New p-adic strings from old dual models

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In the simplest bosonic p-adic string, a central role is played by an S_3 symmetry in the four-particle amplitude. Using an S_3 analysis of four-point dual amplitudes, we exhibit a general class of p-adic string amplitudes. All are shown to satisfy a factorization (adelic) property for the infinite product over p-adic amplitudes. Explicit examples are worked through for a p-adic string with spin-0 and spin-2 poles and no tachyon.

I. INTRODUCTION

Superstring theory is currently undergoing a phase of consolidation with efforts being devoted to both second-quantized and first-quantized formalisms. In the former, string field theory has not yet fulfilled its initial promise. In the latter, conformal field theory has led to the present attempts to classify all possible vacua of the string.

It is appropriate at this time to pursue an entirely new approach to string theory in order, perhaps, to progress in a completely different direction. Such an approach is the p-adic string.[1] The idea is to replace real numbers by p-adic numbers at some point (e.g., the string coordinate) and, in this way, remarkable formulas for p-adic string tree amplitudes have been obtained.[2–5] The physical interpretation of the p-adic string has been studied using an effective field theory[4,5] and other methods.[6,7] For the moment, the name "p-adic string" refers to a set of amplitudes rather than to a physical picture, just as in the early period of the dual resonance models.

To proceed further, it is necessary to establish for what class of stringlike amplitudes the p-adic counterpart exists. In particular, the adelic formula has been established only for the simplest bosonic string.[2,3,5,8–10] In the present paper we show that new p-adic strings which generalize the adelic formula can be arrived at using the 1973 symmetry-group approach[11,12] to generalize dual resonance models. Here we shall show how the amplitudes arising from that old approach are precisely such that the p-adic counterparts have properties in parallel with those of the Veneziano model. Using the p-adic techniques, we may be able to establish for the case of a real variable a geometrical interpretation, perhaps not a string but at least based on Lagrangians.

In the four-point function, a key role is played by the quantity $\gamma=\alpha(s)+\alpha(t)+\alpha(u)+1$ related to the mass-shell condition for the external particles. The Veneziano model requires $\gamma=0$ and has a tachyon ground state at $m^2=-2$ (we use units with $\alpha'=\frac{1}{2}$). The new p-adic strings we shall arrive at correspond to $\gamma=6n$ with n=non-negative integer, and have ground state at $m^2=-2+12n$. There is no tachyon for nonzero n.

The outline of the paper is as follows: in Sec. II the p-adic string for the Veneziano model is reviewed; in Sec. III the new p-adic strings with $\gamma\neq0$ are developed. In Sec. IV explicit examples for $\gamma=6$ are worked out. Section V is a discussion. Appendixes A and B are devoted to technical questions which arise in the text.

II. p-ADIC VERSION OF SIMPLEST DUAL MODEL (VENEZIANO)

For the simple bosonic string, we write the four-particle amplitude as

$$A_4=\int_{-\infty}^{\infty}dx\,|x|^{-\alpha(t)-1}|1-x|^{-\alpha(s)-1} \tag{2.1}$$

$$=B(-\alpha(s),-\alpha(t))+B(-\alpha(t),-\alpha(u))+B(-\alpha(u),-\alpha(s))\ . \tag{2.2}$$

In order to express A_4 in terms of p-adic gamma functions, let us define

$$\Gamma_p(z)=\frac{1-p^{z-1}}{1-p^{-z}} \tag{2.3}$$

for finite prime number p and for real variables let us define[13]

$$\Gamma_\infty(z)=2(2\pi)^{-z}\cos\frac{\pi z}{2}\Gamma(z)\ , \tag{2.4}$$

where $\Gamma(z)$ is the Euler gamma function. From Eqs. (2.3) and (2.4) one finds the adelic formula

$$\Gamma_\infty(z)\prod_p\Gamma_p(z)=1\ . \tag{2.5}$$

Equation (2.5) has, in general, no region of convergence but may be regularized by the technique discussed in Ref. 10. From Eq. (2.2) we find, using (for the present case $n=0$, but we write the general form for later use)

$$\gamma=\alpha(s)+\alpha(t)+\alpha(u)+1=2n\ , \tag{2.6}$$

$$\sum_{Q=s,t,u}\sin\pi\alpha(Q)=(-1)^{n+1}4\prod_Q\cos\frac{\pi\alpha(Q)}{2}\ , \tag{2.7}$$

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin\pi z}\ , \tag{2.8}$$

that

$$A_4=\frac{4}{\pi}\prod_Q \Gamma(-\alpha(Q))\cos\frac{\pi\alpha(Q)}{2} \tag{2.9}$$

$$=\prod_Q \Gamma_\infty(-\alpha(Q)) , \tag{2.10}$$

using the definition in Eq. (2.4).

From Eqs. (2.3), (2.5), and (2.10) we see that defining

$$A_4^{(p)}=\prod_{Q=s,t,u}\left[\frac{1-p^{-\alpha(Q)-1}}{1-p^{\alpha(Q)}}\right] , \tag{2.11}$$

which is the result of using p-adic integration in Eq. (2.1), then

$$A_4\prod_p A_4^{(p)}=1 . \tag{2.12}$$

The definition, Eq. (2.11), is the p-adic amplitude given in Ref. 2. Equation (2.12) is the adelic formula; its regularization was discussed in Ref. 10.

Note especially the mechanism by which $A_4^{(p)}$ of Eq. (2.11) avoids double poles: at, e.g., $\alpha(s)=\alpha(t)=0$, one has $-\alpha(u)-1=0$ because $\gamma=0$; for any $\gamma\neq 0$ the numerator $1-p^{-\alpha(u)-1}$ would not have the required zero.

III. NEW p-ADIC STRINGS

To discuss our new p-adic strings, it will require some recall of old dual-resonance models, in the classification of Ref. 11. Although that paper speculated about N-point functions, the most solid part focused on the case $N=4$ which is of greatest interest. The s-t-u symmetry of the p-adic $A_4^{(p)}$ will require that we consider a stringlike amplitude

$$A_4=A_{st}+A_{tu}+A_{us} , \tag{3.1}$$

where

$$A_{st}=\int_0^1 dx\, x^{-\alpha(t)-1}(1-x)^{-\alpha(s)-1}(1-x+x^2)^{\gamma/2}\phi . \tag{3.2}$$

The requirement that A_{st} have no odd daughters (i.e., poles corresponding to odd units of spin below the parent trajectory) leads uniquely to the form of Eq. (3.2) for arbitrary $\gamma=\alpha(s)+\alpha(t)+\alpha(u)+1$. In the case $\gamma=0$ and $\phi=1$ we have the Veneziano amplitude. For $\gamma=0$ and $\phi=\alpha(s)\beta_s+\alpha(t)\beta_t+\alpha(u)\beta_u$ with $\beta_s=x(1-x+x^2)^{-1}$, $\beta_t=(1-x)(1-x+x^2)^{-1}$, and $\beta_u=-x(1-x)(1-x+x^2)^{-1}$ one arrives at the Neveu-Schwarz amplitude.

For general γ and ϕ the form of Eq. (3.2) is uniquely dictated also by the requirement of summability: that Eq. (3.1) takes the form of Eq. (3.4) below.

Note that for γ unequal to an even integer, the integrand of Eq. (3.2) would have branch points at $x=\exp(\pm i\pi/3)$; these two points are fixed points under the permutation group S_3 which takes $x\to(1-x)\to -x(1-x)^{-1}$. In the multiparticle generalization[11] the relevant group for the N-point function becomes an S_N acting simultaneously on the Koba-Nielsen variables and external momenta. In fact, the function ϕ is restricted to depend on x, s, and t in the following special way:

$$\phi=\phi\left[\begin{matrix}\alpha(s) & \alpha(t) & \alpha(u)\\ \beta_s(x); & \beta_t(x); & \beta_u(x)\end{matrix}\right] , \tag{3.3}$$

where ϕ is invariant under the simultaneous transformations $s\to t\to u\to s$ and $x\to(1-x)\to -x/(1-x)\to x$. In other words, ϕ is S_3 invariant under permutations of the three argument pairs $\alpha(i),\beta_i(x)$ with $i=s,t,u$.

This ensures that one may write

$$A_4=\int_{-\infty}^{\infty} dx|x|^{-\alpha(t)-1}|1-x|^{-\alpha(s)-1}\times(1-x+x^2)^{\gamma/2}\phi \tag{3.4}$$

and that A_4 has poles only at $\alpha(i)=0,2,4,6,\ldots$. The residues of these poles correspond to only even spins. The S_3 symmetry thus removes all the odd daughter trajectories in A_{st}; one then removes all the odd spins by adding terms as in A_4 of Eq. (3.1).

In order to construct the p-adic counterpart of A_4 one replaces, at some stage, the real integration variable by a p-adic variable. One possibility is to replace the factor such as $(1-x+x^2)$ in Eq. (3.2) by its p-adic norm directly; this is discussed in Appendix A. Our procedure will instead be to write first the integral as one of the Euler β-function type, by extraction of a kinematic factor, before making the replacement of the real variable by a p-adic variable. This latter procedure will lead to an adelic formula for the amplitude; whether or not the former procedure leads to an adelic formula is less certain (see Appendix A) and this is why we relegate that discussion to an appendix. In any case, the important point is that there exists a definite ambiguity, *a priori,* in proceeding to the p-adic amplitude with potentially quite different answers. In order to resolve this ambiguity we have given weight to the possibility of writing explicitly an adelic formula. But this is for the moment only a matter of preference and is not based on any fundamental principle.

In the amplitude of Eq. (3.2) we must impose restrictions both on γ and on the form of the function ϕ. As already mentioned, the amplitude A_4 has poles only at $\alpha(i)=0,2,4,6,\ldots$. Let ϕ be chosen such that the lowest pole is at $\alpha(i)=2n$, n=non-negative integer. Then the relevant Euler β function will lead to a p-adic amplitude of the form analogous to Eq. (2.11): namely,

$$\hat{A}_4^{(p)}=\prod_{Q=s,t,u}\frac{1-p^{-\alpha(Q)-1+2n}}{1-p^{\alpha(Q)-2n}} . \tag{3.5}$$

This will have a doule pole at, e.g., $\alpha(s)=\alpha(t)=2n$ unless $\alpha(u)=2n-1$ corresponding to $\gamma=\alpha(s)+\alpha(t)+\alpha(u)+1=6n$. This suggests that γ must be a multiple of 6. In fact, only in such a case is it possible (see Appendix B) to write

$$A_{st}=K_4B(2n-\alpha(s),2n-\alpha(t)) , \tag{3.6}$$

where K_4 is an s-t-u symmetric kinematic factor.

Using Eqs. (2.7) and (3.6), and $\gamma=6n$, then permits us to rewrite

$$A_4=K_4\prod_Q \Gamma_\infty(2n-\alpha(Q)) . \tag{3.7}$$

Using the adelic property, Eq. (2.5), then permits us to define the p-adic amplitude

$$A_4^{(p)} = K_4 \hat{A}_4^{(p)} , \tag{3.8}$$

where $\hat{A}_4^{(p)}$ is given by Eq. (3.5) and which has the same spin structure at the ground-state pole $\alpha(i)=2n$ as the string amplitude A_4.

The general principles are thus: choose ϕ such that the poles at $\alpha(i)=0,2,4,\ldots,(2n-2)$ are removed from A_4; choose $\gamma=6n$; calculate the kinematic factor which is s-t-u symmetric by virtue of Appendix B. The p-adic amplitude is then given by Eq. (3.8).

The adelic formula now relates

$$\hat{A}_4 = K_4^{-1} A_4 \tag{3.9}$$

to $\hat{A}_4^{(p)}$ according to

$$\hat{A}_4 \prod_p \hat{A}_4^{(p)} = 1 \tag{3.10}$$

by virtue of Eq. (2.7).

The reader may ask, why not choose $\gamma=6n$ and use general ϕ? In such a case, the string amplitude has lower-mass poles than the one at $\alpha(i)=2n$, namely the poles at $\alpha(i)=0,2,4,\ldots,(2n-2)$. These lower-mass poles will appear not in $\hat{A}_4^{(p)}$ but in the kinematic factor K_4. This would mean that $A_4^{(p)}$ defined by Eq. (3.8) would have two different types of poles. The ones at $\alpha(i)=2n+2\pi i(\ln p)^{-1}M$ are similar to those of the p-adic Veneziano string, but the poles arising from K_4 are only on the real axis. While such a generalization may itself be very interesting, we wish to pursue a generalization even more closely parallel to the simplest possible case. In particular, we shall not allow noncanceled poles from the kinematic factor K_4.

IV. EXPLICIT EXAMPLES

For the general amplitude given by Eq. (3.2) the simplest case is with $\gamma=0$ and $\phi=1$ corresponding to the amplitude discussed in the review in Sec. II.

The first nontrivial new p-adic string occurs with $\gamma=6$. For this case, according to the discussion given in Sec. III, one must choose ϕ as a function of $\alpha(i),\beta_i$ such that the $\alpha(i)=0$ pole is absent in the string amplitude A_4. The simplest choice, already considered in Ref. 11, is to take

$$\phi = \sum_{i=s,t,u} \alpha(i)\beta_i . \tag{4.1}$$

A detailed calculation gives the kinematic factor K_4 as

$$K_4 = -3\left[\frac{4-\alpha(s)\alpha(t)}{1-\alpha(u)} + \frac{4-\alpha(t)\alpha(u)}{1-\alpha(s)} + \frac{4-\alpha(u)\alpha(s)}{1-\alpha(t)} + 4\right] . \tag{4.2}$$

The poles at $\alpha(i)=1$ in K_4 are immediately canceled by the numerator of $\hat{A}_4^{(p)}$. All that we insist is that no pole of K_4 survives in A_4. This is therefore true for

$$A_4^{(p)} = K_4 \prod_{Q=s,t,u} \Gamma_p(2-\alpha(Q)) . \tag{4.3}$$

The following remarks apply to this A_4 example.

(i) There is an adelic formula of the kind

$$A_4 = K_4 \hat{A}_4 , \tag{4.4a}$$

$$A_4^{(p)} = K_4 \hat{A}_4^{(p)} , \tag{4.4b}$$

$$\hat{A}_4 \prod_p \hat{A}_4^{(p)} = 1 . \tag{4.4c}$$

The symmetric group has been used to remove the $\alpha(i)=0$ pole, and thus this seems to be a natural generalization of the Veneziano case.

(ii) The real pole at $\alpha(s)=2$ has residue proportional to $[\alpha^2(t)-3\alpha(t)+4]$ in both the string amplitude A_4 and in the p-adic amplitude $A_4^{(p)}$. Hence the kinematic factor K_4 is playing the key role of providing the correct spin structure at the p-adic pole.

(iii) Potential double poles at, e.g., $\alpha(s)=\alpha(t)=2$ are avoided by the zero of $(1-p^{-\alpha(u)+1})$ in the numerator; this stems from the choice of $\gamma=\alpha(s)+\alpha(t)+\alpha(u)+1=6$ which is in any case necessitated by the s-t-u symmetry of K_4. [Note that for the complex poles at, e.g., $\alpha_s=2+2\pi i(\ln p)^{-1}M$ the residue is nonpolynomial and there exist double poles such as $\alpha_t=2-2\pi i(\ln p)^{-1}M$ and $\alpha(u)=1$.]

Other examples for $\gamma=6$ are straightforward to compute. We may take, instead of Eq. (4.1),

$$\phi = \alpha(s)\beta_t\beta_u + \alpha(t)\beta_u\beta_s + \alpha(u)\beta_s\beta_t , \tag{4.5}$$

which leads to the kinematic factor

$$K_4 = -5 + \sum_{Q=s,t,u} \frac{[3-\alpha(Q)][2-\alpha(Q)][1-\alpha(Q)]\alpha(Q)}{[1-\alpha(s)][1-\alpha(t)][1-\alpha(u)]} . \tag{4.6}$$

A third example is

$$\phi = \alpha(s)\beta_s^3 + \alpha(t)\beta_t^3 + \alpha(u)\beta_u^3 , \tag{4.7}$$

leading to

$$K_4 = \frac{[2-\alpha(s)][2-\alpha(t)]}{1-\alpha(u)} + \frac{[2-\alpha(t)][2-\alpha(u)]}{1-\alpha(s)} + \frac{[2-\alpha(u)][2-\alpha(s)]}{1-\alpha(t)} . \tag{4.8}$$

One may proceed further to generalize, e.g., to cases with the poles at both $\alpha(Q)=0$ and 2 removed. One such choice is $\gamma=12$ and

$$\begin{aligned}\phi = {} & \alpha(s)[1-\alpha(s)][2-\alpha(s)]\beta_s^3 \\ & + \alpha(t)[1-\alpha(t)][2-\alpha(t)]\beta_t^3 \\ & + \alpha(u)[1-\alpha(u)][2-\alpha(u)]\beta_u^3\end{aligned} \tag{4.9}$$

in Eq. (3.2). We do not attempt here to classify all such possibilities systematically.

V. DISCUSSION

If the p-adic approach is taken seriously and one insists on some adelic formula for at least the four-particle tree amplitude, then the amplitudes treated in this paper appear to us as the natural generalization of the simplest bosonic string. One interesting question is how to pursue such a generalization to the higher-point functions.

If the p-adic string is related to the existence of a building block more fundamental than a string, then such a broad class of amplitudes may possess a geometrical interpretation that is more accessible by p-adic techniques than by other methods. On the other hand, the absence of massless particles may mean a physical interpretation not as a unifying "theory of everything" but instead as, e.g., a theory of hadrons or as a possible new phase of the unifying string at energies beyond the Planck scale.

In any case, among the work on dual models in the period which ended over ten years ago, our results suggest the approach of Ref. 11 as worthy of further study since it has not, to our knowledge, been studied for over a decade.

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APPENDIX A: ALTERNATIVE p-ADIC VERSION

In this appendix we will discuss an alternative p-adic version of the amplitude Eq. (3.1) and its generalization. Let us consider the integral[14]

$$A_{st}=\int_0^1 dx\, x^{-\alpha(t)-1}(1-x)^{-\alpha(s)-1}(1-x+x^2)^{\gamma/2}\ , \tag{A1}$$

with $\gamma=\alpha(s)+\alpha(t)+\alpha(u)+1$. Here we do not restrict to the $\gamma=6n$ case. Because of S_3 generated by $x\to(1-x)$, $x\to(-x)/(1-x)$, the total amplitude $A_4=A_{st}+A_{tu}+A_{us}$ can be expressed as

$$A_4=\int_{-\infty}^{\infty} dx|x|^{-\alpha(t)-1}|1-x|^{-\alpha(s)-1}|1-x+x^2|^{\gamma/2}\ . \tag{A2}$$

The p-adic version of the above amplitude can be defined by the usual procedure; replacing the real integral and the norm by the p-adic counterparts:

$$A_4^{(p)}\equiv\int_{Q_p} dx|x|_p^{-\alpha(t)-1}|1-x|_p^{-\alpha(s)-1}\times|1-x+x^2|_p^{\gamma/2}\ . \tag{A3}$$

In order to calculate the above integral it is standard to split $A_4^{(p)}$ according to the following four integral regions: (i) $|x|_p<1$, (ii) $|x|_p>1$, (iii) $|x|_p=1$, $|1-x|_p<1$, and (iv) $|x|_p=|1-x|_p=1$. Calculations for (i)–(iii) are easily done and give the results

$$A_{4(\mathrm{i})}^{(p)}\equiv\int_{|x|_p<1}dx\,|x|_p^{-\alpha(t)-1}|1-x|_p^{-\alpha(s)-1}|1-x+x^2|_p^{\gamma/2}=\int_{|x|_p<1}dx|x|_p^{-\alpha(t)-1}=\frac{(1-p^{-1})p^{\alpha(t)}}{1-p^{\alpha(t)}}\ , \tag{A4}$$

$$A_{4(\mathrm{ii})}^{(p)}\equiv\int_{|x|_p>1}dx|x|_p^{-\alpha(t)-1}|1-x|_p^{-\alpha(s)-1}|1-x+x^2|_p^{\gamma/2}=\int_{|x|_p>1}dx|x|_p^{\alpha(u)-1}=\frac{(1-p^{-1})p^{\alpha(u)}}{1-p^{\alpha(u)}}\ , \tag{A5}$$

$$A_{4(\mathrm{iii})}^{(p)}\equiv\int_{|x|_p=1,\ |1-x|_p<1}dx|x|_p^{-\alpha(t)-1}|1-x|_p^{-\alpha(s)-1}\times|1-x+x^2|_p^{\gamma/2}=\int_{|x|_p=1,\ |1-x|_p<1}dx|1-x|_p^{-\alpha(s)-1}\equiv\frac{(1-p^{-1})p^{\alpha(s)}}{1-p^{\alpha(s)}}\ . \tag{A6}$$

Here we have used the fact $|1-x+x^2|_p=1$, $|x^2|_p$, and 1 for (i), (ii), and (iii), respectively. Notice that the above three terms are independent of γ and give the same form of "propagator" as in the case of Veneziano model.[3-5]

The (iv) term gives the contact term which depends on the value γ in general. We define this as $C_p(\gamma)$:

$$C_p(\gamma)\equiv\int_{|x|_p=|1-x|_p=1}dx|x|_p^{-\alpha(t)-1}|1-x|_p^{-\alpha(s)-1}\times|1-x+x^2|_p^{\gamma/2}=\int_{|x|_p=|1-x|_p=1}dx|1-x+x^2|_p^{\gamma/2}\ . \tag{A7}$$

For $p=2$

$$C_2(\gamma)=0\ , \tag{A8}$$

since the integration region is empty. For the $p=3$ case, using the change of the variable $x=2+3y$ ($|y|\le1$),

$$C_3(\gamma)=\int_{x=2+3y}dx|1-x+x^2|_p^{\gamma/2}=3^{-1}\int_{|y|_p\le1}dy|3[1+3(y+1)]|_p^{\gamma/2}=3^{-1-\gamma/2}\ . \tag{A9}$$

For $p>3$, since x can be expressed as $x=a+py$ ($a=2,3,\ldots,p-1$, $|y|_p\le1$), $C_p(\gamma)$ becomes

$$C_p(\gamma)=\sum_{a=2}^{p-1}\int_{x=a+py}dx\,|1-x+x^2|_p^{\gamma/2}=\sum_{a=2}^{p-1}p^{-1}\int_{|y|_p\le1}dy|(1-a+a^2)+p(2a-1)y+p^2y^2|_p^{\gamma/2}\ . \tag{A10}$$

In order to evaluate the norm of the integrand, we have to know whether the equation $1-a+a^2=0\ (\mathrm{mod}\ p)$ has solutions for $a=2,3,\ldots,p-1$. It can be shown that if $p-1$ is divisible by 3 the above equation has two solutions and otherwise it does not have any solution. Therefore, if $p-1$ is not divisible by 3,

$$C_p(\gamma)=\frac{p-2}{p}\ . \tag{A11}$$

If $p-1$ is divisible by 3, separating two terms in which $1-a+a^2=m(a)p$ [$m(a)$ integer]

$$C_p(\gamma)=\frac{p-4}{p}+\sum_{2\text{ terms}} p^{-1-\gamma/2}\int_{|y|_p\le 1} dy\,|m(a)+(2a-1)y+py^2|_p^{\gamma/2}\,. \quad \text{(A12)}$$

Now let us define an integral J as

$$J=\int_{|y|_p\le 1} dy|m(a)+(2a-1)y+py^2|_p^{\gamma/2}\,. \quad \text{(A13)}$$

Writing $y=b+pz$ ($b=0,1,\ldots,p-1$, $|z|_p\le 1$),

$$J=\sum_{b=0}^{p-1} p^{-1}\int_{|z|_p\le 1} dz\,|m(a)+(2a-1)b+p[(2a-1)z+(b+pz)^2]|_p^{\gamma/2}\,. \quad \text{(A14)}$$

To calculate this integral it is enough to notice that the equation $m(a)+(2a-1)b=0$ (mod p) has always one solution in $\{b\}=\{0,1,\ldots,p-1\}$. Separating that term and using $m(a)+(2a-1)b=n(a,b)p$ [$n(a,b)$ integer]

$$J=\frac{p-1}{p}+p^{-1-\gamma/2}\int_{|z|_p\le 1}|n(a,b)+b^2+(2a-1)z+p(2bz+pz^2)|_p^{\gamma/2}\,. \quad \text{(A15)}$$

The above procedure can be repeated and J can be obtained as an infinite series:

$$J=\frac{p-1}{p}+p^{-1-\gamma/2}\left[\frac{p-1}{p}+p^{-1-\gamma/2}\left[\frac{p-1}{p}+\cdots\right]\right]=\frac{p-1}{p}\,\frac{1}{1-p^{-1-\gamma/2}}\,. \quad \text{(A16)}$$

Substituting this into Eq. (A12), we get

$$C_p(\gamma)=\frac{p-4}{p}+2\frac{p-1}{p}\,\frac{1}{p^{1+\gamma/2}-1}\,. \quad \text{(A17)}$$

Combining Eqs. (A4)–(A6), (A8), (A9), (A11), and (A16) we can summarize the result

$$A_4^{(p)}=\sum_{Q=s,t,u}\frac{(1-p^{-1})p^{\alpha(Q)}}{1-p^{\alpha(Q)}}+C_p(\gamma)\,, \quad \text{(A18)}$$

where

$$C_2(\gamma)=0\,, \quad \text{(A19)}$$

$$C_3(\gamma)=3^{-1-\gamma/2}\,, \quad \text{(A20)}$$

$$C_{p>3}(\gamma)=\frac{p-2}{p},\quad p-1 \text{ is not divisible by } 3\,, \quad \text{(A21)}$$

$$=\frac{p-4}{4}+2\frac{p-1}{p}\,\frac{1}{p^{1+\gamma/2}-1}\,,\quad p-1 \text{ is divisible by } 3\,. \quad \text{(A22)}$$

Whether the product of the above amplitudes over all prime numbers gives some simple result is not so clear since the formula involves the divisibility by 3. Therefore the possibility of a new adelic formula based on this amplitude is still an open question.

APPENDIX B: PROOF OF *s-t-u* SYMMETRY

In Eq. (3.6) of the text, an assertion was made without proof. Namely, that if we choose to write the amplitude A_{st} as a particular β function multiplied by a kinematic factor (K_4) then the factor K_4 which is obviously *s-t* symmetric actually has a larger symmetry under *s-t-u* interchange. In particular it was asserted that, for any $n=$integer≥ 0, if

$$A_{st}=\int_0^1 dx\,x^{-\alpha(t)-1}(1-x)^{-\alpha(s)-1}\times(1-x+x^2)^{\gamma/2}\phi(\alpha(i),\beta_i) \quad \text{(B1)}$$

$$=K_4B(2n-\alpha(s),2n-\alpha(t))\,, \quad \text{(B2)}$$

then if and only if $\gamma=6n$ is the kinematic factor K_4 *s-t-u* symmetric.

To prove this, let $\gamma=6n+\delta$ and let $\hat\alpha(Q)=[\alpha(Q)-2n]$ so that $\hat\gamma=\hat\alpha(s)+\hat\alpha(t)+\hat\alpha(u)+1=\delta$. Then, using Eqs. (B1) and (B2) we have

$$B(-\hat\alpha(s),-\hat\alpha(t))=\int_0^1 dx\,x^{-\hat\alpha(t)-1}(1-x)^{-\hat\alpha(s)-1}\hat\phi\,, \quad \text{(B3)}$$

with

$$\hat\phi=K_4^{-1}(1-x+x^2)^{\delta/2}(\beta_s\beta_t\beta_u)^{-n}\phi(\hat\alpha(i)+2n,\beta_i)\,. \quad \text{(B4)}$$

For $B(-\hat\alpha(s),-\hat\alpha(t))$ of Eq. (B3) to have the same residues of the poles at $\hat\alpha(s)=0,1,2,3\ldots$ as it would had we replaced $\hat\phi$ by unity, then $\hat\phi$ must possess the S_3 symmetry under $x\to(1-x)\to -x(1-x)^{-1}$ and $s\to t\to u$.

Now, in Eq. (B4), the factor $(\beta_s\beta_t\beta_u)^{-1}$ is S_3 symmetric as, by assumption, is ϕ. There remain two factors, K_4^{-1} which depends only on $\alpha(s)$, $\alpha(t)$, and $\alpha(u)$ (not on β_i) and $(1-x+x^2)^{\delta/2}$ which depends only on x [not on $\alpha(i)$]. The only possibility is thus that $\delta=0$ and that K_4 is *s-t-u* symmetric. This corresponds to $\gamma=6n$ as stated in the text.

As an illustration of the $\hat\phi$ in Eq. (B3), for the first example of Sec. IV one finds

$$\hat\phi=\frac{1}{3}\left[\frac{4-\alpha(s)\alpha(t)}{1-\alpha(u)}+\frac{4-\alpha(t)\alpha(u)}{1-\alpha(s)}+\frac{4-\alpha(u)\alpha(s)}{1-\alpha(t)}+4\right]^{-1}\times\frac{(1-x+x^2)^2}{x^3(1-x)^3}[\alpha(s)x+\alpha(t)(1-x)-\alpha(u)x(1-x)]\,. \quad \text{(B5)}$$

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