PHYSICAL REVIEW D VOLUME 39, NUMBER 4 15 FEBRUARY 1989

Renormalized electromagnetic stress tensor in Schwarzschild spacetime

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(Received 23 August 1988)

The renormalized quantum stress tensor for an electromagnetic field in the Hartle-Hawking state in Schwarzschild spacetime is calculated using a mode sum which is evaluated numerically. The results shed light on stress-tensor-approximation methods that have been proposed in recent years.

I. INTRODUCTION

Four years ago, Howard[1] performed the numerical calculation of the renormalized one-loop stress tensor for a conformally invariant scalar field in thermal equilibrium with a Schwarzschild black hole and found remarkably good agreement with a simple analytic expression found earlier by Page.[2] Since then there has been considerable interest in Page's approximation for scalar and higher-spin fields.[3–7] In Ref. 5 it was shown that there is a problem in applying this approximation scheme to the case of an electromagnetic field in Schwarzschild spacetime, in that the value of the energy density on the horizon disagrees with the value calculated by Elster.[8,9] In a separate publication,[10] we have shown that this published value was in error and gave the corrected value

$$-\langle T_t{}^t\rangle(r=2M)=-\frac{19}{30\pi^2}\kappa^4\ , \qquad (1.1)$$

where $\kappa\equiv 1/4M$. Even with this correction, however, the Page approximation does not agree with the exact result. More recently Frolov and Zel'nikov[7] suggested an extended version of the Page approximation which gives a conserved tensor with the trace anomaly and also necessarily gives the correct value of the energy density on the horizon. However, the main drawback with the Page approximation is that it is not known in what sense it is valid as an *approximation* except in certain simple cases—the major problem being that it is too local; i.e., it is not clear in what way it reflects the global choice of the state. The only concrete justification for interest in Page's approximation for scalar fields in Schwarzschild spacetime is Howard's numerical calculation. We have undertaken the corresponding numerical calculation for the case of electromagnetism.

Howard separated the scalar stress tensor

$$\langle T_\mu{}^\nu\rangle=\langle T_\mu{}^\nu\rangle_{\text{analytic}}+\frac{4\kappa^4}{3\pi^2}\Delta_\mu{}^\nu\ ,$$

where $\Delta_\mu{}^\nu$, consisting of a mode sum evaluated numerically, is a small correction to Page's approximate tensor $\langle T_\mu{}^\nu\rangle_{\text{analytic}}$. We have made a similar split for the electromagnetic case. In this case, however, the approximate tensor corresponds neither to the Page tensor nor that of Frolov and Zel'nikov.

In this paper we have followed the same general method of Howard but have taken advantage of techniques introduced by Leaver[11] (for a calculation of quasinormal modes of Schwarzschild black holes) to streamline the program. The decrease in calculation time was sufficient to enable us to perform the numerical calculation on a small personal computer.

II. CALCULATION OF $\langle T_\mu{}^\nu\rangle$

As our methods closely follow those of Refs. 1 and 8, we will simply give an outline here, relegating the details of the calculation to the appendixes.

The quantum stress tensor for the electromagnetic field may be written

$$\langle T_\mu{}^\nu\rangle=\langle T_\mu{}^\nu\rangle_M+\langle T_\mu{}^\nu\rangle_{\text{GB}}+\langle T_\mu{}^\nu\rangle_{\text{gh}}\ , \qquad (2.1)$$

where $\langle T_\mu{}^\nu\rangle_M$ is the expectation value of the operator version of the classical Maxwell stress tensor

$$\langle T_\mu{}^\nu\rangle_M=\langle \hat F_{\mu\alpha}\hat F^{\nu\alpha}-\tfrac{1}{4}g_\mu{}^\nu\hat F^{\alpha\beta}\hat F_{\alpha\beta}\rangle \qquad (2.2)$$

which is built of the field operators $\hat A_\mu(x)$ in the usual way:

$$\hat F^{\mu\nu}=\hat A^{\mu;\nu}-\hat A^{\nu;\mu}\ . \qquad (2.3)$$

The tensor operators $\langle T_\mu{}^\nu\rangle_{\text{GB}}$ and $\langle T_\mu{}^\nu\rangle_{\text{gh}}$ are the contributions due to the gauge-breaking and ghost terms in the action. Classically, one can show that

$$\langle T_\mu{}^\nu\rangle_{\text{GB}}+\langle T_\mu{}^\nu\rangle_{\text{gh}}=0 \qquad (2.4)$$

and this equation is respected by any gauge-invariant renormalization procedure.

We can write Eq. (2.2) as

$$\langle T_\mu{}^\nu\rangle_M=\lim_{x'\to x}\tau_\mu{}^{\nu\alpha\beta'}G_{\alpha\beta'}(x,x')\ , \qquad (2.5)$$

where

$$G_{\alpha\beta'}(x,x')=\langle \hat{A}_\alpha(x)\hat{A}_{\beta'}(x')\rangle \tag{2.6}$$

is the electromagnetic two-point function, and the differential operator τ is given by

$$\tau^{\mu\nu\alpha\beta'}=D^{\mu\nu\alpha\beta'}-\tfrac{1}{4}g^{\mu\nu}D_\gamma{}^{\gamma\alpha\beta'} \tag{2.7}$$

with

$$D^{\mu\nu\alpha\beta'}=(g^{\beta'}_{\gamma}g^{\nu}_{\delta'}\nabla^{\delta'}-g_{\gamma\delta'}g^{\nu\beta'}\nabla^{\delta'})(g^{\gamma\alpha}\nabla^{\mu}-g^{\mu\alpha}\nabla^{\gamma}) . \tag{2.8}$$

Expression (2.5) diverges in the limit and so must be renormalized. We feel that the neatest prescription for this is Hadamard renormalization,[6,12] as demonstrated by Kirsten.[13] In the present case, however, we will follow Howard and Elster and employ the Christensen geodesic point-separation method.[14,15] This procedure consists of taking the limit

$$\langle T_\mu{}^\nu\rangle_{\rm ren}=\lim_{x'\to x}[\tau_\mu{}^{\nu\alpha\beta'}G_{\alpha\beta'}(x,x') - \langle T_\mu{}^\nu\rangle_{\rm subtract}(x;x')] , \tag{2.9}$$

where $\langle T_\mu{}^\nu\rangle_{\rm subtract}$ is a quantity based on the Schwinger-DeWitt short-distance expansion of the Green's function. In Ref. 15 Christensen gives geometrical expressions for the quartic, quadratic, logarithmic, and finite divergent terms of $\langle T_\mu{}^\nu\rangle_{\rm subtract}$. There is also a linearly divergent term which Christensen eliminates, directing users to average over σ^α and $-\sigma^\alpha$. In explicit calculations it is inconvenient to perform this averaging, and failing to do it while ignoring this term can lead to wrong answers.[10] In a Ricci-flat spacetime this term is given by

$$\langle T_\mu{}^\nu\rangle_{\rm div,lin}=\frac{1}{24\pi^2}C_{\mu\alpha}{}^{\nu}{}_{\beta;\gamma}\frac{\sigma^\alpha\sigma^\beta\sigma^\gamma}{\sigma^2} . \tag{2.10}$$

Here $C_{\alpha\beta\gamma\delta}$ is the Weyl tensor, and $\sigma^\alpha\equiv\sigma^{;\alpha}$, where $\sigma(x,x')$ is one-half the square of the geodesic distance between x and x'.

III. THE HARTLE-HAWKING STRESS TENSOR

The electromagnetic two-point function appropriate to the Hartle-Hawking boundary conditions has been given by Elster:

$$G_{\mu\nu'}(x,x')=\frac{i}{8\pi M}\sum_{n=-\infty}^{\infty}e^{in\kappa(\tau-\tau')}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\sum_{a,b=1}^{4}{}^{ab}R_l{}^m(r,r')\,{}^a_\mu Y_l{}^m(\theta,\phi)\,{}^b_{\nu'}Y_l{}^{m*}(\theta',\phi') . \tag{3.1}$$

Here we are using the Schwarzschild space coordinates r,θ,ϕ and imaginary Schwarzschild time τ; n and l are the (imaginary) energy and angular momentum eigenvalues, respectively (note that this is opposite to the usage of Elster). The functions ${}^a_\mu Y_l{}^m(\theta,\phi)$ are the vector spherical harmonics.[16] The ${}^{ab}R_l{}^m(r,r')$, which are examined by Elster, are not expressible in terms of known functions. Fortunately, their contribution to $\langle T_\mu{}^\nu\rangle$ may be reduced to that of a single two-point function, which shall be described below and in Appendix A.

Taking points separated in the time direction ($it'=it+i\epsilon$, $\mathbf{x}'=\mathbf{x}$), we proceed as in Elster and find, for the $\theta\theta$ component of expression (2.5),

$$\tau_\theta{}^{\theta\alpha\beta'}G_{\alpha\beta'}(x,x')=\frac{\kappa^4}{\pi^2}\frac{16}{(\xi+1)^2}\sum_{n=1}^{\infty}\cos n\kappa\epsilon\left[\sum_{l=1}^{\infty}(l+\tfrac{1}{2})l(l+1)\left[\frac{1}{n}\,{}_1p_l^n(\xi)\,{}_1q_l^n(\xi)-2\,{}_1p_l^0(\xi)\,{}_1q_l^0(\xi)\right]\right] . \tag{3.2}$$

Here we have switched to the dimensionless variable $\xi=(r/M)-1$. The radial functions ${}_1p_l^n(\xi)$ and ${}_1q_l^n(\xi)$ are the spin-one solutions of the Schwarzschild radial wave equation (A3). (We subsequently drop the spin index.) Their definitions and general properties are given in Appendix A and Ref. 17. We have incorporated the $n=0$ term into the sum from $n=1$ to infinity in Eq. (3.2) by use of Eq. (3.7d) below.

The Christensen subtraction term for points separated in imaginary time is

$$\langle T_\theta{}^\theta\rangle_{\rm subtract}=\frac{1}{\pi^2\epsilon^4}\left[\frac{\xi+1}{\xi-1}\right]^2+\frac{1}{6\pi^2M^2\epsilon^2}\frac{2\xi-1}{(\xi+1)^2(\xi-1)^2}-\frac{1}{720\pi^2M^4}\frac{16\xi^2-76\xi+49}{(\xi+1)^6(\xi-1)^2}+\frac{1}{4\pi^2M^4}\frac{1}{(\xi+1)^6} . \tag{3.3}$$

The last term in this expression arises from the "linearly divergent" term (2.10) which was omitted by previous authors. With our choice of point separation in time it gives rise simply to a nondivergent contribution, but is essential for the conservation of the renormalized stress tensor.

Subtracting Eq. (3.3) from Eq. (3.2), we have

$$\langle T_\theta{}^\theta\rangle=\langle T_\theta{}^\theta\rangle_{\rm analytic}+\frac{\kappa^4}{\pi^2}\Delta_\theta{}^\theta , \tag{3.4}$$

where

$$\langle T_\theta{}^\theta\rangle_{\rm analytic}=\frac{\kappa^4}{720\pi^2}(1+2w+3w^2+44w^3-305w^4+66w^5-579w^6) \tag{3.5}$$

with $w\equiv 2M/r=2/(\xi+1)$ and

$$\Delta_\theta{}^\theta = \frac{16}{(\xi+1)^2} \lim_{\epsilon\to 0} \sum_{n=1}^{\infty} \cos n\kappa\epsilon \left[\sum_{l=1}^{\infty} (l+\tfrac{1}{2}) l(l+1) \left[\frac{1}{n} p_l^n(\xi) q_l^n(\xi) - 2 p_l^0(\xi) q_l^0(\xi) \right] - \frac{n^3}{96} \frac{(\xi+1)^4}{(\xi-1)^2} + \frac{n}{6} \left[\frac{1}{(\xi-1)^2} + \frac{2}{\xi-1} \right] - \frac{1}{(\xi+1)^2} \right] . \tag{3.6}$$

In passing from Eq. (3.1) to Eq. (3.4) we have employed the following distribution theoretic identities valid for $\epsilon \neq 0$:

$$\sum_{n=1}^{\infty} n^3 \cos n\kappa\epsilon = \frac{1}{\kappa^4 \epsilon^4} , \tag{3.7a}$$

$$\sum_{n=1}^{\infty} n^2 \cos n\kappa\epsilon = 0 , \tag{3.7b}$$

$$\sum_{n=1}^{\infty} n \cos n\kappa\epsilon = \frac{1}{\kappa^2 \epsilon^2} , \tag{3.7c}$$

$$\sum_{n=1}^{\infty} \cos n\kappa\epsilon = -\tfrac{1}{2} , \tag{3.7d}$$

which enabled us to incorporate the ϵ^{-4} and ϵ^{-2} pieces of $\langle T_\mu{}^\nu \rangle_{\text{subtract}}$ directly into the sum over the energy eigenvalue n. We note that here we were also required to use Eq. (3.7d) to *remove* a term (proportional to w^4) from the n sum and incorporate it into $\langle T_\theta{}^\theta \rangle_{\text{analytic}}$ (see Appendix B and Sec. V). This term, arising from the second term on the right-hand side of Eq. (C1), has no analogue in Howard's calculation. With these adjustments made, one can show (see Appendix B) that the sum $\Delta_\theta{}^\theta$ given by Eq. (3.6) is convergent for $\xi > 1$, and we are free to take the limit $\epsilon \to 0$ inside the summation.

IV. NUMERICAL EVALUATION OF $\Delta_\theta{}^\theta$

As outlined in Appendix B, we may rewrite the n summand in Eq. (3.6) by adding and subtracting the first three WKB approximants to the product $(1/n) p_l^n(\xi) q_l^n(\xi)$, which are given in Table I. The resulting expression for $\Delta_\theta{}^\theta$ divides naturally into two terms:

$$\Delta_\theta{}^\theta = \frac{16}{(\xi+1)^2} \sum_{n=1}^{\infty} (\Delta_n^\Sigma + \Delta_n^I) ,$$

where

TABLE I. The WKB approximants to $(1/n)\, {}_s p_l^n(\xi)\, {}_s q_l^n(\xi)$.

$$W_l^{(1)n} = \frac{1}{\chi}$$

$$W_l^{(2)n} = \frac{1}{8\chi^3} \left[[1+8s^2(\xi-1)] - 2(2\xi^2-6\xi+7)\frac{\chi_0^2}{\chi^2} + 5(\xi-2)^2 \frac{\chi_0^4}{\chi^4} \right]$$

$$W_l^{(3)n} = \frac{1}{32\chi^5} \left[\tfrac{1}{4}\{(16\xi^2+11) + 16s^2(\xi-1)[(2\xi^2-6\xi+7)+12s^2(\xi-1)]\} + \frac{\chi_0^2}{\chi^2}\{-(16\xi^4-60\xi^3+88\xi^2-70\xi+171) - 40s^2(\xi-1)(\xi^2-3\xi+5)\} + \frac{\chi_0^4}{\chi^4}\{\tfrac{7}{2}[(56\xi^4-320\xi^3+773\xi^2-1020\xi+666)+40s^2(\xi-1)(\xi-2)^2]\} + \frac{\chi_0^6}{\chi^6}[-231(2\xi^2-6\xi+7)(\xi-2)^2] + \frac{\chi_0^8}{\chi^8}[\tfrac{1155}{4}(\xi-2)^4] \right]$$

where $\chi^2 = (l+\tfrac{1}{2})^2(\xi^2-1) + \chi_0^2$, $\chi_0^2 = \dfrac{n^2}{16}(\xi+1)^4$

$$\Delta_n^{\Sigma}=\sum_{l=1}^{\infty}(l+\tfrac{1}{2})l(l+1)\left[\frac{1}{n}p_l^n(\xi)q_l^n(\xi)-W_l^{(1)n}-W_l^{(2)n}-W_l^{(3)n}\right],$$

$$\Delta_n^{I}=\sum_{l=1}^{\infty}(l+\tfrac{1}{2})l(l+1)[W_l^{(1)n}+W_l^{(2)n}+W_l^{(3)n}-2p_l^0(\xi)q_l^0(\xi)]-\frac{n^3}{96}\frac{(\xi+1)^4}{(\xi-1)^2}+\frac{n}{6}\left[\frac{1}{(\xi-1)^2}+\frac{2}{(\xi-1)}\right]-\frac{1}{(\xi+1)^2}. \tag{4.1}$$

The numerical evaluation of the function $(1/n)p_l^n(\xi)q_l^n(\xi)$ is carried out as follows: following Leaver,[11] we write, for general spin s,

$${}_sp_l^n(\xi)=e^{n(\xi-1)/4}(\xi-1)^{n/2}\sum_{k=0}^{\infty}a_k\left[\frac{\xi-1}{\xi+1}\right]^k, \tag{4.2}$$

where $a_0=1$ and a_k satisfies

$$k(k+n)a_k+[s^2-3(k-1)^2-n(k-2)-l(l+1)]a_{k-1}-[2s^2-3(k-2)^2-l(l+1)]a_{k-2}+[s^2-(k-3)^2]a_{k-3}=0, \tag{4.3}$$

where $a_k\equiv 0$ for $k<0$. The sum in Eq. (4.2) converges on the half-open interval $\xi\in[1,\infty)$. ${}_sq_l^{\gamma}(\xi)$ may be calculated by the integral of the Wronskian

$${}_sq_l^n(\xi)=2n\,{}_sp_l^n(\xi)\int_{\xi}^{\infty}\frac{d\xi'}{(\xi'^2-1)\,{}_sp_l^n(\xi')^2}. \tag{4.4}$$

The product appearing in Δ_n^{Σ} [Eq. (4.1)] can therefore be written ($s=1$ implied)

$$\frac{1}{n}p_l^n(\xi)q_l^n(\xi)=2\int_{\xi}^{\infty}\frac{d\xi'}{\xi'^2-1}\left[\frac{p_l^n(\xi)}{p_l^n(\xi')}\right]^2. \tag{4.5}$$

A variable-step-size Simpson's-rule integration routine was used to evaluate (4.5) numerically, with $p_l^n(\xi)$ in the integrand generated by Eq. (4.2).

To evaluate Δ_n^I it is convenient to use the Watson-Sommerfeld technique of converting the summation to an integration in the complex λ plane. The details of this procedure are described in Appendix B. The λ integral was easily evaluated by a Romberg integration program using as a contour in the complex λ plane the half-circle centered on the branch point at $\lambda=\frac{1}{4}n^2(1+\xi)^2(\xi^2-1)^{-1/2}$ (see Fig. 3 of Ref. 18).

As the sum over n converges rapidly, we were able to terminate the sum after $n=4$ and retain 1% accuracy in $\Delta_\theta{}^\theta$. The upper limit on the sum over l in Δ_n^{Σ} ranged from 25 to 200, depending on ξ and n. Numerical evaluation of Δ_n^{Σ} on the horizon is impossible as more and more terms in the sum are required for convergence as $\xi\to 1$. However, numerical experiments showed that it tended to zero as the horizon was approached. Using our improved algorithms described above we were able to obtain all our numerical results using a small personal computer.

The results for $\langle T_\theta{}^\theta\rangle$, $\langle T_t{}^t\rangle$, and $\langle T_r{}^r\rangle$ are plotted in Figs. 1–3 which also show the contributions to the full

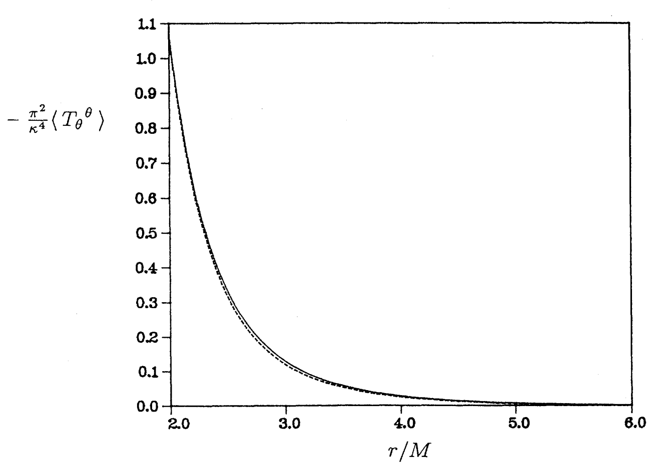

FIG. 1. The angular pressure as a function of r/M. The solid line is the numerically calculated value; the dashed line is the analytic approximation.

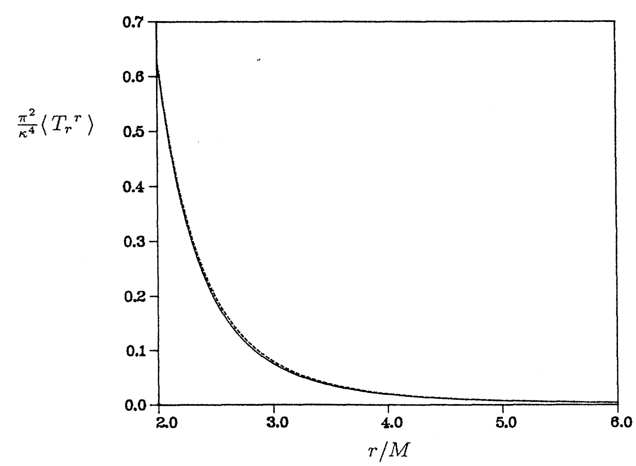

FIG. 2. The radial pressure.

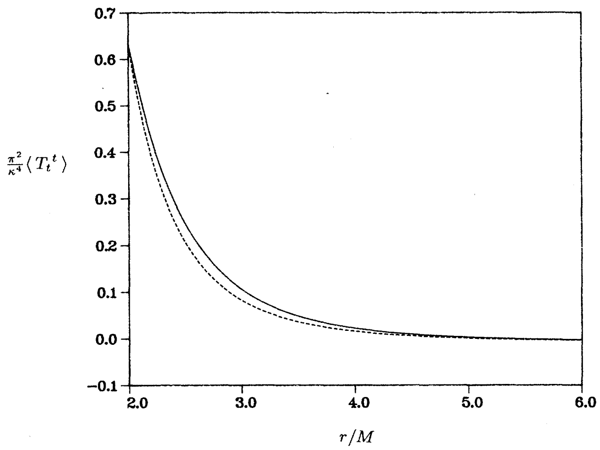

FIG. 3. The energy density.

answer from $\langle T_\mu{}^\nu\rangle_{\text{analytic}}$ and $\Delta_\theta{}^\theta$. The results for $\langle T_t{}^t\rangle$ and $\langle T_r{}^r\rangle$ have been obtained from $\langle T_\theta{}^\theta\rangle$ using conservation and the trace anomaly, which give[19]

$$\langle T_r{}^r\rangle = \frac{1}{r(r-2M)}\int_{2M}^{r}[M\langle T_\mu{}^\mu\rangle(r') + 2(r'-3M)\langle T_\theta{}^\theta\rangle(r')]dr' , \tag{4.6a}$$

$$\langle T_t{}^t\rangle = \langle T_\mu{}^\mu\rangle - \langle T_r{}^r\rangle - 2\langle T_\theta{}^\theta\rangle . \tag{4.6b}$$

Here $\langle T_\mu{}^\mu\rangle$ is the state-independent trace anomaly given by

$$\langle T_\mu{}^\mu\rangle = -\frac{13}{60\pi^2}\frac{M^2}{r^6} . \tag{4.7}$$

V. APPROXIMATIONS

The approximation of renormalized stress tensors in Schwarzschild spacetime has a long history. The program was initiated by Whiting who proposed a simple approximation for $\langle\varphi^2\rangle$ for the conformal scalar field φ, which was found to fit very closely the exact expression calculated numerically by Fawcett.[20] Page[2] then observed that this formula could be obtained by making a conformal transformation to the ultrastatic optical metric where it could be simply approximated using heat-kernel techniques. In addition, Page was able to use this same method to obtain an approximation for $\langle T_\mu{}^\nu\rangle$ for conformally invariant scalar fields in static Einstein spacetimes. Interest in this approximation increased when it was shown numerically by Howard[1] to be in good agreement with the exact tensor for the Hartle-Hawking state of a conformally invariant scalar field in Schwarzschild spacetime. The Page approximation was extended to conformally invariant fields of arbitrary spin in Refs. 4 and 5 by relating it to the conformal transformation law for the renormalized one-loop effective action. However, it was also shown in Ref. 5 that this approximation method could not work for the electromagnetic field in the Hartle-Hawking state, giving a severe discrepancy on the horizon or at infinity, which can be calculated explicitly,[8–10] according to whether point-separation, zeta-function, or dimensional renormalization is used. A possible resolution of this problem was proposed by Frolov and Zel'nikov,[7] who consider adding to the anomalous term in the renormalized effective action the most general conformally invariant action that could be constructed from the geometry and the timelike Killing vector and which has the correct scaling properties. This yielded one extra conserved tensor which vanished at infinity but not on the horizon, thus enabling them to choose its coefficient to give agreement with the exact results on the horizon and at infinity. However, their approximation suffers from the same drawback as all the other approaches in that it can give no measure of the error it produces.

Specifically, for the electromagnetic field in the Schwarzschild background, Ref. 5 gives

$$\langle T_\theta{}^\theta\rangle_{\text{BOP}} = \frac{\kappa^4}{720\pi^2}(1+2w+3w^2+44w^3 + 55w^4+66w^5-579w^6) \tag{5.1}$$

which gives

$$\langle T_\theta{}^\theta\rangle_{\text{BOP}}(r=2M) = -\frac{17}{30\pi^2}\kappa^4 .$$

Frolov and Zel'nikov propose

$$\langle T_\theta{}^\theta\rangle_{\text{FZ}} = \frac{\kappa^4}{720\pi^2}[(1+2w+3w^2+4w^3 + 5w^4+6w^5-729w^6) + cw^3(4+5w+6w^2+15w^3)] , \tag{5.2}$$

where c is chosen to guarantee that the tensor has the correct value on the horizon at $w=1$. Choosing $c=10$ reproduces Eq. (5.1); Frolov and Zel'nikov chose $c=46$ in order to match Elster's (incorrect) value, while a choice of $c=-2$ gives the correct horizon value of

$$\langle T_\theta{}^\theta\rangle(r=2M) = -\frac{32}{30\pi^2}\kappa^4 ,$$

but does not seem to provide a good approximation elsewhere; see Fig. 4.

While none of these approximations can be said to be valid, they are not irrelevant to our answer. Our tensor may be written in the form

$$\langle T_\theta{}^\theta\rangle = \langle T_\theta{}^\theta\rangle_{\text{BOP}} + \frac{\kappa^4}{\pi^2}\overline{\Delta}_\theta{}^\theta , \tag{5.3}$$

where

$$\overline{\Delta}_\theta{}^\theta = \Delta_\theta{}^\theta - \tfrac{1}{2}w^4 . \tag{5.4}$$

We remind the reader that in the process of rewriting Eq. (2.9) we were forced to remove a term from the "mode sum" part of $\langle T_\theta{}^\theta\rangle$ in order to render the sum convergent in the limit $\epsilon\to 0$. It is precisely this piece, which comes from the $n=0$ (zero-frequency) term in the sum,

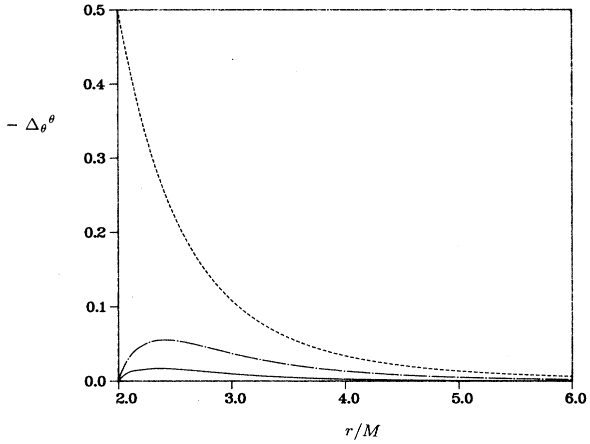

FIG. 4. $\Delta_\theta{}^\theta=(\pi^2/\kappa^4)(\langle T_\theta{}^\theta\rangle-\langle T_\theta{}^\theta\rangle_{\rm approx})$ for $\langle T_\theta{}^\theta\rangle_{\rm approx}=\langle T_\theta{}^\theta\rangle_{\rm analytic}$ (solid curve), $\langle T_\theta{}^\theta\rangle_{\rm BOP}$ (dashed curve), and $\langle T_\theta{}^\theta\rangle_{\rm FZ}$ (dashed-dotted curve).

that spoils the approximation.

For completeness we note all components of $\langle T_\mu{}^\nu\rangle_{\rm analytic}$:

$$\langle T_t{}^t\rangle_{\rm analytic}=\frac{\pi^2}{45}T_H^4(-3-6w-9w^2-12w^3+315w^4-78w^5+249w^6)\ , \tag{5.5}$$

$$\langle T_r{}^r\rangle_{\rm analytic}=\frac{\pi^2}{45}T_H^4(1+2w+3w^2-76w^3+295w^4-54w^5+285w^6)\ , \tag{5.6}$$

$$\langle T_\theta{}^\theta\rangle_{\rm analytic}=\frac{\pi^2}{45}T_H^4(1+2w+3w^2+44w^3-305w^4+66w^5-579w^6)\ , \tag{5.7}$$

where $T_H\equiv\kappa/(2\pi)$ is the Hawking temperature. Here, $\langle T_t{}^t\rangle_{\rm analytic}$ and $\langle T_r{}^r\rangle_{\rm analytic}$ have been obtained from $\langle T_\theta{}^\theta\rangle_{\rm analytic}$ through use of Eq. (4.6).

VI. CONCLUSION

Our calculations have demonstrated that the methods of Howard[1] can be extended to the electromagnetic case and continue to yield an excellent approximation when other methods fail. In addition, the method could readily be extended to other spins and to massive fields, and overcomes the major failing of other methods in providing a concrete way of determining the accuracy of the approximation.

Our investigation also led us to discover the problem of the linearly divergent Christensen subtraction term which had been missed by Elster[8] and Frolov and Zel'nikov.[9] Having obtained a finite result without this term we spent much time calculating $\Delta_\theta{}^\theta$ down to $r=2.01M$ to convince ourselves that it really did tend to zero. Finally, we were led to repeat Christensen's calculation and only then found the problem.

ACKNOWLEDGMENTS

We would like to thank R. P. Brown for her generous donation of computer facilities and B. Allen for helpful comments on this paper. The work of one of us (B.P.J.) was supported in part by Schweizerischer Nationalfonds.

APPENDIX A

It has been shown by Wheeler[21] that the sourceless Maxwell equations in a Schwarzschild background are separable and that the radial dependence of both electric and magnetic fields are governed by a single function $a(r)$ which is a solution of

$$r(r-2M)\frac{d^2a}{dr^2}+2M\frac{da}{dr}-\left[\frac{(i\omega)^2r^3}{r-2M}+l(l+1)\right]a=0\ , \tag{A1}$$

where $l(l+1)$ and $(i\omega)^2$ are the separation constants associated with the angular and time variables. This equation was shown to be a special case of a general real differential equation governing the radial dependence of integer-spin perturbations $(s=0,1,2)$:[22–26]

$$r(r-2M)\frac{d^2a}{dr^2}+2M\frac{da}{dr}-\left[\frac{(i\omega)^2r^3}{r-2M}+l(l+1)-2M\frac{s^2-1}{r}\right]a=0\ . \tag{A2}$$

Note that this equation has three singular points: regular singularities occur at $r=0$ and $r=2M$ and an irregular singularity at $r=\infty$. Equations of this type, which also occur in problems in atomic physics, are discussed by Leaver.[27]

For our purposes, it is more convenient to switch to the variable $\xi=(r/M)-1$. Then Eq. (A2) becomes

$$\frac{d}{d\xi}\left[(\xi^2-1)\frac{df}{d\xi}\right]-\left[\frac{\nu^2}{16}\frac{(\xi+1)^4}{\xi^2-1}+l(l+1)-\frac{2s^2}{\xi+1}\right]f=0\ , \tag{A3}$$

where we have set $a(r)=(\xi+1)f(\xi)$ and $i\omega=-\nu\kappa$ $(\kappa=1/4M)$. We choose independent solutions ${}_sp_l^\nu(\xi)$ and ${}_sq_l^\nu(\xi)$ defined by

$$\left.\begin{array}{l}{}_sp_l^\nu(\xi)\sim(\xi-1)^{\nu/2}\ ,\\ {}_sq_l^\nu(\xi)\sim(\xi-1)^{-\nu/2},\end{array}\right.\quad \xi\to1\ , \tag{A4}$$

and ${}_sq_l^\nu(\xi)\to0$ as $\xi\to\infty$. The analytic properties of ${}_0p_l^\nu(\xi)$ and ${}_0q_l^\nu(\xi)$ as functions of ξ, l, and ν are discussed in Ref. 17: these results may be trivially extended to the higher spin functions. Their Wronskian is given by

$$W[{}_sp_l^\nu(\xi),{}_sq_l^\nu(\xi)]={}_sp_l^\nu\,{}_sq_l'^\nu-{}_sp_l'^\nu\,{}_sq_l^\nu=\begin{cases}-(\xi^2-1)^{-1}, & \nu=0\ ,\\ -2\nu(\xi^2-1)^{-1}, & \nu\neq0\ .\end{cases} \tag{A5}$$

The solutions of Eq. (A3) for $\nu=0$ are given by

$$
\begin{aligned}
{}_s p_l^0(\xi) &= \left[\frac{(l-s)!}{(l+s)!}\right](\xi+1)^s \frac{d^s}{d\xi^s}\left[\left[\frac{\xi-1}{\xi+1}\right]^{s/2} P_l^s(\xi)\right] , \\
{}_s q_l^0(\xi) &= \left[\frac{(l-s)!}{(l+s)!}\right](\xi+1)^s \frac{d^s}{d\xi^s}\left[\left[\frac{\xi-1}{\xi+1}\right]^{s/2} Q_l^s(\xi)\right] ,
\end{aligned}
\tag{A6}
$$

where P_l^s and Q_l^s are the associated Legendre functions.[28] Useful alternative expressions for the spin-1 zero-frequency functions are

$$
\begin{aligned}
{}_1 p_l^0(\xi) &= P_l(\xi) - \frac{\xi-1}{l(l+1)}\frac{dP_l}{d\xi}(\xi) , \\
{}_1 q_l^0(\xi) &= Q_l(\xi) - \frac{\xi-1}{l(l+1)}\frac{dQ_l}{d\xi}(\xi) .
\end{aligned}
\tag{A7}
$$

APPENDIX B

We may write the n summand in Eq. (3.5),

$$
\sum_{l=1}^{\infty}(l+\tfrac{1}{2})l(l+1)\left[\frac{1}{n}p_l^n(\xi)q_l^n(\xi)-2p_l^0(\xi)q_l^0(\xi)\right]
-\frac{n^3}{96}\frac{(\xi+1)^4}{(\xi-1)^2}+\frac{n}{6}\left[\frac{1}{(\xi-1)^2}+\frac{2}{\xi-1}\right]-\frac{1}{(\xi+1)^2} ,
\tag{B1}
$$

as two sums:

$$
\begin{aligned}
\Delta_n^{\Sigma} &= \sum_{l=1}^{\infty}(l+\tfrac{1}{2})l(l+1)\left[\frac{1}{n}p_l^n(\xi)q_l^n(\xi)-W_l^{(1)n}-W_l^{(2)n}-W_l^{(3)n}\right] , \\
\Delta_n^{I} &= \sum_{l=1}^{\infty}(l+\tfrac{1}{2})l(l+1)[W_l^{(1)n}+W_l^{(2)n}+W_l^{(3)n}-2p_l^0(\xi)q_l^0(\xi)]-\frac{n^3}{96}\frac{(\xi+1)^4}{(\xi-1)^2}+\frac{n}{6}\left[\frac{1}{(\xi-1)^2}+\frac{2}{\xi-1}\right]-\frac{1}{(\xi+1)^2} ,
\end{aligned}
\tag{B2}
$$

where the W's are the first-, second-, and third-order WKB approximants to $(1/n)p_l^n(\xi)q_l^n(\xi)$, as given in Table I. The first sum is $O(n^{-3})$ for large n. We show here that, despite its appearance, the second sum is similarly $O(n^{-3})$. Using the formula (C1),

$$
\sum_{l=1}^{\infty}\left[2(l+\tfrac{1}{2})l(l+1)p_l^0(\xi)q_l^0(\xi)-\frac{l(l+1)}{(\xi^2-1)^{1/2}}-\frac{8\xi-7}{8(\xi^2-1)^{3/2}}\right]=\frac{8\xi-7}{8(\xi^2-1)^{3/2}}-\frac{1}{(\xi+1)^2} .
$$

Δ_n^I becomes

$$
\begin{aligned}
\Delta_n^I = \sum_{l=1}^{\infty}&\left[(l+\tfrac{1}{2})l(l+1)(W_l^{(1)n}+W_l^{(2)n}+W_l^{(3)n})-\frac{l(l+1)}{(\xi^2-1)^{1/2}}-\frac{8\xi-7}{8(\xi^2-1)^{3/2}}\right] \\
&-\frac{8\xi-7}{8(\xi^2-1)^{3/2}}-\frac{n^3}{96}\frac{(\xi+1)^4}{(\xi-1)^2}+\frac{n}{6}\left[\frac{1}{(\xi-1)^2}+\frac{2}{\xi-1}\right] .
\end{aligned}
\tag{B3}
$$

For the purposes of asymptotic analysis, it will be more convenient to write the above sum as

$$
\Delta_n^I = \sum_{l=0}^{\infty}\left[(l+\tfrac{1}{2})l(l+1)(W_l^{(1)n}+W_l^{(2)n}+W_l^{(3)n})-\frac{l(l+1)}{(\xi^2-1)^{1/2}}-\frac{8\xi-7}{8(\xi^2-1)^{3/2}}\right]-\frac{n^3}{96}\frac{(\xi+1)^4}{(\xi-1)^2}+\frac{n}{6}\left[\frac{1}{(\xi-1)^2}+\frac{2}{\xi-1}\right] .
\tag{B4}
$$

We employ the Sommerfeld-Watson technique of rewriting the sum as a contour integral, i.e., for $F(l)$ any function which is analytic in the right-hand half-plane:

$$
\sum_{l=0}^{\infty}F(l)=\mathrm{Re}\, i\int_{-1/2}^{\infty}dl\,\cot\pi l\,F(l)=\int_0^{\infty}d\lambda\,F(-\tfrac{1}{2}+\lambda)-\mathrm{Re}\, i\int_0^{\infty}d\lambda\frac{2}{1+e^{2\pi\lambda}}F(-\tfrac{1}{2}+i\lambda)
\tag{B5}
$$

(see Ref. 1 for details). We further split our expression as

$$
\Delta_n^I = I_1(n,\xi)-I_2(n,\xi)-\frac{n^3}{96}\frac{(\xi+1)^4}{(\xi-1)^2}+\frac{n}{48}\frac{3\xi^2+16\xi-11}{(\xi-1)^2}-\frac{1}{4}\left[I_3(n,\xi)+I_4(n,\xi)+\frac{n}{4}\frac{\xi+1}{\xi-1}\right] ,
\tag{B6}
$$

where

$$\begin{aligned}
I_1(n,\xi)&=\int_0^\infty d\lambda\left[\lambda^3(W^{(1)n}_{-1/2+\lambda}+W^{(2)n}_{-1/2+\lambda}+W^{(3)n}_{-1/2+\lambda})-\frac{\lambda^2}{(\xi^2-1)^{1/2}}-\frac{8\xi-7}{8(\xi^2-1)^{3/2}}+\frac{n^2(\xi+1)^4}{32(\xi^2-1)^{3/2}}\right],\\
I_2(n,\xi)&=2\,\mathrm{Re}\int_0^\infty d\lambda\frac{1}{1+e^{2\pi\lambda}}\lambda^3(W^{(1)n}_{-1/2+i\lambda}+W^{(2)n}_{-1/2+i\lambda}+W^{(3)n}_{-1/2+i\lambda}),\\
I_3(n,\xi)&=\int_0^\infty d\lambda\left[\lambda(W^{(1)n}_{-1/2+\lambda}+W^{(2)n}_{-1/2+\lambda}+W^{(3)n}_{-1/2+\lambda})-\frac{1}{(\xi^2-1)^{1/2}}\right],\\
I_4(n,\xi)&=2\,\mathrm{Re}\int_0^\infty d\lambda\frac{1}{1+e^{2\pi\lambda}}\lambda(W^{(1)n}_{-1/2+i\lambda}+W^{(2)n}_{-1/2+i\lambda}+W^{(3)n}_{-1/2+i\lambda}).
\end{aligned}\tag{B7}$$

Note that we have added a term proportional to n^2 to ensure convergence of I_1. Since, for $\epsilon\neq 0$, $\sum_1^\infty n^2\cos n\kappa\epsilon=0$, this term does not contribute to the final sum over n. We may now perform the integrations I_1 and I_3, and isolate the leading behavior in n of I_2 and I_4 by expanding the integrands about $\lambda=0$. The terms in the integrals of order n^3 and n cancel those explicit in (B6), and the terms of order n^{-1} cancel among themselves, resulting in an expression that is of order n^{-3} for large n. We refer the interested reader to Refs. 1 and 18 where similar calculations are performed in more detail.

Inserting the expression for the W's from Table I, and performing all integrations that can be carried out analytically, our final expression is

$$\begin{aligned}
\Delta_n^I=&-\frac{\xi-1}{120n^3(1+\xi)^7}[(-17\xi+63)(\xi+1)+80(11\xi-25)]\\
&-2\int_0^\infty d\lambda\frac{\lambda^3+\frac14\lambda}{1+e^{2\pi\lambda}}(W^{(1)n}_{-1/2+i\lambda}-W^{(1)n}_{-1/2}+W^{(2)n}_{-1/2+i\lambda}+W^{(3)n}_{-1/2+i\lambda}).
\end{aligned}\tag{B8}$$

To see that this expression is zero at $\xi=1$, we expand the WKB approximants in the integral in powers of λ/Λ, and $\Lambda^2=\chi_0^2/(\xi^2-1)$ and χ_0 is defined in Table I. We then integrate the resulting series term by term to arrive at an asymptotic series in Λ^{-2}. As the constant term in this series is zero at $\xi=1$, and the rational function in Δ_n^I is linear in $(\xi-1)$, we may conclude that $\Delta_n^I(\xi=1)=0$.

APPENDIX C

The purpose of this appendix is to establish the identity

$$\sum_{l=1}^\infty\left[2(l+\tfrac12)l(l+1)p_l^0(\xi)q_l^0(\xi)-\frac{l(l+1)}{(\xi^2-1)^{1/2}}-\frac{8\xi-7}{8(\xi^2-1)^{3/2}}\right]=\frac{8\xi-7}{8(\xi^2-1)^{3/2}}-\frac{1}{(\xi+1)^2}.\tag{C1}$$

We begin by considering the sum

$$\sum_{l=0}^\infty\left[2(l+\tfrac12)l(l+1)p_l^0(\xi)q_l^0(\xi)-\frac{l(l+1)}{(\xi^2-1)^{1/2}}-\frac{8\xi-7}{8(\xi^2-1)^{3/2}}\right].\tag{C2}$$

Substituting the definitions (A7) and distributing the summation, the expression (C2) becomes

$$\begin{aligned}
&\sum_{l=0}^\infty\left[2(l+\tfrac12)l(l+1)P_l(\xi)Q_l(\xi)-\frac{l(l+1)}{(\xi^2-1)^{1/2}}-\frac{1}{8(\xi^2-1)^{3/2}}\right]\\
&\quad+(\xi-1)\sum_{l=0}^\infty\left[(2l+1)\left[\frac{dP_l}{d\xi}(\xi)Q_l(\xi)+P_l(\xi)\frac{dQ_l}{d\xi}(\xi)\right]+\frac{\xi}{(\xi^2-1)^{3/2}}\right]\\
&\quad+(\xi-1)^2\sum_{l=0}^\infty\left[\frac{2l+1}{l(l+1)}\frac{dP_l}{d\xi}(\xi)\frac{dQ_l}{d\xi}(\xi)+\frac{1}{(\xi^2-1)^{3/2}}\right].
\end{aligned}\tag{C3}$$

The first two sums in (C3) were examined by Howard[1] who found

$$\sum_{l=0}^\infty\left[2(l+\tfrac12)l(l+1)P_l(\xi)Q_l(\xi)-\frac{l(l+1)}{(\xi^2-1)^{1/2}}-\frac{1}{8(\xi^2-1)^{3/2}}\right]=0,\tag{C4a}$$

$$\sum_{l=0}^\infty\left[(2l+1)\frac{dP_l}{d\xi}(\xi)Q_l(\xi)+\frac{\xi}{2(\xi^2-1)^{3/2}}\right]=0,\tag{C4b}$$

$$\sum_{l=0}^{\infty}\left[(2l+1)P_l(\xi)\frac{dQ_l}{d\xi}(\xi)+\frac{\xi}{2(\xi^2-1)^{3/2}}\right]=0\ . \tag{C4c}$$

The last sum in (C3) may be evaluated by considering the identity

$$\sum_{l=0}^{\infty}\left[(2l+1)(-1)^m P_l^{-m}(\xi)Q_l^m(\xi)-\frac{1}{(\xi^2-1)^{1/2}}\right]=-\frac{m}{\xi^2-1}\ , \tag{C5}$$

proved in the Appendix of Ref. 18. Noting that

$$P_l^{-1}(\xi)=\frac{(\xi^2-1)^{1/2}}{l(l+1)}\frac{dP_l}{d\xi}(\xi),\quad Q_l^1(\xi)=(\xi^2-1)^{1/2}\frac{dQ_l}{d\xi}(\xi)\quad (l>0)$$

and

$$P_0^{-1}(\xi)=\frac{\xi-1}{(\xi^2-1)^{1/2}},\quad Q_0^1(\xi)=\frac{-1}{(\xi^2-1)^{1/2}}\ ,$$

Eq. (C5) may be written for $m=1$ as

$$(\xi^2-1)\sum_{l=1}^{\infty}\left[\frac{2l+1}{l(l+1)}\frac{dP_l}{d\xi}(\xi)\frac{dQ_l}{d\xi}(\xi)+\frac{1}{(\xi^2-1)^{3/2}}\right]+P_0^{-1}(\xi)Q_0^1(\xi)+\frac{1}{(\xi^2-1)^{1/2}}=\frac{1}{\xi^2-1} \tag{C6}$$

or

$$(\xi-1)^2\sum_{l=1}^{\infty}\left[\frac{2l+1}{l(l+1)}\frac{dP_l}{d\xi}(\xi)\frac{dQ_l}{d\xi}(\xi)+\frac{1}{(\xi^2-1)^{3/2}}\right]=\frac{\xi}{(\xi+1)^2}-\frac{(\xi-1)^2}{(\xi^2-1)^{3/2}}\ . \tag{C7}$$

Inserting the results (C4) and (C7) into expression (C3), and moving all $l=0$ terms to the right-hand side of the resulting equation, we are left with the identity (C1).

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