Reflectionless symmetric potentials from vertex operators

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The construction of symmetric reflectionless potentials with an arbitrary bound-state spectrum is compared using supersymmetric quantum mechanics and using vertex operators. The simplicity of each method may have useful consequences for the other.

It is relatively easy to construct the unique symmetric reflectionless potential $V(x)=V(-x)$ possessing an arbitrary bound-state spectrum. In part for this reason, several of us have employed such potentials over the past ten years as a means of describing arbitrary potentials such as those encountered in quarkonium systems.[1–6]

In this Brief Report we note that techniques associated with vertex operators[7] allow an extremely concise derivation of a form previously obtained[3] for such potentials. By comparison with results obtained[5,8] using supersymmetric quantum mechanics,[9] we find an alternative representation of the vertex operator in terms of Schrödinger wave functions. Finally, we comment on the fact that the same level cannot be inserted twice in constructing symmetric reflectionless potentials. This extends a previous observation,[4] and makes contact with the supersymmetric nature of the transformation which adds a level to a potential.

We seek a potential $V(x)$ in the one-dimensional Schrödinger equation

$$\left[-\frac{\hbar^2}{2\mu}\frac{d^2}{dx^2}+V(x)\right]\psi(x)=E\psi(x) \tag{1}$$

with the following properties.

(a) The potential is symmetric: $V(x)=V(-x)$. The odd-parity solutions of Eq. (1) then are the appropriate reduced radial wave functions $u(r)=rR(r)$, satisfying $u(0)=0$, for S waves in a central potential $V(r)$.

(b) The bound states in Eq. (1) occur at energies

$$E_n=-\hbar^2\kappa_n^2/2\mu\ . \tag{2}$$

Henceforth we shall set $\hbar=2\mu=1$.

(c) The potential $V(x)$ leads to scattering without reflection. As a result, the construction of $V(x)$ is possible on the basis of its bound states alone.[10] Scattering data, normally required in the inverse-scattering formalism,[11] may be dispensed with.

The result for $V(x)$ satisfying conditions (a)–(c) above, obtained in Ref. 3, is that

$$V(x)=-2\frac{d^2}{dx^2}\ln D(x)\ , \tag{3}$$

where

$$D(x)=\sum_S \Pi(S,\bar{S})\cosh\left[x\left[\sum_{m\in S}\kappa_m-\sum_{n\in\bar{S}}\kappa_n\right]\right]\ . \tag{4}$$

Here we define the order of levels such that

$$\kappa_n>\kappa_{n-1}>\cdots>\kappa_1 \tag{5}$$

and

$$\Pi(S,\bar{S})\equiv\sum_{i\in S;j\in\bar{S}}\left|\frac{\kappa_i+\kappa_j}{\kappa_i-\kappa_j}\right|\ . \tag{6}$$

When the set S or $\bar{S}$ is empty, $\Pi(S,\bar{S})\equiv 1$. As an example, for one level, at $E=-\kappa^2$,

$$D^{(1)}(x)=\cosh\kappa x\ , \tag{7}$$

corresponding to

$$V^{(1)}(x)=-2\kappa^2/\cosh^2\kappa x\ , \tag{8}$$

while for two levels,[4]

$$D^{(2)}(x)=2[\cosh(\kappa_1+\kappa_2)x+g_{12}\cosh(\kappa_1-\kappa_2)x]\ , \tag{9}$$

where $g_{12}\equiv(\kappa_1+\kappa_2)/(\kappa_2-\kappa_1)$.

The potential $V(x)$ in the Schrödinger equation may be regarded as an instantaneous "snapshot" of a family of solitons of various nonlinear equations. The Korteweg–de Vries equation[12] is the simplest of these:

$$u_t=u_{xxx}+6uu_x\ . \tag{10}$$

The multisoliton solutions of Eq. (10) are just the families of reflectionless potentials:

$$V(x)=-u(x,t=0)\ . \tag{11}$$

The one-soliton solution of Eq. (10), related to the one-level potential (8), is

$$u(x,t)=2\kappa^2/\cosh^2\kappa(x-x_0+4\kappa^2t)\ . \tag{12}$$

In general there will be equations of order $2n+1$ in x, and first order in t, whose one-soliton solutions are of the form

$$u(x,t)=2\kappa^2/\cosh^2\kappa[x-x_0+(2\kappa)^{2n}t] \ . \tag{13}$$

One may regard the solitons as evolving according to infinitely many time variables, labeled as $t_3,t_5,t_7,\ldots$, in such a way that, for a single soliton,

$$u(x,t_3,t_5,\ldots)=2\kappa^2/\cosh^2\kappa\left[x-x_0+\sum_{n=1}^{\infty}(\kappa)^{2n}t_{2n+1}\right] . \tag{14}$$

(Here we have absorbed some powers of 2 into definitions of t_{2n+1}.) It has then been shown[7,13,14] that the general N-soliton solution is of the form

$$u_n=2\frac{\partial^2}{\partial x^2}\ln\tau_N \ , \tag{15}$$

with

$$\tau_n=[A_nX(\kappa_n)+B_nX(-\kappa_n)]\tau_{n-1} \tag{16}$$

and

$$X(\kappa)\equiv\exp(\kappa x+\kappa^3t_3+\kappa^5t_5+\cdots)\times\exp\left[-\frac{1}{\kappa}\frac{\partial}{\partial x}-\frac{1}{3\kappa^3}\frac{\partial}{\partial t_3}-\cdots\right] . \tag{17}$$

The operator $X(\kappa)$ is known as the *vertex operator*.[15] It has appeared previously in the dual resonance model, the forerunner of string theory.[16] The function τ_0 may be defined to be 1. The ratio of the coefficients A_n and B_n in Eq. (16) is a free parameter for each n, while multiplication of τ_n by an overall factor does not change u_n.

The one-soliton solution which is symmetric in x at $t_3=t_5=\cdots=0$ is generated by

$$t_1^{\rm symm}=[X(\kappa)+X(-\kappa)]1 \ . \tag{18}$$

At $t_3=t_5=\cdots=0$, this agrees, up to an irrelevant overall factor, with the result (7).

A useful identity[17] is

$$X(\kappa)X(\kappa')1=\left|\frac{\kappa-\kappa'}{\kappa+\kappa'}\right|^{1/2}\exp(\kappa x+\kappa^3t_3+\cdots)\times\exp(\kappa'x+\kappa'^3t_3+\cdots) . \tag{19}$$

This is most easily proved (let us take $\kappa>\kappa'$) by noting that

$$\exp\left[-\frac{1}{p\kappa^p}\frac{\partial}{\partial t_p}\right]$$

is a shift operator:

$$\exp\left[-\frac{1}{p\kappa^p}\frac{\partial}{\partial t_p}\right]f(t_p)=f\left[t_p-\frac{1}{p\kappa^p}\right] . \tag{20}$$

Then

$$\begin{aligned}X(\kappa)X(\kappa')&=\exp(\kappa x+\kappa^3t_3+\cdots)\exp\left[\kappa'\left[x-\frac{1}{\kappa}\right]+\kappa'^3\left[t_3-\frac{1}{3\kappa^3}\right]+\cdots\right]\\&=\exp\left[-\sum_{k=0}^{\infty}\frac{(\kappa'/\kappa)^{2\kappa+1}}{2k+1}\right]\exp(\kappa x+\kappa^3t_3+\cdots)\exp(\kappa'x+\kappa'^3t_3+\cdots) .\end{aligned} \tag{21}$$

But

$$\sum_{k=0}^{\infty}\frac{(\kappa'/\kappa)^{2k+1}}{2k+1}=\frac{1}{2}\ln\frac{1+\kappa'/\kappa}{1-\kappa'/\kappa} \ , \tag{22}$$

leading to (19).

One finds, first of all, that $X(\kappa)X(\kappa)1=0$. Physically this corresponds to the fact that one cannot add the same level to a potential twice. There is no degeneracy of bound states in one-dimensional quantum mechanics. (For an earlier discussion in the context of the inverse-scattering method, see Ref. 4.)

The two-level symmetric, reflectionless potential may be obtained from

$$\begin{aligned}\tau_2^{\rm symm}|_{t_3=t_5=\cdots=0}&=[X(\kappa_2)+X(-\kappa_2)][X(\kappa_1)+X(-\kappa_1)]1\\&=2\left[\left|\frac{\kappa_2-\kappa_1}{\kappa_2+\kappa_1}\right|^{1/2}\cosh(\kappa_1+\kappa_2)x+\left|\frac{\kappa_2+\kappa_1}{\kappa_2-\kappa_1}\right|^{1/2}\cosh(\kappa_2-\kappa_1)x\right] .\end{aligned} \tag{23}$$

Up to an inessential overall factor, this is just (9).

The construction of the symmetric, reflectionless n-level potential is now straightforward. We define

$$t_n^{\rm symm}=[X(\kappa_n)+X(-\kappa_n)]t_{n-1}^{\rm symm} \tag{24}$$

and

$$V_n(x)=-2\frac{\partial^2}{\partial x^2}\tau_n^{\rm symm}\bigg|_{t_3=t_5=\cdots=0} . \tag{25}$$

The shift operators contained in $X(\kappa_n)$, acting to the right, generate products

$$\prod_{j=1}^{n-1}\left|\frac{\kappa_n \pm \kappa_j}{\kappa_n+\kappa_j}\right|^{1/2}$$

when acting on the exponentials in $\tau_{n-1}^{\rm symm}$. If we divide out all the factors of the form

$$\left|\frac{\kappa_n - \kappa_j}{\kappa_n+\kappa_j}\right|^{1/2}$$

we just obtain the result (4).

The product of N vertex operators $X(\kappa_N)\cdots X(\kappa_1)$ $(\kappa_N \geq \cdots \geq \kappa_1)$ acting on 1 will vanish when any two levels are made to coalesce, because of the ubiquitous factors

$$\left|\frac{\kappa_i - \kappa_j}{\kappa_i+\kappa_j}\right|^{1/2} .$$

Again, degeneracy of levels in one-dimensional quantum mechanics is forbidden. (A slightly more involved proof applies to the symmetrized version leading to τ_N.) The vertex operators act like fermions in this respect, obeying an exclusion principle. There is thus a further connection between the vertex operator (which performs a Bäcklund transformation,[18] adding a soliton) and the supersymmetry transformation in supersymmetric quantum mechanics, which adds a level[19] to a potential. (These connections were noted in Ref. 20.)

In practice for large n there turns out to be a more useful way to evaluate $V_n(x)$ than to compute $D(x)$ in Eq. (4). One solves the chain of equations

$$\begin{aligned} &f_n^2 - f_n' = V_{n-1}(x) + \kappa_n^2 \quad [V_0(x) \equiv 0] ,\\ &f_n^2 + f_n' = V_n(x) + \kappa_n^2, \quad n = 1, \ldots, N , \end{aligned} \tag{26}$$

easily derived using the methods of supersymmetric quantum mechanics.[5,8] Symmetric potentials are obtained if one imposes the boundary condition $f_n(0)=0$. The equations for $f_n^2 - f_n'$ are Riccati equations which may be linearized using the substitution

$$f_n = -w_n'/w_n , \tag{27}$$

so that

$$w_n''/w_n = V_{n-1}(x) + \kappa_n^2 . \tag{28}$$

But this is just the Schrödinger equation for w_n in the potential $V_{n-1}(x)$. Since $-\kappa_n^2$ is below all bound-state energies in V_{n-1}, the wave function $w_n(x)$ [obeying $w_n'(0)=0$ in order that $f_n(0)=0$] is even in x, nodeless, and diverges exponentially at $x=\pm\infty$. It has been encountered previously in related studies.[21] The ground-state Schrödinger wave function in $V_n(x)$, with bound-state energy $-\kappa_n^2$, turns out to be just $1/w_n(x)$ (Refs. 5 and 8).

Numerically in Eqs. (26) one only has to perform a single "sweep" of the integration region, passing information obtained on $w_j(x)$ at the kth integration step back to the $(k-1)$th step in the determination of $w_{j+1}(x)$ (Ref. 5).

Taking differences of pairs of Eqs. (26), one finds

$$V_n - V_{n-1} = 2f_n' = -2\frac{\partial^2}{\partial x^2}\ln w_n , \tag{29}$$

which implies

$$D_N(x) = \prod_{n=1}^{N} w_n(x) . \tag{30}$$

In particular

$$\tau_n|_{t_3=t_5=\cdots=0} = w_n(x)\tau_{n-1}|_{t_3=t_3=\cdots=0} . \tag{31}$$

If $V(x)$ is replaced by $-u(x,t_3,t_5,\ldots)$ and the corresponding $w_n(x)$ by $w_n(x,t_3,t_5,\ldots)$, it is in fact true[7,14] that

$$\frac{\tau_n}{\tau_{n-1}} - \frac{X(\kappa_n)\tau_{n-1}}{\tau_{n-1}} = w_n \tag{32}$$

for arbitrary times $t_3, t_5, \ldots$.

We have thus shown one example in which a problem associated with a vertex operator is most directly solved by recourse to the *linear* problem, involving determination of the "fake" Schrödinger wave functions $w_n(x)$. We would expect this result to have useful generalizations.

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