Conservation laws, Korteweg–de Vries and sine-Gordon systems, and the role of supersymmetry

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It is shown that the eigenvalue problem of the L operator for the sine-Gordon equation can be put in a supersymmetric form. We comment on the connection between the conserved quantities of the Korteweg–de Vries and sine-Gordon systems.

Perhaps the simplest of all integrable systems is the Korteweg–de Vries (KdV) equation

$$\frac{\partial u}{\partial t}=6u\frac{\partial u}{\partial x}-\frac{\partial^3 u}{\partial x^3} . \tag{1}$$

This equation has been studied[1] widely in the literature and is known to possess an N-soliton solution.

The conservation laws in (1) may be derived [1(a),2] from a simple consideration of the Poisson brackets. Expressing (1) in two equivalent ways ($D\equiv\partial/\partial x$),

$$\frac{\partial u}{\partial t}=Mu,\quad M=-D^3+2(Du+uD) , \tag{2a}$$

$$\frac{\partial u}{\partial t}=D(3u^2-D^2u) , \tag{2b}$$

the following definitions of Poisson brackets,

$$\{F[u],G[u]\}_1=\int dx\frac{\delta F[u]}{\delta u(x)}M\frac{\delta G[u]}{\delta u(x)} , \tag{3a}$$

$$\{F[u],G[u]\}_2=\int dx\frac{\delta F[u]}{\delta u(x)}D\frac{\delta G[u]}{\delta u(x)} , \tag{3b}$$

where F and G are functionals of u, may be seen to lead to

$$\frac{\partial u}{\partial t}=\{u(x),H_1\}_1=\text{right-hand side (RHS) of (2a)} ,$$

$$H_1=\int dx\, u^2(x) , \tag{4a}$$

$$\frac{\partial u}{\partial t}=\{u(x),H_2\}_2=\text{RHS of (2b)} ,$$

$$H_2=\int dx\,[u^3+\tfrac{1}{2}(Du)^2] . \tag{4b}$$

On account of (3) and (4), one can immediately write down the connection

$$D\frac{\delta H_{n+1}}{\delta u(x)}=M\frac{\delta H_n}{\delta u(x)},\quad n=0,1,2,\ldots , \tag{5}$$

where $H_0=\int dx\, u(x)$, H_1,H_2,H_3, etc., play the role of conserved quantities. It is obvious from the Poisson-brackets structure of (4) that the conserved quantities are actually a sequence of Hamiltonians each generating its own evolution equation.

To derive the conservation laws in other integrable systems, the transition from the KdV equation to its modified form (MKdV, for short) is naturally the next obvious step. The MKdV equation is not only interesting in its own right but has a subtle role to play in bringing out the connection between supersymmetric quantum mechanics and the construction of the N-soliton solution of the KdV equation. To appreciate this, we need to note that the quantities

$$u_\pm=f^2\pm f_x-k^2 , \tag{6a}$$

$$k=\text{const} \tag{6b}$$

are automatic solutions in (1), provided that f satisfies

$$f_t+6(k^2-f^2)f_x+f_{xxx}=0 . \tag{7}$$

Equation (7) is recognized in the literature as the MKdV equation and one of solutions (6) (when $k=0$) is similar[3] to Miura's transformation.

The combinations in (6) are nothing but the so-called partner potentials in supersymmetric quantum mechanics. Physically this means that if we define

$$f_-=f^2-f'=k^2 \tag{8a}$$

then the positivity of k^2 prevents f_- from having any bound state. On the contrary, it can be shown[4,5] that the f_+ given by

$$f_+=f^2+f'=k^2 \tag{8b}$$

possesses a zero-energy bound state. Thus one can carry out the construction of reflectionless potentials as indeed the form[4] of the S matrix for the f_+ reveals. Furthermore, employing appropriate boundary conditions on f, the solutions u_+ and u_- may be identified with the $N+1$ and N solitons of (1), respectively.

The conserved quantities for the MKdV system may be worked out from those of the KdV using (6). These turn out to be

$$H_0=\int dx\, f^2,\quad H_1=\int dx[f^4-6k^2f^2+(Df)^2]\,, \tag{9}$$

etc. However, one can also obtain (9) by expressing the MKdV equation in a pair of equivalent ways as was in the case of the KdV. In this case, the quantity corresponding to M in (2a) is $[-D^3+\frac{3}{2}(Df^2+f^2D)-6k^2D]$.

The purpose of this work is to show how supersymmetric transformations may be used to relate the conserved quantities in the KdV system and those which follow from the sine-Gordon equation. Actually, as will be seen in the following, a remarkable property of the Lax form for the sine-Gordon equation is that it is endowed with a supersymmetric structure if we look at the eigenvalue problem of the L operator. The relation between the conserved quantities in the KdV and sine-Gordon systems emerges as a trivial consequence. It is not quite obvious why a link should exist at all between the KdV system which is a nonrelativistic nonlinear equation and the sine-Gordon system which is a relativistic one. However, as should become clear from the arguments given below, supersymmetric transformations do not really transform the sine-Gordon equation bodily into the KdV equation: only the eigenvalue equation of the L operator (and as a consequence, the conserved quantities) for the two are mapped onto each other.

For the KdV equation, the L and B operators are given by[6]

$$L=-\frac{\partial^2}{\partial x^2}+u(x,t)\,, \tag{10a}$$

$$B=4\frac{\partial^3}{\partial x^3}-3\left[u\frac{\partial}{\partial x}+\frac{\partial}{\partial x}u\right]. \tag{10b}$$

These enable one to express (1) in the Lax form

$$\dot{L}=[L,B]\,. \tag{11}$$

The spectrum of L is conserved and the conserved quantities may be obtained from the definitions[7]

$$L(x,y)=-\frac{\partial^2}{\partial x^2}\delta(x-y)+u(x)\delta(x-y) \tag{12a}$$

along with

$$\mathrm{Tr}L=\int dx\, dy\, \delta(x-y)L(x,y)\,. \tag{12b}$$

On the other hand, the sine-Gordon equation when expressed in light-cone coordinates $x_+=\frac{1}{2}(x+t)$, and $x_-=\frac{1}{2}(x-t)$ reads

$$\frac{\partial^2\psi}{\partial x_+\partial x_-}=-\sin\psi\,, \tag{13a}$$

i.e.,

$$\dot{\psi}'=-\sin\psi\,, \tag{13b}$$

where the overdot (prime) denotes a partial derivative with respect to x_+ (x_-). As the work of Ablowitz, Kaup, Newell, and Segur[8] has shown, like the KdV equation, (13b) also admits to the Lax form (11) with

$$L=2\sigma_3\frac{\partial}{\partial x_-}+\sigma_2\psi'\,, \tag{14a}$$

$$B=(\sigma_3\cos\psi+\sigma_2\sin\psi)L^{-1}\,. \tag{14b}$$

The use of light-cone coordinates reflects that the conserved quantities will be the x_- integrals of appropriate functions of ψ.

Because of the presence of the Pauli matrices σ_2 and σ_3 in (14), the operators L and B possess a (2×2) matrix structure. Hence the spectral problem for L looks like

$$L\chi=\xi\chi\,, \tag{15a}$$

where ξ is a constant and χ is given by

$$\chi=\begin{bmatrix}\chi_1\\ \chi_2\end{bmatrix}. \tag{15b}$$

Inserting (15b) in (15a) and using the form of L in (14a), the resulting equations become

$$2\chi_1'-i\psi'\chi_2=\xi\chi_1\,, \tag{16a}$$

$$i\psi'\chi_1-2\chi_2'=\xi\chi_2\,. \tag{16b}$$

Interestingly (16) may be put in a supersymmetric form. This may be achieved if we introduce the notation

$$\chi_+=\chi_1+\chi_2\,, \tag{17a}$$

$$\chi_-=\chi_1-\chi_2 \tag{17b}$$

along with $W=i\psi'/2$. We obtain as a consequence

$$-\chi_+''-(-W^2-W')\chi_+=-(\xi^2/4)\chi_+\,, \tag{18a}$$

$$-\chi_-''-(-W^2+W')\chi_-=-(\xi^2/4)\chi_-\,. \tag{18b}$$

The supersymmetric Hamiltonian H^s which acts on the two-component column

$$\begin{bmatrix}\chi_+\\ \chi_-\end{bmatrix}$$

may be expressed as

$$H^s=\begin{bmatrix}H_+ & 0\\ 0 & H_-\end{bmatrix}, \tag{19}$$

where $H_\pm=-d^2/dx^2+U_\pm$ and U_+ and U_- represent the so-called bosonic (+) and fermionic (−) potentials

$$U_\pm=W^2\pm W'\,. \tag{20}$$

Indeed factorizability of H^s enables us to define nilpotent fermionic operators Q and Q^+ such that

$$Q=\begin{bmatrix}0 & 0\\ A & 0\end{bmatrix},\quad Q^\dagger=\begin{bmatrix}0 & A^\dagger\\ 0 & 0\end{bmatrix}, \tag{21}$$

where

$$A=-\frac{d}{dx}+W\,, \tag{22a}$$

$$A^\dagger=\frac{d}{dx}+W\,. \tag{22b}$$

In terms of Q and $Q^{\dagger}$, H^s is

$$H^s=Q^{\dagger}Q+QQ^{\dagger} \tag{23}$$

with H^+ and H^- corresponding to the pieces $A^{\dagger}A$ and $AA^{\dagger}$, respectively.

Since $Q^2=Q^{\dagger 2}=0$, both Q and $Q^{\dagger}$ commute with H^s. In supersymmetric quantum mechanics, this means[9] that if ϕ_n is an eigenvector then, with the exception of the $n=0$ case, $Q\phi_n$ is one also. Since $Q\phi_n$ is zero trivially, one concludes that the ground state is associated with the bosonic Hamiltonian H_+ only. However, all other states are paired, thereby yielding two towers of degenerate eigenvalues.

Such a natural embedding of supersymmetry in the eigenvalue problem of the L operator for the sine-Gordon equation does not seem to have been noticed before. To the best of our knowledge, supersymmetry has been so far developed to construct the general N-soliton solutions of the KdV equation.

It is straightforward to relate the conservation laws in the KdV and sine-Gordon systems. To this end, we write down the eigenvalue equation of the L operator for the KdV system. Using (10a), this reads

$$\left[-\frac{d^2}{dx^2}+u\right]\chi=\hat{\xi}\chi \ . \tag{24}$$

It then follows that either of Eqs. (18) is identical to (24) if u transforms as

$$u\to(W^2\pm W') \ , \tag{25a}$$

$$\hat{\xi}\to-(\xi^2/4) \ . \tag{25b}$$

Effectively, this implies that given the set of conserved quantities H_1, H_2, etc. of the KdV system, the corresponding ones for the sine-Gordon system may be written down using the mapping (25) and the link between W and ψ. These are

$$\begin{aligned} Q_0&=-\tfrac{1}{4}\int \psi'^2 dx_- \ , \\ Q_1&=\tfrac{1}{16}\int(\psi'^4-4\psi''^2)dx_- \ , \\ Q_2&=-\tfrac{1}{64}\int(\psi'^6-20\psi'^2\psi''^2+8\psi'''^2)dx_- \ , \end{aligned} \tag{26}$$

etc. It may be noted that to express the Q's in the above form, we have integrated by parts[10] and discarded the total derivatives. It should be emphasized the the Q's do not depend on the arbitrariness of the sign in (25).

To bring out the role of supersymmetry in the above correspondence let us distinguish the two cases in (25) by $\bar{u}_+$ and $\bar{u}_-$. Our preceding discussion on supersymmetry now tells us that if suitable boundary conditions are imposed on ψ (and consequently W) $\bar{u}_+$ may be interpreted as an $(N+1)$-soliton solution if $\bar{u}_-$ corresponds to the N-soliton solution.

Elimination of W yields

$$\frac{\partial}{\partial x_-}(\bar{u}_+ +\bar{u}_-)^{1/2}=\tfrac{1}{2}(\bar{u}_+ -\bar{u}_-) \ , \tag{27}$$

which provides a relation between $\bar{u}_+$ and $\bar{u}_-$. Expectedly, this relation is similar to what one would have obtained if f was eliminated between (6a) and (6b).

In this way the connection between the infinite sequence of conserved quantities in the KdV system and those of the sine-Gordon system may be established. But, unlike the transformation (6) which implied that if f was a solution of the MKdV then $u_\pm$ were those of the KdV, the transformations (25) only provide a convenient connection between the conservation laws in the KdV and sine-Gordon systems. However, it is not true that if ψ satisfies the sine-Gordon equation then the right-hand side of (25) will satisfy the KdV equation.

To conclude, it may be remarked that during a literature survey we came across the work of Chodos[7] in which a connection between the conserved quantities in the KdV and sine-Gordon systems had been noticed. Although a formal proof has been given to show that the coefficient of i in the transformation (25) did not contribute to the sine-Gordon conservation laws, Chodos did not notice that, as the sine-Gordon equation is invariant under $\psi\to-\psi$, the transformation $u\to-\frac{1}{4}(\psi'^2+2i\psi'')$ could also be an equally acceptable choice and so it was not surprising that the conserved quantities could not really depend on the imaginary part. In fact, one can check through simple integration by parts that one can do away with the imaginary quantities in (26). Nevertheless, Chodos's work appears to be the first attempt[11] to relate the conservation laws of the KdV and sine-Gordon systems. That this author refrained from drawing any conclusions regarding the closeness of his transformation to those of supersymmetry is perhaps because the discovery of supersymmetry in nonrelativistic mechanics came about[12] later.

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[10]For instance, $\int \psi'^2\psi'' dx_- = 0$, $\int \psi'^4\psi'' dx_- = 0$, and $\int (\psi''^3 + 2\psi'\psi''\psi''')dx_- = 0$.

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