

Determination of phases of generalized Kobayashi-Maskawa matrix elements from their magnitudes

James D. Bjorken

Fermi National Accelerator Laboratory, P.O. Box 500, Batavia, Illinois 60510

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For an N -generation Kobayashi-Maskawa mixing matrix having the property that matrix elements decrease rapidly with increasing generation change, we show by construction that knowledge of the moduli of all elements is sufficient, up to a discrete ambiguity, to determine all phase information.

There has recently been interest in the fact that knowledge of only the magnitudes of the elements of the three-generation Kobayashi-Maskawa (KM) mixing-matrix elements is sufficient to determine all phase information.¹ This implies that, in principle, decay-width information alone can determine the presence or absence of CP -violating couplings. (This inference of course assumes the correctness of the KM description, in particular that the mixing matrix is unitary.)

It is natural to ask whether this feature can be generalized to the case of N generations. A general argument appears at best complicated;¹ the case of four generations has, by a brute force calculation, recently been argued to be correct.² However, counterexamples also have been shown to exist.³

Our goal here is more modest but, hopefully, practical. It is likely that a generalized $N \times N$ KM matrix will share the property seen for three generations: namely, that the matrix elements rapidly get smaller as the generation change increases. We shall show by construction that this restriction alone allows one, up to a discrete ambiguity, to reconstruct the phase information from the magnitudes of the elements.

We assume that all $|V_{ij}|$ are known and that, if $j \geq i$, then

$$|V_{ji}| \gg |V_{j+1,i}|, \quad (1)$$

$$|V_{ji}| \gg |V_{j,i-1}|, \quad (2)$$

$$|V_{ij}| \gg |V_{i,j+1}|, \quad (3)$$

and

$$|V_{ij}| \gg |V_{i-1,j}|. \quad (4)$$

This implies that $|V_{ii}| \sim 1$ because all off-diagonal elements are small. We also assume that quark phases are chosen⁴ such that V_{ii} and $V_{i,i+1}$ are real and non-negative.

Then the argument proceeds by induction. Suppose all elements V_{ij} for $|i-j| < n$ have been determined. Now for $j-i = n$ use the unitarity restriction

$$\sum_{k=1}^N V_{ik} V_{jk}^* = 0. \quad (5)$$

From this sum, there are two special terms $k=i$ and j which contribute, to good approximation, the quantity $V_{ij} + V_{ji}^*$. For $k < i$ and $k > j$ it is easy to see that relative

to $|V_{ij}|$ or $|V_{ji}^*|$ the individual magnitudes of the terms are very small: e.g.,

$$|V_{ij}| \gg |V_{i,i-1} V_{j,i-1}^*| \gg |V_{i,i-2} V_{j,i-2}^*| \gg \dots \quad (6)$$

Thus the significant terms in the sum come from $i \leq k \leq j$ and we get the main result:

$$V_{ij} + V_{ji}^* \sim - \sum_{k=i+1}^{j-1} V_{ik} V_{jk}^*. \quad (7)$$

The right-hand side, by induction, is already determined, so that a triangle construction in the complex plane can be used. Since the lengths of the sides of the triangle and the orientation of the base (the right-hand side) are known, all phase angles are determined up to a twofold ambiguity obtained by reflection about the base.

Some special cases are worth noting.

(1) $j = i + 1$. In this case there is no sum on the right-hand side, so that

$$V_{i+1,i}^* \simeq -V_{i,i+1} = -V_{i,i+1}^*. \quad (8)$$

This gives, as is very well known,

$$V_{cd} \simeq -V_{us}, \quad V_{ts} \simeq -V_{cd} \quad (9)$$

and for four generations, in obvious notation,

$$V_{Tb} \simeq -V_{tB}. \quad (10)$$

(2) $j = i + 2$. The one interesting case for three generations is the well-known triangle relation⁵

$$(V_{ub} + V_{td}^*) \simeq -V_{us} V_{ts} \simeq V_{us} V_{cb}. \quad (11)$$

For four generations there is, in addition, the relation

$$V_{cB} + V_{Ts}^* \simeq -V_{cb} V_{Tb}. \quad (12)$$

(3) $j = i + 3$. This first occurs for four generations and leads in that case to one constraint, sufficient to complete the determination of the matrix elements:

$$V_{uB} + V_{Td}^* \simeq -(V_{us} V_{Ts}^* + V_{ub} V_{Tb}^*). \quad (13)$$

For the first time, the right-hand side becomes a complex vector, leading to a generalized orientation of the unitarity triangle.

It is unclear to this author what range of validity is applicable to this construction. Within our assumptions a perturbative scheme can most certainly be used to

refine the formulas given above. But its radius of convergence has not been investigated, nor the detailed connection to the counterexamples found in Ref. 3. Whatever the connection, we may see that the construction we have presented would be in trouble were the right-hand side of Eq. (7) ever "accidentally" small.

It is interesting to speculate on the pattern of magnitudes of four-generation KM matrices consistent with our input assumptions and see whether the unitarity triangles become nontrivial, i.e., nondegenerate, in the level of approximation we have considered. For example, suppose

$$V_{\text{KM}} \sim \begin{pmatrix} 1 & \theta & \theta^3 & \theta^4 \\ \theta & 1 & \theta^2 & \theta^3 \\ \theta^3 & \theta^2 & 1 & \theta \\ \theta^4 & \theta^3 & \theta & 1 \end{pmatrix}, \quad (14)$$

where the entries θ^n indicate only the order of magnitude of the elements, as expressed in terms of powers of the Cabibbo angle. Then we immediately see that all three unitarity-triangle relations [Eqs. (11), (12), and (13)]

remain nontrivial, the sides being of order θ^3 , θ^3 , and θ^4 , respectively. On the other hand, were we to have

$$V_{\text{KM}} \sim \begin{pmatrix} 1 & \theta & \theta^3 & \theta^5 \\ \theta & 1 & \theta^2 & \theta^4 \\ \theta^3 & \theta^2 & 1 & \theta^3 \\ \theta^5 & \theta^4 & \theta^3 & 1 \end{pmatrix} \quad (15)$$

then the first and third unitarity triangles would be nontrivial (of size $\sim \theta^3$ and θ^5 , respectively), while the second "triangle," Eq. (12), becomes approximately degenerate.

We have alluded to a discrete ambiguity in the above construction. This was missed by the author and pointed out to him by A. Martin. For an $N \times N$ matrix there are $n = (N-1)(N-2)/2$ triangle constructions; hence a 2^n -fold ambiguity, including that associated with overall complex conjugation $V_{ij} \leftrightarrow V_{ij}^*$.

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