Gaussian-improved one-loop effective potential

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A new type of effective potential is proposed. It combines the ordinary one-loop effective potential and a Gaussian approximation within the method of the constraint effective potential. The potential contains the one-loop term and formally bounds the exact effective potential from above. $\lambda\phi^4$ field theory in four dimensions is reconsidered and the new potential for this theory is calculated.

I. INTRODUCTION

A well-known method to study the behavior of quantum-field-theory models (spontaneous symmetry breaking, triviality, stability, phase structure, etc.) is to use the effective action or the effective potential $V_{\text{eff}}(\phi_0)$ (Ref. 1). The (exact) effective potential $V_{\text{eff}}(\phi_0)$ is rather difficult to compute in practice; so a reliable and calculationally relatively simple approximate effective potential is needed. In the physical literature the most popular one is the one-loop (or, in general, L-loop) effective potential. The one-loop effective potential is a computationally rather simple object and is believed to be a good approximation. Since it has been shown that the one-loop effective potential can be obtained by minimizing the expectation value of the shifted linearized Hamiltonian in a Gaussian state[2] a nonlinearity is lost in a sense from the point of view of the so-called Gaussian (Hartree-Fock) approximation. Recently, the Gaussian effective potential $V_{\text{GEP}}(\phi_0)$, which takes into account the full nonlinear Hamiltonian, has attracted much attention.[3] The Gaussian effective potential $V_{\text{GEP}}(\phi_0)$ is the minimum expectation value of the Hamiltonian density $\mathcal{H}$ in a Gaussian state $|s\rangle$ wherein the field operator ϕ has a constant expectation value ϕ_0. It is an essentially nonperturbative object and by virtue of the definition it formally satisfies the highly nontrivial inequality

$$V_{\text{eff}}(\phi_0) \le V_{\text{GEP}}(\phi_0) . \tag{1}$$

Inequality (1), however potentially very interesting, may be spoiled by renormalization procedure.[4,5]

The third so-called constraint effective potential[6,7] $V_{\text{con}}(\phi_0)$ [see Eq. (3)] was at first introduced as a computational tool in the analysis of the exact effective potential $V_{\text{eff}}(\phi_0)$ permitting us to avoid the inconvenient Legendre transform. But later on it appeared that it was related to similar definitions in statistical mechanics and it had something to do with the Wilson effective potential extensively used by constructivists.[8] The constraint effective potential $V_{\text{con}}(\phi_0)$ tends to the true effective potential $V_{\text{eff}}(\phi_0)$ in the infinite-volume limit,[6] i.e.,

$$\lim_{\Omega\to+\infty} V_{\text{con}}(\phi_0;\Omega)=V_{\text{eff}}(\phi_0) . \tag{2}$$

The aim of our paper is to propose a new, hybrid, Gaussian-improved one-loop effective potential $V_{\text{GL}}(\phi_0)$. This effective potential consists of three parts: the classical potential $V(\phi_0)$, the usual one-loop contribution $V_{1L}(\phi_0)$, and a Gaussian term $V_G(\phi_0)$. It should be stressed that this compound appears in a very natural way as a result of some simple procedure performed on the functional integral representation for the effective potential. As a starting point we have used the concept of the constraint effective potential $V_{\text{con}}(\phi_0)$ and some earlier ideas presented in Ref. 4.

Introducing a new object needs some justification. In the first place, proposing a compromise effective potential $V_{\text{GL}}(\phi_0)$ we aim to calm the controversy between the "one-loop traditionalists" and the "followers" of the Gaussian-effective-potential method. In the second place, we would like to satisfy for our effective potential $V_{\text{GL}}(\phi_0)$ an inequality analogous to inequality (1). Thus we hope to retain advantages following from both of these approaches. Our method is defined for the Euclidean version of quantum field theory and it works only for boson fields.

The plan of the paper is as follows. Section II reviews the definition of the constraint effective potential $V_{\text{con}}(\phi_0)$ and introduces the notion of the Gaussian-improved one-loop effective potential $V_{\text{GL}}(\phi_0)$ for boson field theory in the Euclidean space. As a standard example $\lambda\phi^4$ theory is reconsidered in Sec. III, where a cutoff version of $V_{\text{GL}}(\phi_0)$ for this model is derived. The renormalized Gaussian-improved one-loop effective potential is obtained in Sec. IV. A comparison with other methods and summary is given in Sec. V.

II. GENERAL FORMALISM FOR BOSON FIELDS

The constraint effective potential $V_{\text{con}}(\phi_0)$ corresponding to a classical Euclidean boson action $S(\phi)=S_0(\phi)+S_I(\phi)$ in a $(D+1)$-dimensional space, where $S_0(\phi_0)$ is a free term and $S_I(\phi_0)$ is an interaction, is defined by the following Euclidean path integral:[6]

$$\exp[-\Omega V_{\text{con}}(\phi_0)]\equiv\text{const}\times\int D\phi_1 \exp[-S(\phi_0+\phi_1)] , \tag{3}$$

where Ω is a $(D+1)$-dimensional volume ($\Omega\equiv\int d^{D+1}x$), ϕ_0 is the constant component of the field

ϕ [the zero $(D+1)$-momentum component of the field ϕ in the Fourier expansion], and ϕ_1 is the nonconstant component of the field ϕ [nonzero $(D+1)$-momentum components of the field ϕ in the Fourier expansion]. Obviously $\phi=\phi_0+\phi_1$. Here, the normalization constant we have chosen is

$$\text{const}^{-1}\equiv \int D\phi_1 \exp[-S_0(\phi_1)] \ . \tag{4}$$

One can easily see that Eq. (3) gives for the free action $[S(\phi)=S_0(\phi)]$ the classical potential $V_{\text{con}}(\phi_0)=V(\phi_0)$. Now, using Eq. (4), we can rewrite Eq. (3) as

$$\exp[-\Omega V_{\text{con}}(\phi_0)]=\int d\mu_1 \exp[-S'(\phi_0,\phi_1)] \ , \tag{5}$$

where the Gaussian measure we have introduced is

$$d\mu_1\equiv\text{const}\times D\phi_1 \exp[-S_0(\phi_1)] \ , \tag{6}$$

and

$$S'(\phi_0,\phi_1)=S(\phi_0+\phi_1)-S_0(\phi_1)$$
$$=S_0(\phi_0+\phi_1)-S_0(\phi_1)+S_I(\phi_0+\phi_1) \ .$$

The measure (6) is a probabilistic one; i.e., it is positive and normalized to unity.

We would like to extract the one-loop contribution $V_{1L}(\phi_0)$ performing the expansion

$$S(\phi_0+\phi_1)=S(\phi_0)+S_2(\phi_0,\phi_1)+S_R(\phi_0,\phi_1) \ , \tag{7}$$

where $S_2(\phi_0,\phi_1)$ is a term quadratic in field ϕ_1, and $S_R(\phi_0,\phi_1)$ is a rest. $S_R(\phi_0,\phi_1)$ can contain quadratic terms of "quantum origin," e.g., terms included into normal-ordering procedure. The term linear in field ϕ_1 vanishes because the volume integral of the nonzero $(D+1)$-component of ϕ_1 vanishes:[6] namely,

$$S_1(\phi_0,\phi_1)=\int \mathcal{L}_1(\phi_0)\phi_1(x)d^{D+1}x=\mathcal{L}_1(\phi_0)\int \phi_1(x)d^{D+1}x$$
$$=\mathcal{L}_1(\phi_0)\int \left[\phi(x)-\Omega^{-1}\int \phi(x)d^{D+1}x\right]d^{D+1}x=0 \ .$$

Collecting Eqs. (4) and (7), and inserting unity, we can rearrange Eq. (3) in the following manner:

$$\begin{aligned}\exp[-\Omega V_{\text{con}}(\phi_0)]&=\left[\int D\phi_1 \exp[-S_0(\phi_1)]\right]^{-1}\int D\phi_1 \exp[-S(\phi_0)-S_2(\phi_0,\phi_1)-S_R(\phi_0,\phi_1)]\\ &=\exp[-S(\phi_0)]\left[\int D\phi_1 \exp[-S_0(\phi_1)]\right]^{-1}\int D\phi_1 \exp[-S_2(\phi_0,\phi_1)]\\ &\qquad\times\left[\int D\phi_1 \exp[-S_2(\phi_0,\phi_1)]\right]^{-1}\\ &\qquad\times\int D\phi_1 \exp[-S_2(\phi_0,\phi_1)-S_R(\phi_0,\phi_1)]\\ &=\exp[-S(\phi_0)-S'_{1L}(\phi_0)]\int d\mu_1(\phi_0)\exp[-S_R(\phi_0,\phi_1)] \ ,\end{aligned} \tag{8}$$

where

$$S'_{1L}(\phi_0)\equiv\text{const}\times\int D\phi_1 \exp[-S_2(\phi_0,\phi_1)] \ , \tag{9}$$

and $d\mu_1(\phi_0)$ is a normalized Gaussian measure with a background-field covariance

$$d\mu_1(\phi_0)\equiv\left[\int D\phi_1 \exp[-S_2(\phi_0,\phi_1)]\right]^{-1}D\phi_1 \times\exp[-S_2(\phi_0,\phi_1)] \ .$$

It is easy to check that in the infinite-volume limit $S'_{1L}(\phi_0)$ tends to the one-loop contribution $V_{1L}(\phi_0)$:

$$\lim_{\Omega\to+\infty}\Omega^{-1}S'_{1L}(\phi_0)=V_{1L}(\phi_0) \ .$$

Using the Jensen inequality for the convex function "exp" in Eq. (8) we get the estimation

$$\exp[-\Omega V_{\text{con}}(\phi_0)]\geq\exp\left[-\Omega V(\phi_0)-S'_{1L}(\phi_0) - \int d\mu_1(\phi_0)S_R(\phi_0,\phi_1)\right] \tag{10}$$

and according to Eq. (2) we obtain

$$V_{\text{eff}}(\phi_0)\leq V_{\text{GL}}(\phi_0)\equiv V(\phi_0)+V_{1L}(\phi_0)+V_G(\phi_0) \ , \tag{11}$$

where the Gaussian term $V_G(\phi_0)$ is given by the expression

$$V_G(\phi_0)\equiv\lim_{\Omega\to+\infty}\Omega^{-1}\int d\mu_1(\phi_0)S_R(\phi_0,\phi_1) \ . \tag{12}$$

It is worth noting that $V_G(\phi_0)$ contains a part of the higher-loop contributions.

From inequality (11) it follows that the true effective potential is bounded from above by $V_{\text{GL}}(\phi_0)$ [the sum of the classical potential $V(\phi_0)$, the one-loop term $V_{1L}(\phi_0)$, and the Gaussian term $V_G(\phi_0)$]. Unfortunately, the inequality is rather formal [compare with (1)], and it may be spoiled by the renormalization procedure. We are not able to decide when the renormalization preserves inequality (11) because the exact bare coupling constants are unknown.[4] Instead, we must perform some simpler explicit renormalization procedure to cancel infinities. In any case, provided the one-loop effective potential $V_{1L}(\phi_0)$ gives a reliable approximation the Gaussian-improved one-loop effective potential $V_{\text{GL}}(\phi_0)$ should also give a reliable approximation.

III. CALCULATION OF $V_{GL}(\phi_0)$ FOR $\lambda\phi^4$ THEORY

We will illustrate our method with the help of the simple and widely tested $\lambda\phi^4$ model in $3+1$ dimensions, where the Euclidean action has the form

$$S(\phi)=\int d^4x\left[\tfrac{1}{2}\phi(-\Delta+m_0^2)\phi+\frac{1}{4!}\lambda_0\phi^4\right]. \tag{13}$$

Now

$$V(\phi_0)=\tfrac{1}{2}m_0^2\phi_0^2+\frac{1}{4!}\lambda_0\phi_0^4\,, \tag{14a}$$

and

$$S'_{1L}(\phi_0)=\tfrac{1}{2}\ln\det(-\Delta+m_0^2+\tfrac{1}{2}\lambda_0\phi_0^2)-\tfrac{1}{2}\ln\det(-\Delta+m_0^2)\,, \tag{14b}$$

where the first term on the right-hand side in (14b) is coming from the Gaussian integral in the numerator of Eq. (9), and the second one is coming from the Gaussian integral in the denominator [contained in "const," see Eq. (4)] of Eq. (9). According to Eq. (12),

$$V_G(\phi_0)=\frac{1}{4!}\lambda_0\int d\mu_1(\phi_0)(\phi^4+4\phi_0\phi^3)=\frac{1}{8}\lambda_0\left[\int d\mu_1(\phi_0)\phi^2\right]^2. \tag{14c}$$

The expression (14c) represents a part of the two-loop contribution to the effective potential $V_{\rm eff}(\phi_0)$. Thus, one can implement standard techniques for the effective potential in background fields to settle our problem. The one-loop contribution (14b) can be explicitly calculated:

$$\begin{aligned}V_{1L}(\phi_0)&=\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\ln\left[1+\frac{\lambda_0\phi_0^2/2}{k^2+m^2}\right]\\&=\tfrac{1}{4}(4\pi)^{-2}m^4\left[\chi^2\ln\left[1+\frac{\rho}{1+\chi}\right]-(1+\rho)^2\ln\left[1+\frac{\chi}{1+\rho}\right]+\chi\rho+\ln(1+\chi)\right],\end{aligned} \tag{15a}$$

where $\rho=\lambda_0\phi_0^2/2m^2$ and χ denotes the dimensionless ultraviolet cutoff $\chi=(\Lambda/m)^2$. Similarly (14c) is expressed by

$$\begin{aligned}V_G(\phi_0)&=\tfrac{1}{8}\lambda_0\left[\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m^2+\lambda_0\phi_0^2/2}\right]^2\\&=\tfrac{1}{8}\lambda_0m^4(4\pi)^{-4}\left[\chi-(1+\rho)\ln\left[1+\frac{\chi}{1+\rho}\right]\right]^2.\end{aligned} \tag{15b}$$

Collecting (15a) and (15b) we obtain

$$\begin{aligned}V_{GL}(\phi_0\,,\chi)=m^4\Bigg\{&\frac{1}{\lambda_0}(\rho+\tfrac{1}{6}\rho^2)+(8\pi)^{-2}\left[\chi^2\ln\left[1+\frac{\rho}{1+\chi}\right]-(1+\rho)^2\ln\left[1+\frac{\chi}{1+\rho}\right]+\chi\rho+\ln(1+\chi)\right]\\&+\tfrac{1}{8}\lambda_0(4\pi)^{-4}\left[(1+\rho)\ln\left[1+\frac{\chi}{1+\rho}\right]-\chi\right]^2\Bigg\}.\end{aligned} \tag{16}$$

Formula (16) gives the cutoff version of the Gaussian-improved one-loop effective potential $V_{GL}(\phi_0)$ for $\lambda\phi^4$ theory in four dimensions.

IV. RENORMALIZATION

Equation (16) is full of ultraviolet divergences and needs renormalization. The renormalization procedure can be performed analogously to the method used in Ref. 9 for the two-loop effective potential. Choosing the standard normalization conditions

$$V''_{GL}(0)=m^2\,,\quad V^{(IV)}_{GL}(0)=\lambda\,,$$

we replace (15a) and (15b) by

$$V_{1L}(\phi_0)=\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\left[\ln\left[1+\frac{\lambda\phi_0^2/2}{k^2+m^2}\right]-\frac{\lambda\phi_0^2/2}{k^2+m^2}+\frac{(\lambda\phi_0^2/2)^2}{2(k^2+m^2)^2}\right] \tag{17a}$$

and

$$V_G=\tfrac{1}{8}\lambda\left[\int\frac{d^4k}{(2\pi)^4}\left[\frac{1}{k^2+m^2+\lambda\phi_0^2/2}-\frac{1}{k^2+m^2}-\frac{\lambda\phi_0^2/2}{(k^2+m^2+\lambda\phi_0^2/2)^2}\right]\right]^2. \tag{17b}$$

The first counterterms in (17a) and (17b) can account for the normal ordering in the interaction term, i.e., $\phi^4\to:\phi^4:$. The second terms follow from the renormalization of the bare coupling constant λ_0. After some standard calculations we finally obtain[9]

$$\begin{aligned}V_{GL}(\rho)=&\,m^4\lambda^{-1}\rho+m^4(6\lambda)^{-1}\rho^2\\&+(8\pi)^{-2}m^4[(1+\rho)^2\ln(1+\rho)-\rho-\tfrac{3}{2}\rho^2]\\&+2(8\pi)^{-4}m^4\lambda[(1+\rho)\ln(1+\rho)-\rho]^2\,,\end{aligned} \tag{18}$$

where we have performed "almost two-loop" perturbative renormalization, and $\rho=\lambda\phi_0^2/2m^2$.

In the case of the pure Yang-Mills theory, the situation seems to be more satisfactory because of the possibility of introducing a nonperturbative (in the sense explained below) and more accurate renormalization procedure ("little above leading-logarithm" accuracy). Gauge symmetry (Ward identities) imposes a condition on the renormalization constant of the wave function and of the gauge coupling which leads directly from the one-loop approximation to the leading-logarithm one without using the renormalization-group equation.[10] This renormalization is nonperturbative in the sense that one inserts the renormalization constants not bothering about keeping the proper order in an expansion parameter (the number of loops).

V. SUMMARY AND DISCUSSION

Combining the concept of the constraint effective potential $V_{\rm con}(\phi_0)$ and the method of the Jensen inequality we have succeeded in obtaining a Gaussian-improved one-loop effective potential $V_{\rm GL}(\phi_0)$ which possesses the following properties.

(1) $V_{\rm GL}(\phi_0)$ is an enlargement of the one-loop effective potential and in the case of $\lambda\phi^4$ theory it is contained in the two-loop effective potential.

(2) $V_{\rm GL}(\phi_0)$ formally satisfies inequality (11), in the same sense as the Gaussian effective potential satisfies inequality (1). This permits us to bound the effective potential from above which in turn gives a sufficient condition for a theory to be unstable (inconsistent).

(3) In principle, the functional integral in Eq. (8) can be calculated perturbatively and our Gaussian contribution is only the first term in the cumulant expansion.

Now we would like to recapitulate briefly our results concerning $\lambda\phi^4$ theory and compare them with those obtained in other approaches.

(1) The cutoff version of $\lambda\phi^4$ theory as well as its renormalized version is free of inconsistences (instabilities) for large field ϕ_0 encountered in Ref. 3 or in large-N analysis of Ref. 11.

(2) There is no counterpart of the "precarious" or "autonomous" phase of Ref. 12. The only phase visible in our approach is the perturbative one. Nevertheless, the spontaneous broken phase in the cutoff version of the theory cannot be excluded.

(3) There are no symptoms of the triviality of the renormalized $\lambda\phi^4$ theory in four dimensions.

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