot{\phi}$ in the final state is still only three times larger than the initial amplitude. This I consider as an argument for the hypothesis that the growth is due to a fortuitous superposition of many modes rather than any systematic instability: since in this case the number of cusps is three times larger than previously any systematic increase should be more pronounced. On the other hand, a lucky interference is equally likely in both cases. One cusp seems sufficient to excite high-frequency modes, all subsequent ones just mix phases of the already excited modes.

Finally, the laboriously derived formula for the back reaction of the electromagnetic radiation should be used to check whether the chaos can be kept under control. The main difficulty with the back reaction is that it brings in the third derivatives of the dynamic variables and therefore leads to the well-known runaway solutions. The cure for this problem was suggested by Burke:[17] the electromagnetic field of the back reaction should be treated as a source term in the equation of motion and should be evaluated using the retarded variables only. This indeed eliminates the runaway solutions and makes the integration of the equation of motion with the back reaction as easy or difficult as without it. However, the results are disappointing: over the time of four periods the effects of the back reaction are minimal and are rather difficult to discern. The evolution is chaotic and the value of the Liapunov exponent is not affected by the back reaction. As one will see shortly, this behavior is consistent with the power of electromagnetic radiation form the loop. Unfortunately, the accuracy of the code is not sufficient to see any local effects of the back reaction on the cusps and to see any secular effects one must follow the evolution of the loop for a much longer period of time.

The emitted energy of the electromagnetic radiation can be computed with the help of the formula (2.13). Unfortunately, the integral over the angles cannot be factored out so for every direction and frequency one has to evaluate six double integrals. Since the integrand oscillates and is costly to evaluate the required time seriously limits the angular and spectral resolution that can be obtained. In practice I computed the radiation emitted during the last two periods of motion of six frequencies, from $\omega=2\pi/L$ to $\omega=100\pi/L$. For each frequency first I

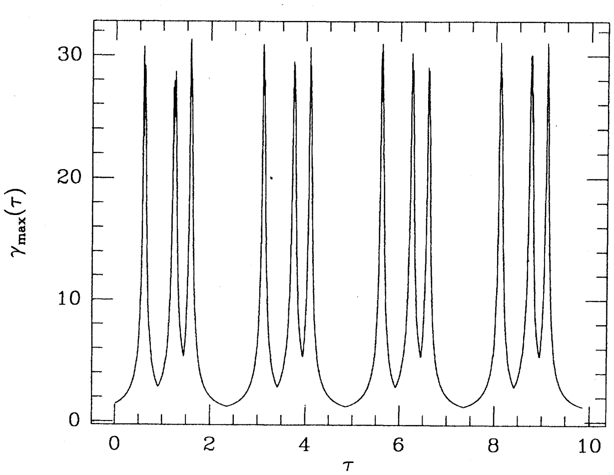

FIG. 8. The maximum of the Lorentz factor $\gamma(\sigma)$ as a function of time for the initial data C.

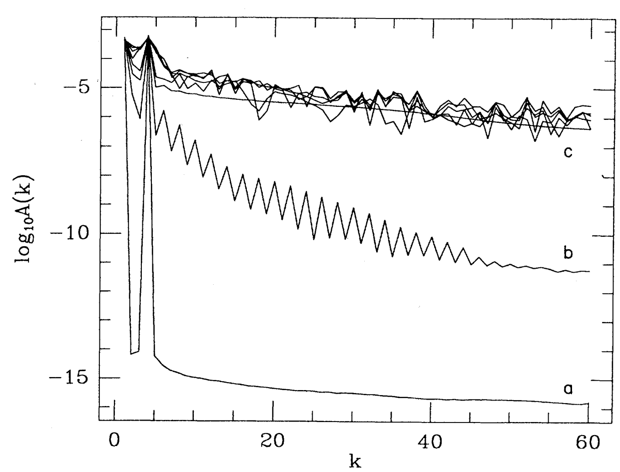

FIG. 9. The norm of the Fourier modes of the scalar field for several times. The curves a, b, and c correspond to $\tau=\frac{1}{16}P$, $\frac{3}{16}P$, $\frac{5}{16}P$, and the others were taken every $\frac{5}{8}P$ later. Initial data C.

TABLE II. The energy radiated at the given frequency $\omega_n = 2\pi n / L$ by the string that was initially in the configuration B or in the C. E_{tot} is the total energy, E_{peak} the energy emitted within a narrow peak.

n	B, E_{tot}	B, E_{peak}	C, E_{tot}	C, E_{peak}
1	0.159×10^{-11}	0.210×10^{-13}	0.111×10^{-11}	0.175×10^{-13}
2	0.103×10^{-9}	0.154×10^{-11}	0.189×10^{-9}	0.396×10^{-11}
5	0.396×10^{-10}	0.151×10^{-11}	0.147×10^{-9}	0.366×10^{-11}
10	0.789×10^{-9}	0.265×10^{-10}	0.189×10^{-8}	0.543×10^{-10}
30	0.791×10^{-8}	0.123×10^{-9}	0.139×10^{-7}	0.236×10^{-9}
50	0.154×10^{-7}	0.216×10^{-9}	0.247×10^{-7}	0.416×10^{-9}

evaluate the emitted radiation in 30×15 uniformly distributed directions and use the results to compute the total energy emitted at this frequency. Next I find the direction where the intensity is the highest and I calculate the radiation within a narrow cone centered at this direction using 121 rays within the cone. The angular width of the cone is 7.75×10^{-2} srad. The results are given in Table III below. The entries in the table should be multiplied by the factor 0.4×10^{51} $(L/10^8)$ s GeV to obtain the energy emitted during this time. The energy emitted during one period is very small in comparison with the energy of the loop that is $8\times10^{52}(L/10^8)$ GeV.

Unfortunately, the angular resolution is not good enough to draw a decent contour plot covering the whole sphere. At low frequency the radiation has a broad dipole pattern, so only small part of the total energy goes in the direction of maximal intensity. On the other hand, at high frequency the pattern of radiation is very irregular with several very intense peaks so none of them is particularly strong in comparison with the total energy emitted at that frequency. These peaks fit well within the narrow cone used to evaluate them and the intensity of radiation in the direction of the maximum is from 8 to 15 times stronger than at the edge of the cone. The isocurves of intensity are nearly circular inside the cone and become irregular at the edge. At the frequency $60\pi/L$ the strongest peak subtends the solid angle $\Omega\approx3\times10^{-2}$ srad, at $\omega=100\pi/L$ the strongest peak is narrower-$\Omega\approx1.5\times10^{-2}$ srad (see Table II).

Finally, I would like to discuss the validity of the mechanical approximation. As I demonstrated above the string remembers its initial shape during the evolution and the amplitudes of the high-frequency oscillations are small. Therefore, it is tempting to use the mechanical approximation and consider the evolution of the current on the string described by an analytic solution obtained for an ordinary string, that is neglecting any influence of the current and electromagnetic field on the string. This approach was used in Ref. 12. However, it is quite obvious that the high-frequency oscillations of the string, while irrelevant for the low-frequency oscillations of the current are very important for the high-frequency modes. Consequently, in this approximation the evolution of the current should be significantly smoother than in the real case and the amplitudes of the high-frequency modes should be systematically underestimated. To test this reasoning I computed the evolution of the same initial configurations as before using the mechanical approximation. As expected, the high-frequency modes are significantly underestimated. A good measure is provided by the parameter B describing the slope of the curve $\log_{10}A(k)$. This parameter is roughly twice as big as before: $B=-0.09$ for the initial conditions B and $B=-0.05$ for the initial conditions C.

The results described here are in clear contradiction with the results obtained by Spergel, Press, and Sherrer[12] who computed a transformation matrix giving the amplitudes of the different Fourier modes of the current after one period of motion. They used the mechanical approximation and assumed that the current does not change much during one period of evolution. They found that high-frequency modes of the current are very strongly damped for all initial configurations considered and that certain initial configuration lead to an exponential growth of the low-frequency modes. I have already showed that, according to this work, there is no damping of the high-frequency modes over the time interval of four periods. This result is in agreement with the very low power of the electromagnetic radiation.

The question of stability has not been touched yet because both initial configurations that I have considered so far are stable according to Ref. 12. To make this comparison I computed the evolution of the string that was initially described by the Chen solution with $\theta=\pi/5$, $\phi=\pi/9$, and $\eta=17\pi/18$; this is an example of an unstable configuration according to Ref. 12. Neither in the

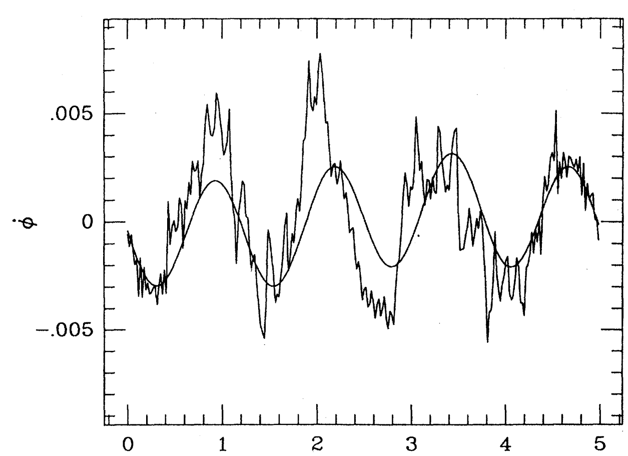

FIG. 10. The time derivative of the scalar field as a function of σ for $\tau=0.125P$ and $\tau=3.125P$. Initial data C.

mechanical approximation nor solving the full set of equation I could detect any trace of instability. It also seems to me that the rates at which the low-frequency oscillations are excited and the high-frequency modes are damped according to the authors of Ref. 12 are not compatible with their assumption that the change of the current during one period is small in comparison of the current itself and would imply a very large power of the electromagnetic radiation.

V. PARTICLE PRODUCTION

In the previous chapter I showed that the electromagnetic field due to the current in the loop may become very strong. In such case one should expect that the vacuum becomes unstable with respect to particle production. The relevant dimensionless scale is $m^2\pi/eE$. Rescaling the field as before one gets $m^2\pi/eE = (0.6/E)(L/10^{11}\text{ cm})$, so electron position pairs should be copiously produced if the rescaled field is larger then one.

The computation of the particle production rate due to the vacuum instability is a rather difficult exercise. In practice it is hard to go beyond the result obtained by Schwinger,[19] who computed the rate of particle creation in a static and homogeneous external field. This result can be used in a general setting provided that the external field does not change rapidly on scales comparable with the electron Compton wavelength. This condition should be satisfied for the field due to the superconducting loop since its geometry is characterized by the macroscopic scale L. The problem with this argument is that the field due to the cusps changes rather rapidly so the field is hardly static. However, the time variation is still slow in comparison with the relevant time scale for particle creation. Also, the time variation of the external field is likely to enhance the rate of particle production so one can use the result of Schwinger to obtain a safe lower limit.

According to Schwinger, the one-loop effective action is

$$W=\frac{-1}{8\pi^2}\int d^4x\int_0^\infty \frac{ds}{s^3}e^{-m^2s}\times\left[(es)^2\mathcal{G}\frac{\mathrm{Re}[\cosh es\sqrt{2(\mathcal{F}+i\mathcal{G})}]}{\mathrm{Im}[\cosh es\sqrt{2(\mathcal{F}+i\mathcal{G})}]}-1\right], \tag{5.1}$$

where m is the mass of an electron and

$$\mathcal{G}=\tfrac{1}{8}\epsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu}=\mathbf{E}\cdot\mathbf{B},\quad \mathcal{F}=\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}=\tfrac{1}{2}(B^2-E^2)\ .$$

The rate of particle production per unit volume (called just rate later) is given by the imaginary part of W:

$$\mathcal{P}=\frac{2\,\mathrm{Im}W}{VT}=\frac{1}{4\pi}\int_0^\infty\frac{ds}{s^3}e^{-m^2s}(es)^2\mathcal{G}\,\mathrm{Re}[\cosh es\sqrt{2(\mathcal{F}+i\mathcal{G})}]\times\delta(\mathrm{Im}[\cosh es\sqrt{2(\mathcal{F}+i\mathcal{G})}])\ . \tag{5.2}$$

This formula is correct in general but it requires a careful treatment of the limit $\mathcal{G}\to 0$. In this case it is advantageous to go back to the formula (5.1) and use the relation

$$\frac{(es)^2\mathcal{G}}{\mathrm{Im}[\cosh es\sqrt{2(\mathcal{F}+i\mathcal{G})}]}=\frac{esf_+}{\sinh esf_+}\frac{esf_-}{\sinh esf_-}\ ,$$

where

$$f_\pm=\frac{1}{\sqrt{2}}(\sqrt{\mathcal{F}+i\mathcal{G}}\pm\sqrt{\mathcal{F}-i\mathcal{G}})\ .$$

When $\mathcal{G}\to 0$ then $f_-\to 0$ and the limit can be taken easily. The rate of particle production becomes

$$\mathcal{P}=\frac{\theta(E^2-B^2)e^2(E^2-B^2)}{4\pi^3}\times\sum_{n=1}^\infty\frac{1}{n^2}\exp\left[-\frac{m^2n\pi}{e\sqrt{E^2-B^2}}\right]. \tag{5.3}$$

In the general case formula (5.2) should be used. After some tedious algebra one gets

$$\mathcal{P}=\frac{e^2|\mathbf{E}\cdot\mathbf{B}|}{4\pi^2}\sum_{n=1}^\infty\frac{1}{n}\coth n\pi\left[\frac{r+B^2-E^2}{r-B^2+E^2}\right]^{1/2}\times\exp\left[-\frac{m^2n\pi\sqrt{2}}{e\sqrt{r-B^2+E^2}}\right], \tag{5.4}$$

where $r=\sqrt{(B^2-E^2)^2+4(\mathbf{E}\cdot\mathbf{B})^2}$.

These results may be used to compute the rate of particle production around the loop. It is convenient to rescale the electromagnetic field as before. Upon doing this, one gets a factor T/L^2 in front of the expression for $\mathcal{P}$ and a factor $L/\sqrt{T}$ in the argument of the exponential. The crucial factor determining the efficiency of particle production is $m^2\pi L/\sqrt{T}$; for $L=1.5\times10^{10}$ cm this factor is equal to one, for smaller loops the particle production is more effective.

To compute the number of particles created per unit time I set up a grid of $(11)^3$ points filling the cube of the size $2L$, compute the rate at all points, and integrate over the volume of the cube. The calculation was repeated several times during the evolution of the loop, for several L. The results are given in Table III. The important point is that even for the current much smaller than the critical current for the breakdown of the superconductivity particle production is very important if the size of the loop is smaller than 10^8 cm. Particles are produced because the electric field of the loop is strongly enhanced in the vicinity of the cusps. This is the reason why the rates are much stronger for the initial conditions C: in this case there are more cusps. We also see that the energy loss during one period can easily exceed the energy of the loop. This means that the present estimates become invalid for the small loop and the back reaction of particle production on the loop must be taken into account. The back reaction should limit the velocity of the would be cusps and make the evolution of the loop smoother.

TABLE III. The rate of particle production inside the cube of the size $2L$ centered at the center of the loop for initial conditions B and C. The rates are computed for three values of $L=1.5\times10^7$, 1.5×10^8, 1.5×10^9 cm. ΔE is the energy of the particles that would be created at this rate during one period of motion. The total energy of the loop is $1.2\times10^{53}(L\ \mathrm{cm}/10^7)$ GeV.

$t(P)$	$\mathcal{P}\times(2L)^3(\mathrm{s}^{-1})$ (B)	ΔE (GeV) (B)	$\mathcal{P}\times(2L)^3(\mathrm{s}^{-1})$ (C)	ΔE (GeV/s) (C)
3.125	4.1×10^{22}	3.1×10^{26}	3.4×10^{55}	2.5×10^{59}
3.125	0	0	1.0×10^{44}	7.7×10^{48}
3.125	0	0	2.6×10^{-80}	1.9×10^{-74}
3.25	3.1×10^{53}	2.3×10^{57}	1.6×10^{56}	1.2×10^{60}
3.25	2.1×10^{27}	1.5×10^{32}	3.3×10^{48}	2.4×10^{53}
3.25	0	0	1.9×10^{-37}	1.3×10^{-31}
3.375	1.5×10^{54}	1.1×10^{58}	1.3×10^{55}	1.0×10^{59}
3.375	1.4×10^{35}	1.1×10^{40}	2.3×10^{42}	1.7×10^{47}
3.375	0	0	8.1×10^{-95}	6.0×10^{-89}
3.50	4.2×10^{56}	3.1×10^{60}	3.1×10^{57}	2.3×10^{61}
3.50	2.1×10^{51}	1.6×10^{56}	9.1×10^{54}	6.7×10^{59}
3.50	1.0×10^{-10}	7.8×10^{-5}	3.4×10^{21}	2.5×10^{27}
3.625	3.5×10^{54}	2.6×10^{58}	7.6×10^{56}	5.6×10^{60}
3.625	4.3×10^{39}	3.2×10^{44}	1.2×10^{51}	0.9×10^{56}
3.625	0	0	3.9×10^{-16}	2.9×10^{-10}
3.75	6.5×10^{47}	4.9×10^{51}	4.2×10^{54}	3.1×10^{58}
3.75	5.2×10^{-21}	3.9×10^{-16}	5.1×10^{39}	3.8×10^{44}
3.75	0	0	0	0
3.875	1.4×10^{45}	1.0×10^{49}	2.6×10^{52}	2.0×10^{56}
3.875	2.8×10^{-44}	2.1×10^{-39}	2.6×10^{21}	2.0×10^{26}
3.875	0	0	0	0
4.0	6.2×10^{52}	4.7×10^{56}	1.0×10^{54}	7.7×10^{57}
4.0	6.2×10^{24}	4.7×10^{29}	1.5×10^{34}	1.1×10^{39}
4.0	0	0	0	0

VI. CONCLUSIONS

The main conclusion of this work is that the evolution of a superconducting cosmic loop is chaotic. The occurrence of the near cusps causes the excitation of high-frequency modes of the string and of the current. The resulting alternating current is very irregular. As the number of the excited modes grows the amplitude of the current grows also but as soon as all modes are excited the growth ceases and only more or less random peaks occur. The largest Liapunov exponent is positive, as it should for a chaotic system.

The electromagnetic radiation is not very effective in dissipating the energy of the loop. At high frequency the angular distribution of radiation is very irregular with several intense peaks; the angular width of the peaks increases with the frequency.

Particle production is very efficient in dissipating the energy of loops smaller than 10^8 cm. The back reaction of particle production should smooth out the cusps.

The problem of computing the final state of the loop is not tractable since it requires evolving the string for a time many times longer than it is possible. However, the chaotic behavior seen here most likely precludes the occurrence of the stable smooth rings.

The future of the numerical investigations of this kind is not particularly good. For small loops the interactions with the environment must be included; this is particularly true for the back reaction of particle production. Since very high-frequency modes are rapidly excited even if there were none in the initial state the spatial resolution must be significantly improved and this is very time consuming. Both problems are difficult. The present code can achieve a better resolution provided that the computation of the retarded electromagnetic field is significantly improved.

The formula or the back reaction of the electromagnetic radiation that has been derived here requires further elaboration. In particular the question of absorbing the divergent terms should be further studied by including the corrections to the Nambu action. The work on this problem is in progress and will be reported later.

Of course, the main problem is the serious disagreement with the results of Spergel, Press, and Scherrer. It seems clear that the controversy is due to the different results obtained for the electromagnetic field at the cusps: I got a very large electric field, they do not see anything special there.[20] This controversy is likely to remain unresolved until a third code appears.

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