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Perturbative QCD corrections to the ratio R for τ decay

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The perturbative QCD correction to the semihadronic decay rate of a heavy lepton is expressed as an expansion in α_s. In the case of the τ lepton, the ratio of the semihadronic and electronic decay rates is $R=3[1+\alpha_s/\pi+5.20(\alpha_s/\pi)^2+104.0(\alpha_s/\pi)^3]$. The use of R to give a precise determination of α_s is advocated.

The inclusive semihadronic decay rate of the τ lepton is conveniently expressed in terms of the ratio

$$R=\frac{\Gamma(\tau^-\to\nu_\tau+\text{hadrons})}{\Gamma(\tau^-\to\nu_\tau e^-\bar{\nu}_e)} \ . \tag{1}$$

A naive estimate of R is obtained by approximating the numerator by the decay rate into quark-antiquark pairs. Including $d\bar{u}$ and $s\bar{u}$, we obtain[1]

$$R\simeq N_c(\cos^2\theta_C+\sin^2\theta_C)=3 \ . \tag{2}$$

The corrections to this naive prediction can be classified into three categories: (1) perturbative QCD; (2) nonperturbative QCD; and (3) electroweak corrections, and they are all reviewed in Ref. 2. The electroweak corrections are enhanced by a large logarithm[3] but are still relatively small, increasing the prediction by about 2.4%. Although the decay is a timelike process, the nonperturbative QCD corrections can be treated systematically using the operator-product expansion.[4–6] These corrections are also estimated to be small, decreasing R by a few percent. The largest corrections by far are the perturbative QCD corrections. The purpose of this paper is to express these corrections as a simple power series in the strong coupling constant α_s.

The ratio R can be expressed as an integral over the invariant mass of the hadrons:

$$R=\frac{2}{\pi}\int_0^{M^2}\frac{ds}{M^2}\left[1-\frac{s}{M^2}\right]^2\left[\left[1+2\frac{s}{M^2}\right]\text{Im}\Pi_T(s+i\epsilon)-\text{Im}\Pi_L(s+i\epsilon)\right] , \tag{3}$$

where M is the mass of the heavy lepton. The functions $\Pi_T(s)$ and $\Pi_L(s)$ are the transverse and longitudinal components of the hadronic part of the W-boson self-energy function, with an overall factor of $e^2/96\pi^2\sin^2\theta_W$ removed for convenience. Perturbative QCD can be used to approximate $\Pi_T(s)$ and $\Pi_L(s)$ for large spacelike s, but it is not applicable in (3) as it stands, because s is timelike and the integral extends down to small s. However, as shown by Lam and Yan,[7] the analytic properties of the Π_T and Π_L allow R to be expressed as a contour integral in the complex s plane:

$$R=\frac{1}{i\pi}\int_C\frac{ds}{M^2}\left[1-\frac{s}{M^2}\right]^2\left[\left[1+2\frac{s}{M^2}\right]\Pi_T(s)-\Pi_L(s)\right] , \tag{4}$$

where the contour C runs clockwise around the circle of radius $|s|=M^2$. This contour avoids the small-s region, and furthermore, as pointed out in Ref. 5, the factor $(1-s/M^2)^2$ suppresses the contribution from the timelike region. Thus, provided M^2 is sufficiently large, the functions Π_T and Π_L can be reliably approximated using perturbative QCD.

In the case of the τ lepton, the mass M is sufficiently small that one must consider the possibility of large nonperturbative QCD corrections. Because the timelike region of the s contour is suppressed, the operator-product expansion can be used to expand $\Pi_T(s)$ and $\Pi_L(s)$ in powers of $1/s$. This expansion systematically organizes all nonperturbative effects into matrix elements of local operators, and makes reliable estimates of the nonperturbative corrections to R possible. The best available estimates[6] place them between -1% and -3%. While there is a large uncertainty in the magnitude of the corrections, there is no uncertainty in the sign because all the operators of dimensions 6 or less contribute with the same negative sign.[5]

We now proceed to calculate the perturbative QCD corrections to R. Because R can be expressed as an integral along the contour $|s|=M^2$ as in (4), it is clear that the appropriate expansion parameter will be $\alpha_s(M)$. Using integration by parts, (4) can be rewritten in the form

$$R=\frac{1}{2\pi i}\int_c \frac{ds}{s}\left[1-2\frac{s}{M^2}+2\frac{s^3}{M^6}-\frac{s^4}{M^8}\right]s\frac{d}{ds}\Pi_T(s)$$
$$=\frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta(1+2e^{i\theta}-2e^{3i\theta}-e^{4i\theta})$$
$$\times s\frac{d}{ds}\Pi_T(s=-M^2e^{i\theta}) \; . \quad (5)$$

We have dropped the Π_L term since the perturbative corrections do not contribute to the longitudinal self-energy function. The logarithmic derivative $s(d/ds)\Pi_T$ can be extracted from the recent calculation[8] of the ratio R for e^+e^- annihilation to order α_s^3. It is equal to the function $D(s)$ calculated in Ref. 8, except that $3\sum Q_f^2$ should be replaced by $3\sum|V_{ff'}|^2$ and $(\sum Q_f)^2$ should be set to 0. Here, $V_{ff'}$ is a Kobayashi-Maskawa matrix element and the sum is over pairs of quarks which couple to the W and are light enough to be produced in the decay of the heavy lepton. In the case of the τ lepton, the sum is $|V_{ud}|^2+|V_{us}|^2\simeq 1$. The resulting expansion for $s(d/ds)\Pi_T$ is

$$s\frac{d}{ds}\Pi_T=3\sum|V_{ff'}|^2\left[1+\frac{\alpha_s(-s)}{\pi}+K_1\left[\frac{\alpha_s(-s)}{\pi}\right]^2+K_2\left[\frac{\alpha_s(-s)}{\pi}\right]^3+\cdots\right],$$
$$K_1=1.986-0.115f \; , \quad (6)$$
$$K_2=95.87-4.22f+0.086f^2 \; .$$

In the case of $f=3$ flavors, the coefficients are $K_1=1.641$ and $K_2=83.98$. The coupling constant α_s is the $\overline{\text{MS}}$ (modified minimal-subtraction scheme) coupling constant evaluated at the renormalization scale $\mu^2=-s$.

To evaluate the integral, we expand the coupling constant $\alpha_s(-s)$ around the point $s=-M^2$ on the integration contour $s=-M^2e^{i\theta}$, $-\pi<\theta<\pi$. The behavior of $\alpha_s(t)$ as a function of complex t is governed by the β function:

$$t\frac{d}{dt}\alpha_s(t)=\tfrac{1}{2}\beta(\alpha_s(t)) \; .$$

The expansion of the β function[9] is needed to order α_s^3:

$$\frac{1}{\pi}\beta(\alpha_s)=-\beta_0\left[\frac{\alpha_s}{\pi}\right]^2-\beta_1\left[\frac{\alpha_s}{\pi}\right]^3+\cdots \; , \quad (7)$$
$$\beta_0=\frac{33-2f}{6}, \quad \beta_1=\frac{306-38f}{24} \; .$$

In the case of three flavors, these coefficients are $\beta_0=\frac{9}{2}$ and $\beta_1=8$. To order α_s^3, we find

$$\frac{\alpha_s(M^2e^{i\theta})}{\pi}=\frac{\alpha_s(M^2)}{\pi}+i\frac{\theta}{2\pi}\beta(\alpha_s(M^2))-\frac{\theta^2}{8\pi}\beta(\alpha_s(M^2))\beta'(\alpha_s(M^2))+\cdots$$
$$=\frac{\alpha_s(M^2)}{\pi}-\frac{i}{2}\beta_0\theta\left[\frac{\alpha_s(M^2)}{\pi}\right]^2+\left[-\frac{i}{2}\beta_1\theta-\frac{1}{4}\beta_0^2\theta^2\right]\left[\frac{\alpha_s(M^2)}{\pi}\right]^3+\cdots \; . \quad (8)$$

This expansion must be inserted into (6), which in turn must be inserted into (5). The integrals over θ can be evaluated analytically. The result is

$$R=3\sum|V_{ff'}|^2\left[1+\frac{\alpha_s(M^2)}{\pi}+(K_1+\tfrac{19}{24}\beta_0)\left[\frac{\alpha_s(M^2)}{\pi}\right]^2+\left[K_2+\tfrac{19}{12}\beta_0K_1+\tfrac{19}{24}\beta_1+\frac{265-24\pi^2}{288}\beta_0^2\right]\left[\frac{\alpha_s(M^2)}{\pi}\right]^3+\cdots\right] . \quad (9)$$

In the case of three flavors, it reduces to

$$R=3\left[1+\frac{\alpha_s}{\pi}+5.20\left[\frac{\alpha_s}{\pi}\right]^2+104.0\left[\frac{\alpha_s}{\pi}\right]^2+\cdots\right] . \quad (10)$$

This result is consistent with that of Ref. 6, where it is given in the less convenient form of an expansion in powers of $1/\ln(M^2/\Lambda^2)$.

The coefficient of α_s^3 in (10) is large, even larger than in the ratio R for e^+e^- annihilation which for five flavors is[8]

$$R(s)=\frac{11}{3}\left[1+\frac{\alpha_s(s)}{\pi}+1.41\left[\frac{\alpha_s(s)}{\pi}\right]^2+64.8\left[\frac{\alpha_s(s)}{\pi}\right]^2\right] . \quad (11)$$

The order-α_s^3 correction is significant, which gives us reason to worry that the uncalculated order-α_s^4 correction might also be significant. However, in the absence of any understanding of the large coefficient, it is reasonable to accept the correction at face value. In the case of e^+e^- annihilation, the global fit to R over the energy range $2.6<\sqrt{s}<52$ GeV is improved by including the order-α_s^3 correction.[10] This fit yields a precise determination of the strong coupling constant: $\alpha_s(33\text{ GeV})=0.135\pm0.016$. If this coupling constant is evolved down to M_τ, it becomes $\alpha_s(M_\tau)=0.33\pm0.08$. The prediction (10) for the ratio R for τ decay is then $R=3.9\pm0.5$. The error has been magnified by the evolution of α_s from 30 GeV down to M_τ and is significantly larger than the present experimental uncertainty in R.

It is clear that the accuracy of current measurements of α_s will not permit a precise theoretical prediction of the ratio R for τ decay. However, as ponted out in Refs. 3 and 6, this procedure can be inverted to determine α_s. In fact, τ decay is probably the lowest-energy process from which the running coupling constant can be extracted cleanly without hopeless complications from nonperturbative effects. The ratio R is determined experimentally by measuring the branching fraction B_e of the τ into electrons, and inserting it into the formula

$$R=\frac{1-1.973B_e}{B_e} . \tag{12}$$

The direct measurement[11] of B_e yields the ratio $R=3.71\pm0.13$ while an indirect determination of B_e from measuring the lifetime[11] of the τ yields the value $R=3.32\pm0.16$. Using (10) and taking into account the estimated electroweak and nonperturbative QCD corrections, we obtain the values

$$\alpha_s(M_\tau)=0.30\pm0.03 \tag{13}$$

from the direct measurement of B_e and

$$\alpha_s(M_\tau)=0.19\pm0.06 \tag{14}$$

from the indirect measurement. The errors are due to the uncertainty in the experimental measurement only. A rough estimate of the uncertainty due to higher-order corrections is the size of the order-α_s^3 correction, which is 0.07 for (13) and 0.03 for (14). While this correction introduces a significant uncertainty into this determination of α_s, there is no reason to expect it to be less severe for other processes where the calculation of the order-α_s^3 correction is prohibitively difficult. Even including this uncertainty, the ratio R for τ decay remains competitive with other methods of determining α_s. If the reason for the large order-α_s^3 correction can be understood, then high-precision measurements of the τ lifetime would provide by far the most accurate determinations of α_s.

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