Proof of the quantum bound on specific entropy for free fields

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The quantum bound on specific entropy for free fields states that the ratio of entropy S to total energy E of a system with linear dimension R cannot be larger than $2\pi R/\hbar c$. Here we prove this bound for a generic system consisting of a noninteracting quantum field in three space dimensions confined to a cavity of arbitrary shape and topology. $S(E)$ is defined as the logarithm of the number of quantum states (including the vacuum) accessible up to energy E. An integral equation is derived which relates an upper bound on $S(E)$ to the one-particle energy spectrum in the given cavity. The spectrum may always be bounded from above by a power law in energy whose proportionality constant is the ζ function for the spectrum of the cavity. This last is not calculable in the generic case, but it is here proven to be bounded by that for a sphere which circumscribes the actual cavity. Thus the one-particle spectrum for all cavities that fit inside a given sphere is bounded by a generic formula which can be computed given the field. With the help of this result the integral equation is solved for a fictitious system whose entropy must bound that of the actual system. The resulting bound on $S(E)/E$ proves to be smaller than $2\pi R/\hbar c$ with R interpreted as the radius of the enveloping sphere.

I. INTRODUCTION

A common intuitive feeling is that there should be a limit to the entropy that can be placed in a system of finite size whose energy is limited. This is suggested by the limited phase space available to the components of such a system. In trying to formulate a quantitative statement of this intuition a particularly interesting quantity to focus on is the specific entropy, i.e., the entropy to energy ratio S/E. It is a measure of the entropy (or missing information) which has the advantage of being the same for systems differing only in scale. A few years ago one of us[1] conjectured a quantum bound on specific entropy:

$$S/E \le 2\pi R/\hbar c \ . \tag{1.1}$$

Here R stands for the radius of the smallest sphere that circumscribes the system. Bound (1.1) has the status of a supplement to the second law; the latter only affirms that the entropy of a closed system tends to a maximum without saying how large that should be. If true, bound (1.1) would have varied physical consequences.[2] For instance, it is reflected in a limit on the rate at which information may be transferred within a given energy budget.[3]

The original argument for the bound[1] envisaged it as a consistency condition between black-hole thermodynamics and ordinary statistical physics to guarantee that the second law is respected. But as Unruh and Wald showed,[4] the buoyancy of ponderable bodies in the vicinity of a black hole can replace bound (1.1) as a consistency condition (but see Ref. 5). Nevertheless, bound (1.1) turns out to be correct, and it is our purpose here to give a proof of it in the context of the statistical mechanics of noninteracting fields in flat spacetime. This proof was long overdue since it has been clear from the beginning[1] that a bound of form (1.1) must be provable with no recourse to gravitational physics.

Simple statistical arguments show that bound (1.1) is obeyed by ordinary macroscopic systems with orders of magnitude to spare. This is because most of the energy in such systems is in the form of rest energy, and does not contribute to the enlargement of phase space that would tend to make S/E large. The universal validity of (1.1) is thus best put to test in systems involving massless quantum fields. The first extensive attempts to substantiate the bound in this arena used the *canonical* ensemble.[1]

It was found that validity of the bound hinges on the sign and value of the vacuum (Casimir) energy. If this is positive and not very small on the scale of the typical mode frequencies, then (1.1) is obeyed with the maximum S/E occurring at low excitation energy.[1] However, field-theoretic calculations for various cavities and fields frequently show that the vacuum energies are frequently negative.[6–8] Thus if the vacuum energy is included in E, a violation of (1.1) can be arranged by choosing the field system's temperature so that the thermal energy very nearly compensates the Casimir negative energy. Bound (1.1) has been criticized in this connection.[8,9] Even if the vacuum energy vanishes exactly, or if one chooses to interpret the E in (1.1) as excitation energy above the vacuum, violation of (1.1) is possible at low temperature,[5] though the mean energy range over which the violation occurs is extremely narrow. It has been argued that the above problems may be avoided if one regards the walls of the cavity confining the field as part of the system. These walls are essential to define the Casimir energy and inclusion of their mass in E seems to remove the violations of bound (1.1) in cases amenable to detailed analysis.[10]

But there is no gainsaying the conceptual clarity gained when the bound (1.1) is regarded as applying to the field in the cavity and only the field. This motivates an alternative approach to bound (1.1) which abandons the canonical method in favor of the microcanonical one (entropy is the logarithm of the number of available microstates), interprets E as the available energy above the vacuum state, and ignores the walls of the cavity.[11,12] As is well known, the equivalence of canonical and microcanonical ensembles holds good only for "large systems."[13] However, S/E is found to be at its largest value in states of low excitation[12] where the equivalence cannot be relied upon. The second approach mentioned regards the entropy calculated by microcanonical methods as more basic than the canonical one.

This is not just a convenient point of view. The canonical ensemble owes its popularity more to the convenience it affords in calculations (which are always more complicated the microcanonical way), than to the conviction that it gives a more "correct" entropy. Whereas the microcanonical ensemble method relies only on very general assumptions, like ergodicity, the canonical ensemble may be deduced from it only on the basis of additional hypothesis such as the validity of the saddle-point approximation, positivity of the specific heat, etc.[13] Situations where these conditions are not satisfied are not rare in nature, as witness the hydrogen atom which cannot be canonically described. Therefore, microcanonical theory appears to be the more primary theoretical framework. It is thus natural, if difficult, to base a proof of the entropy bound on it.

Early microcanonical numerical calculations of the specific entropy of free quantum fields confined to cavities with simple shapes were carried out by Gibbons.[11] These and later, more extensive ones,[12] support the entropy bound (1.1) in every case. Kahn and Qadir[14] have given an analytic argument in microcanonical ensemble which lends support to bound (1.1). This argument is, however, limited by its use of the continuum approximation in phase-space arguments. Since numerical calculations show that S/E peaks at low excitation energies[12] where one would expect the continuum approximation to be very crude, the Kahn-Qadir argument represents a suggestive piece of evidence, but not a proof of the entropy bound. We note that Kahn and Qadir regard the bound as valid only under such circumstances as also make the continuum approximation a good one. It will be clear from the rest of this paper that this pessimistic assessment is unjustified: the bound is a rigorous quantum bound for free fields.

Here we give, for the first time, a complete analytical proof of bound (1.1) applicable to free fields confined to cavities of arbitrary shape and topology which is based on the microcanonical ensemble. In Sec. II we set a bound on the number of quantum states accessible to a field system with given available energy in terms of the one-particle energy spectrum. In Sec. III the one-particle spectrum for an arbitrary cavity is shown to be bounded by a power law with a coefficient which depends only on the properties of the one-particle spectrum for the sphere which just circumscribes the cavity of interest. In Sec. IV we put all these pieces together to formulate the proof of bound (1.1). Our units are such that $\hbar$ and Boltzmann's constant k are unity.

II. MANY-PARTICLE NUMBER OF STATES FROM ONE-PARTICLE SPECTRUM

If $\Omega(E)$ denotes the number of quantum states accessible to the field system with energy up to and including E, then the microcanonical definition of entropy is

$$S(E)=\ln\Omega(E)\ . \tag{2.1}$$

In all that follows we shall have in mind mostly the boson case; for fermions the exclusion principle reduces $\Omega(E)$ below that corresponding to bosons, other things being equal. Thus an entropy bound for bosons is automatically applicable to the analogous fermion system (when adjustment is made for differences in spin degeneracy factors).

As a first step we express $\Omega(E)$ in terms of $\Omega_n(E)$, the cumulative number of n-particle states with energy up to and including E:

$$\Omega(E)=\sum_{n=0}^{\infty}\Omega_n(E)\ . \tag{2.2}$$

We shall assume a nondegenerate vacuum so that $\Omega_0(E)=1$. For $n\geq 1$ the Ω_n is defined with the aid of the one-sided Heaviside function $\Theta(x)$ (this function vanishes for $x<0$, is unity for $x>0$, and $\frac{1}{2}$ for $x=0$) in terms of the *one-particle* energy spectrum $\{\omega_i\}$ for the appropriate (Neumann or Dirichlet) boundary conditions at the confining cavity wall. For example, for a scalar field the ω_i are eigenvalues of

$$\nabla^2\psi_i=-\omega_i^2\psi_i\ . \tag{2.3}$$

The eigenvalues ω_i may be taken as non-negative due to the positive-definite nature of the operator $-\nabla^2$. In most of our discussion we assume that all eigenvalues are positive; the possibility and significance of zero eigenvalues are discussed in Sec. V.

Let the one-particle levels labeled $i_1, i_2, \ldots$ be ordered by energy so that $\omega_{i_j}\leq\omega_{i_k}$ if $i_j<i_k$ (degenerate levels are to be ordered arbitrarily). In terms of these we have

$$\Omega_n(E)\equiv\sum_{i_1\leq i_2\leq\cdots\leq i_n}\Theta(E-\omega_{i_1}-\omega_{i_2}-\cdots-\omega_{i_n})\ ,\quad n\geq 1\ . \tag{2.4}$$

The disposition of the limits on the summation has the effect of avoiding double counting of states which differ only by an exchange of (identical) particles. The cumulative number of one-particle states,

$$n(E)\equiv\Omega_1(E)=\sum_{i=0}^{\infty}\Theta(E-\omega_i)\ , \tag{2.5}$$

will play a key role in further discussion. In the absence of zero eigenvalues, $n(E)\to 0$ as $E\to 0$.

Let us define the auxiliary quantities

$$\Omega_n^*(E) \equiv \int_0^E dE_1 \int_0^{E_1} dE_2 \cdots \int_0^{E_{n-2}} dE_{n-1} (dn/dE_1)(dn/E_2)\cdots(dn/dE_{n-1})\, n\left[E-\sum_k^{n-1} E_k\right], \quad n \geq 1$$

and (2.6)

$$\Omega_0^*(E) \equiv \Theta(E) .$$

The $\Omega_n^*(E)$ may be easily evaluated from the definition of $n(E)$ by noticing that its derivative is a sum of delta functions, each with support at an eigenvalue ω_i. Therefore,

$$\Omega_n^*(E) \equiv \sum_{i_1 \leq i_2 \leq \cdots \leq i_{n-1}} n(E-\omega_{i_1}-\omega_{i_2}-\cdots-\omega_{i_{n-1}}) , \quad n \geq 1 \tag{2.7}$$

[note that $\Omega_1^*(E)=\Omega_1(E)$]. We see that the ordering by energy of the integrands in (2.6) induces an ordering by levels in the sum (2.7), and thus precludes overcounting of identical configurations.

Substituting (2.5) in (2.7) we have, for $n>1$,

$$\Omega_n^*(E) = \sum_{i_1 \leq i_2 \leq \cdots \leq i_{n-1}} \Theta(E-\omega_{i_1}-\omega_{i_2}-\cdots-\omega_{i_n}) > \sum_{i_1 \leq i_2 \leq \cdots \leq i_n} \Theta(E-\omega_{i_1}-\omega_{i_2}-\cdots-\omega_{i_n}) .$$

The last inequality is due to the fact that the last particle (labeled i_n) is not constrained to be in a level such that $\omega_{i_{n-1}} \leq \omega_{i_n}$. Therefore, there are more terms in the first sum. Comparing with (2.4) we see that for $n>1$, $\Omega_n^*(E)$ overestimates the true number of n-particle configurations $\Omega_n(E)$. Hence, for all n,

$$\Omega_n(E) < \Omega_n^*(E) . \tag{2.8}$$

Now suppose we relax the "energy ordering" in the defining expression for $\Omega_n^*(E)$. We then overcount states so that the new counting functions

$$N_n(E) \equiv \int_0^E dE_1 \int_0^E dE_2 \cdots \int_0^E dE_{n-1} (dn/dE_1)(dn/E_2)\cdots(dn/dE_{n-1})\, n\left[E-\sum_k^{n-1} E_k\right], \quad n \geq 1$$

and

$$N_0 = \Theta(E)$$

trivially satisfy

$$\Omega_n(E) \leq \Omega_n^*(E) \leq N_n(E) , \tag{2.9}$$

the equality occurring for $n=0,1$. Therefore, we have for the cumulative number of quantum states the bound

$$\Omega(E) = \sum_{n=1}^{\infty} \Omega_n(E) < \sum_{n=1}^{\infty} N_n(E) \equiv N(E) . \tag{2.10}$$

Since $n(E)$ vanishes for negative arguments, we may rewrite $N(E)$ for $E \geq 0$ as

$$N(E) \equiv \Theta(E) + \sum_{n=1}^{\infty} \int_0^E dE_1 \int_0^{E-E_1} dE_2 \int_0^{E-E_1-\cdots E_{n-2}} dE_{n-1} (dn/dE_1)(dn/E_2)\cdots(dn/dE_{n-1})\, n\left[E-\sum_k^{n-1} E_k\right]$$

and we see that $N(E)=0$ for $E<0$.

This expression may be recast in a very elegant form as the integral equation

$$N(E) = \Theta(E) + \int_0^E N(E-E')(dn/dE')dE' \tag{2.11}$$

as may be verified by iterating the latter. Therefore, we achieved our goal of relating the number of accessible quantum states of the field to the cumulative number of one-particle states $n(E)$ through Eqs. (2.10) and (2.11). Since $n(E)$ is related to the one-particle energy spectrum which is not known in closed form for an arbitrary cavity, we now turn to investigate general constraints on the latter.

III. THE ONE-PARTICLE NUMBER OF STATES

Calculating $n(E)$ in detail for arbitrary cavity shape is a hopeless task, but is also unnecessary. Instead, we shall establish a simple constraint on the corresponding ζ function. This is defined as

$$\zeta(k) = \sum_{i=1}^{\infty} \omega_i^{-k} , \tag{3.1}$$

where the ω_i are the energy eigenvalues [defined, for example, by Eq. (1.5) with the appropriate boundary conditions]. Degenerate ω_i enter the sum separately. In accordance with our exclusion of zero eigenvalues, all terms in the sum in (3.1) are well defined. Because the number of eigenvalues up to ω increases as ω^3 for large ω, the $\zeta(k)$ converges only for $k>3$. Note that if one multiplies $\zeta(k)$ by E^k, there are a total of $n(E)$ terms in the sum involving a factor $(E/\omega_i)^k$ larger than unity. Since there are further positive terms in the sum we see that $n(E)$ satisfies the ζ inequality:[11,1]

$$n(E) \le \zeta(k) E^k \ . \tag{3.2}$$

This is useful only for $k>3$. We now prove in two different ways that the ζ function for a general cavity is bounded from above by that for its smallest circumscribing sphere.

A. Local theorem on the ζ function

Intuitively we would expect (say, on the basis of the uncertainty principle) that the energy eigenvalues are the larger the smaller the cavity. A precise statement of this is that as a given cavity S is deformed into another one Σ entirely contained within it, all the eigenvalues increase and, therefore, the ζ function is the smaller for Σ. Figure 1 clarifies the relation between the two cavities showing $\boldsymbol{\eta}$, the normal to the surface of the original cavity, as well as ξ, the vector along the tangent to the deformation paths.

Let $\chi_i(\mathbf{x})$ be that normalized eigenfunction [in the sense of (3.3)] in the deformed cavity which develops out of the (normalized) eigenfunction of the original cavity as the latter is deformed. Concentrating on the case of the *scalar* field with Dirichlet boundary condition we may write

$$\begin{aligned} &(\nabla^2+\Omega_i^2)\Psi_i(\mathbf{x})=0, \quad \Psi_i|_{\partial S}=0 \ , \\ &(\nabla^2+\omega_i^2)\chi_i(\mathbf{x})=0, \quad \chi_i|_{\partial \Sigma}=0 \ . \end{aligned} \tag{3.3}$$

Here ∂S means the sphere's exterior boundary, etc. We assume that the definition of $\chi_i(\mathbf{x})$ is smoothly extended to the region between $\partial\Sigma$ and ∂S. Let us multiply the adjoint equation for Ψ_i by χ_i, the equation for χ_i by Ψ_i^*, subtract the results, and integrate over the interior of S. Use of Green's theorem and the boundary condition for Ψ_i leads to the result

$$\int_{\partial S}\chi_i(\mathbf{x})\nabla\Psi_i^*(\mathbf{x})\cdot d\mathbf{S}=(\omega_i^2-\Omega_i^2)\int_S \Psi_i^*(\mathbf{x})\chi_i(\mathbf{x})d^3x \ . \tag{3.4}$$

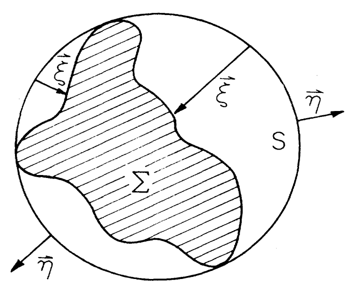

FIG. 1. Illustrating the cavity Σ, its circumscribing cavity S, the normal $\boldsymbol{\eta}$, and the deformation vector ξ.

Next, assuming that the deformation is small, we expand this equation to first order in the displacement vector $\xi(\mathbf{x})$. On the right-hand side (RHS) the factor $(\omega_i^2-\Omega_i^2)$ is already of first order; hence the integral multiplying it may be replaced by unity (normalization). It follows from the boundary condition (see Fig. 1) that

$$\chi_i|_{\partial\Sigma}=\chi_i|_{\partial S}+\xi(\mathbf{x})\cdot\nabla\Psi_i|_{\partial S}+O(\xi^2)=0 \ ,$$

where we have exploited the fact that the difference between $\nabla\Psi_i$ and $\nabla\chi_i$ on ∂S is already of first order. Thus to first order

$$\chi_i|_{\partial S}=-\xi(\mathbf{x})\cdot\nabla\Psi_i|_{\partial S} \ . \tag{3.5}$$

With this result in Eq. (3.4) we have, up to the first order in $\xi(\mathbf{x})$,

$$\omega_i^2-\Omega_i^2=-\int_{\partial S}[\xi(x)\cdot\nabla\Psi_i^*]\nabla\Psi_i\cdot d\mathbf{S}+O(\xi^2) \ . \tag{3.6}$$

But the derivative of Ψ_i tangential to S vanishes, so

$$\omega_i^2-\Omega_i^2=-\int_{\partial S}(\boldsymbol{\eta}\cdot\xi)(\partial\Psi_i/\partial\eta)^2|d\mathbf{S}|+O(\xi^2) \ , \tag{3.7}$$

where $\partial\Psi_i/\partial\eta$ is the normal derivative of Ψ_i at ∂S. If the original cavity is slightly deformed into one entirely contained within it, $\boldsymbol{\eta}\cdot\xi\le 0$ (see Fig. 1) so all the energy eigenvalues grow and the ζ function obviously decreases.

The argument given does not really depend on the sphericity of the original cavity. Hence we can imagine regarding a large deformation of the cavity as composed of many small ones. To each of these we may apply the theorem so that the result applies also to a deformation of arbitrary magnitude, provided the final cavity has the same topology as the initial one.

Some cavities which may not be obtained by deforming a spherical one may be handled by slight modifications of the above argument. Suppose, for example, that Σ is toroidal with its handle nearly centrally located. We would again start with a sphere S circumscribing Σ. As is well known, the scalar eigenfunctions $\Psi_i(\mathbf{x})$ for the sphere may be chosen in spherical coordinates as $j_l(\omega_i r)y_l^m(\theta,\phi)$ where j_l is the spherical Bessel function and y_l^m the spherical harmonics. Now if we exclude from the list of eigenfunctions those with $m=0$, all the remainder vanish along the polar axis $(\theta=0,\pi)$. Thus they satisfy the boundary conditions appropriate for a sphere pierced by a thin tube along its axis. We may deform continuously such a pierced sphere into our torus. Then our theorem may be applied to show that the ω_i for the torus are all larger than the corresponding ones for the pierced sphere. Since the spectra of the whole sphere and the pierced one differ only in that the latter lacks the eigenvalues with $m=0$, we may conclude that $\zeta(k)$ for a torus is smaller than the corresponding $\zeta(k)$ for a circumscribing sphere.

We conclude that the *ζ function with $k>3$ of the scalar field with Dirichlet boundary condition in a cavity is smaller than the corresponding one for a circumscribing spherical cavity if the former cavity is obtainable from a spherical one by shrinking, namely, by a series of deformations such that at each stage the interior of the deformed cavity is completely contained in that of the cavity of the* *previous stage. The same is true if one may start from a spherical cavity pierced through its axis.* The extension of this result to other fields, boundary conditions, and topologies may be had from the following theorem.

B. Global theorem on the ζ function

Consider now any free quantum field which is described by a Hermitian, positive-definite Hamiltonian H. Simple examples are the scalar, electromagnetic, and Dirac fields (the negative eigenvalues for the latter are, of course, interpreted as positive eigenvalues corresponding to antiparticles). We shall suppose it confined to a cavity Σ of arbitrary topology about which we draw the circumscribing sphere S. The confinement is reflected in some boundary conditions specified on $\partial\Sigma$. Analogous conditions may be specified on ∂S. Thus discrete spectra $\{\Omega_i\}$ and $\{\omega_i\}$ appear together with the corresponding sets of eigenfunctions $\{\chi_i(\mathbf{x})\}$ and $\{\Psi_i(\mathbf{x})\}$ (we suppress vector or spinor indices). As clear from the example of the pierced sphere, there is not in general one-to-one correspondence between the two spectra. Thus our approach here will deemphasize comparison of specific eigenvalues in contrast with our previous approach.

In order to handle uniformly both fermion and boson fields, we shall only invoke H^2 (whose spectrum contains squares of frequencies). For example, for the scalar field $H^2=-\nabla^2$ while for a spinor field H^2 would be the square of the usual Dirac Hamiltonian operator. A particular eigenfunction $\chi_k(\mathbf{x})$ is defined over Σ; we extend its definition to the whole of S by stipulating that it vanishes in the complement of Σ within S. It will be important for us that the set of $\Psi_i(\mathbf{x})$ for the outer cavity is complete, i.e., that any function of the appropriate spinorial or tensorial form defined over S may be expanded in these functions. In particular this applies to the $\chi_i(\mathbf{x})$ which we expand as

$$\chi_k(\mathbf{x})=\sum_i A_{ki}\Psi_i(\mathbf{x}) \ . \tag{3.8}$$

From the orthogonality of the $\Psi_k(\mathbf{x})$ (a consequence of Hermiticity of H) it follows that

$$A_{ki}=\int_\Sigma \Psi_i^*(\mathbf{x})\chi_k(\mathbf{x})d^3\mathbf{x} \ . \tag{3.9}$$

Now apply H^{2n} (n is an arbitrary positive integer) to both sides of (3.8) in the interior of Σ. Assuming the eigenfunctions may be differentiated an arbitrary number of times this gives

$$\Omega_k^{2n}\chi_k(\mathbf{x})=\sum_i A_{ki}\omega_i^{2n}\Psi_i(\mathbf{x}) \tag{3.10}$$

(at $\partial\Sigma$ the correct result is to be understood as a limit from the interior of Σ since χ_k is not differentiable at the boundary itself). For any function $F(z)$ whose Taylor expansion converges for all $z>0$ we may thus write

$$F(\mu\Omega_k^2)\chi_k(\mathbf{x})=\sum_i A_{ki}F(\mu\omega_i^2)\Psi_i(\mathbf{x}) \ , \tag{3.11}$$

where μ is any positive constant with dimensions of (energy)$^{-2}$.

Now the functions $\chi_i(\mathbf{x})$ are orthonormal in Σ; hence by multiplying (3.11) by $\chi_k^*(\mathbf{x})$ and integrating over Σ we get

$$F(\mu\Omega_k^2)=\sum_i A_{ki}^* A_{ki}F(\mu\omega_i^2) \ . \tag{3.12}$$

Summing over k gives

$$\sum_k F(\mu\Omega_k^2)=\sum_i\sum_k |A_{ki}|^2F(\mu\omega_i^2) \ , \tag{3.13}$$

where the interchange of the order of summation is predicated on the assumed absolute convergence of the series in the RHS of (3.12) for fixed μ. Of course our result makes sense only if the sum in the left-hand side (LHS) converges, which necessitates that $F(z)$ decrease asymptotically faster than $z^{-3/2}$ (since the number of eigenvalues up to Ω grows as Ω^3 for large Ω).

Now $\{\chi_i\}$ is *not* a complete set for S since the functions are naturally associated only with the subspace Σ. Suppose we try to represent a Ψ_i in Σ in the form

$$\Psi_i(\mathbf{x})=\sum_1 B_{ik}\chi_k(\mathbf{x}), \quad \mathbf{x}\in\Sigma \ . \tag{3.14}$$

The best expansion, the one which differs least from Ψ_i in the least-squares sense,[15] corresponds to choosing $B_{ik}=A_{ki}^*$ [see (3.14)]. By Bessel's inequality[15] the incompleteness of $\{\chi_i\}$ translates into the strict inequality $\Sigma_k|B_{ik}|^2<1$. In view of this and the relation $B_{ik}=A_{ki}^*$, (3.13) is equivalent to

$$\sum_i F(\mu\Omega_i^2)<\sum_i F(\mu\omega_i^2) \ . \tag{3.15}$$

Let us take $F(z)=\exp(-z)$; this function certainly satisfies the conditions of convergence of the Taylor series, and decreases faster than $z^{-3/2}$. Since inequality (3.15) is valid for all $\mu>0$, we may multiply both sides of it by the same positive function of μ, and then integrate over the same range of μ, and the result must preserve the sense of the inequality. For example,

$$\int_0^\infty d\mu\,\mu^{k/2-1}\sum_i \exp(-\mu\Omega_i^2) < \int_0^\infty d\mu\,\mu^{k/2-1}\sum_i \exp(-\mu\omega_i^2) \ ,$$

where k is a positive number to be further constrained presently. Now, as a function of μ, the series of exponentials is uniformly convergent essentially because of the rapid decrease in magnitude of the terms as the eigenvalues grow large. The sum and integral may thus be interchanged. After rescaling and evaluating the integral we get

$$\sum_i \Omega_i^{-k}<\sum_i \omega_i^{-k} \ . \tag{3.16}$$

We thus discover that $k>3$ is a necessary condition, since otherwise the resulting sums do not converge.[16] In view of definition (3.1) the result shows that the ζ function of Σ is smaller than that for S. In particular *the ζ function of order larger than 3 for a field confined to a cavity of arbitrary shape and topology is smaller than the corresponding ζ function for a spherical cavity which com-*

pletely circumscribes it.

In view of the ζ inequality (3.2), our theorem tells us that

$$n(E) < n_k^*(E) \equiv \zeta_{sp}(k) E^k, \quad k > 3 , \tag{3.17}$$

where $\zeta_{sp}(k)$ is the ζ function for the *sphere* just circumscribing the cavity Σ. We shall make use of this important result forthwith.

IV. THE BOUND ON SPECIFIC ENTROPY

Return now to the integral equation (2.11). Suppose that $n(E)$ for the cavity of interest is there replaced by $n_k^*(E)$ defined in (3.17). Then the solution of the modified integral equation, call it $N_k^*(E)$, corresponds to a fictitious system with more one-particle states up to a given energy than the true system. Thus $N(E) < N_k^*(E)$. Therefore,

$$\Omega(E) < N_k^*(E), \quad k > 3 . \tag{4.1}$$

Let us now convert the integral equation for $N_k^*(E)$ into a more convenient, algebraic, equation by taking its Laplace transform. Denoting the Laplace transform of a function $f(E)$ by $\tilde{f}(s)$ we have

$$\tilde{N}_k^*(s) = \frac{1}{s[1 - s\tilde{n}_k^*(s)]} . \tag{4.2}$$

It follows from (3.17) that

$$\tilde{n}_k^*(s) = \Gamma(k+1)\zeta_{sp}(k) s^{-(k+1)} , \tag{4.3}$$

where $\Gamma(k+1)$ is the usual Γ function. In all that follows we limit our remarks to the case of integral k, so that $\Gamma(k+1) = k!$. Substitution of (4.3) into (4.2) gives

$$\tilde{N}_k^*(s) = \frac{s^{k-1}}{s^k - k!\zeta(k)} . \tag{4.4}$$

Now $\tilde{N}_k^*(E)$ is the inverse Laplace transform of $\tilde{N}_k^*(s)$: namely, the integral of $\tilde{N}_k^*(s)\exp(sE)/2\pi i$ along a contour parallel to the imaginary s axis to the right of the k poles of $\tilde{N}_k^*(s)$. These are distributed uniformly along a circle in the complex s plane whose radius is $[k!\zeta_{sp}(k)]^{1/k}$. Their phases correspond to the k distinct kth roots of unity, $\sigma_1, \sigma_2, \ldots, \sigma_k$. It is convenient to translate the contour to large negative s while indenting it to avoid the poles in the manner illustrated in Fig. 2 for the case $k=4$. In this way only the pole residues contribute to the inverse transform; the contribution of the vertical part of the contour, labeled C, vanishes in the limit. In view of all this

$$N_k^*(E) = k^{-1} \sum_{n=1}^{k} \exp\{\sigma_n [k!\zeta_{sp}(k)]^{1/k} E]\} . \tag{4.5}$$

Since the ζ function is only defined for $k > 3$, let us choose $k=4$ and set $X \equiv [4!\zeta_{sp}(4)]^{1/4}$ [some reflection shows that this leads to the tightest bound on $\Omega(E)$]. Then by exploiting various trigonometric and transcendental identities we have

$$\begin{aligned} N_4^*(E) &= \tfrac{1}{4}[\exp(XE) + \exp(-XE) + \exp(iXE) + \exp(-iXE)] \\ &= \tfrac{1}{2}[\cosh(XE) + \cos(XE)] \le \tfrac{1}{2}[\cosh(XE) + 1] = \cosh^2(XE/2) < \exp(XE) = \exp\{[24\zeta_{sp}(4)]^{1/4} E\} . \end{aligned} \tag{4.6}$$

In light of (4.1) we conclude that the microcanonical entropy $S = \ln[\Omega(E)]$ obeys

$$S(E)/E < [24\zeta_{sp}(4)]^{1/4} . \tag{4.7}$$

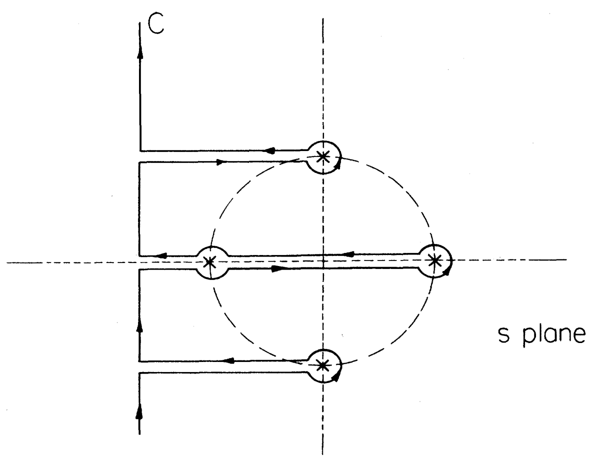

FIG. 2. Illustrating the contour for evaluation of the inverse Laplace transform of $\tilde{N}_k^*(s)$. Crosses mark its poles.

In previous work [Ref. 12, Eq. (37)] we had already suggested that $\zeta(4)^{1/4}$ provides a good estimate of the maximum S/E. What we have shown here is that there is no need to consider each cavity separately; if all that we want is to set an upper bound on S/E, it suffices to look at the spherical cavity. The ζ function $\zeta_{sp}(4)$ has been numerically calculated elsewhere[12] for several massless fields which represent (apart from the question of rest mass) most of the elementary fields seen in nature. The results are given in Table I.

It is clear from Table I that the entropy bound is respected for the three fields considered and for all cavities, simply connected or not, of whatever shape. The addition of rest masses can only reduce the phase space available at a given energy, so that massive fields are subject to bounds tighter than those displayed in Table I. If a variety of fields can exist in the cavity, the appropriate bound is obtained by replacing $\zeta_{sp}(4)$ in (4.7) by the sum of ζ functions of all species concerned. This is because in our argument the identity of the one-particle levels is of no consequence. Since the ζ function appears in the bound to such a low power, it is evident that many elementary fields can be introduced without violating bound

TABLE I. ζ function for free fields confined to a sphere of radius R.

Field	$\zeta_{sp}(4)$	$\frac{[24\zeta_{sp}(4)]^{1/4}}{2\pi R}$
Scalar	$0.0417R^4$	0.159
Electromagnetic	$0.266R^4$	0.253
Spinor	$0.0832R^4$	0.189

(1.1) [for example, up to 244 vector fields or 1564 scalar fields can be accommodated in inequality (4.7)]. Thus, with due reserve it is clear that, in the absence of interactions, we have obtained a fairly general proof of the entropy bound.

V. SUMMARY AND ASSESSMENT

We have proven that the microcanonical entropy of a system of free quantum field governed by fairly general Hamiltonians and confined to a cavity of arbitrary shape and topology is subject to bound (1.1). The strategy has been to first relate the number of quantum configurations accessible up to a given energy to the number of one-particle levels up the same energy via an integral equation. This stage does not depend on the way the field is confined, but only on the assumption that interactions are negligible (except for those which confine it and are expressed as boundary conditions). Next, the one-particle spectrum for the real cavity was bounded by that for the circumscribing sphere with the help of a very general theorem relating the ζ functions of the two cavities. Finally, the integral equation for a fiducial system which is related more to the sphere than to the actual cavity is solved, and a bound set on the entropy. This latter bound is stronger than (1.1) for all fields we meet in nature, and so establishes its validity.

Throughout we have assumed that there are no zero one-particle energy eigenvalues. The existence of such "zero modes" formally causes the ζ function to become undefined, and would seem to invalidate the above procedure. This, however, is *not* a loophole in the proof. It must be realized that a zero mode formally allows one to construct an infinity of different states of the field with like energy by just adding quanta in the zero mode one at a time. In this case the microcanonical entropy seems not to be well defined (infinite). However, zero modes are generally associated with some symmetry of the system (usually translations) and do not represent dynamical degrees of freedom.[17] They are commonly regarded as associated with a "condensate," i.e., the classical part of a quantum field in the theory of symmetry breaking, or the condensate in a superfluid.[18] When a field possessing zero modes is second quantized, creation and anhilation operators are associated only with modes other than the zero modes (above-the-condensate particles). Therefore, in our approach it is justified to ignore zero modes.

Still clamoring for attention are field systems with interactions. Two detailed examples[12,19] show how difficult it is to violate bound (1.1) even with the help of interactions. It is plain that in the presence of interactions the additivity of particle energies used in the first part of our program must be given up. The proof must thus proceed along a rather different path. Progress in this direction will be reported elsewhere.[20]

All the above presupposes flat spacetime. One example[21] has been given showing how the wave character of particles helps to protect the bound in the presence of strong gravitation. It is a pressing matter to convert these insights into a general approach for dealing with strongly gravitating systems. One goal is obviously to clarify the ultimate relation of bound (1.1) to black holes, a relation which first suggested the bound[1] and is still reflected in the numerical factor appearing in it.

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