PHYSICAL REVIEW D VOLUME 39, NUMBER 4 15 FEBRUARY 1989

Gravitational fields of straight and circular cosmic strings: Relation between gravitational mass, angular deficit, and internal structure

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(Received 26 September 1988)

The first part of the paper formulates general conditions (independent of a particular gauge-theoretic model) under which a cylindrical distribution of matter can be joined to a vacuum exterior with a conical geometry and exhibits the relation between angular deficit and internal structure. To bring out the relation to gravitational mass, the second part is devoted to a detailed study of solutions of the initial-value problem for circular loops of string at a moment of time symmetry.

I. INTRODUCTION

In general relativity, cosmic strings have gravitational fields with distinctly non-Newtonian characteristics.[1] Their idiosyncrasies have been a snare for the unwary ever since strings were first introduced into cosmology by Kibble, Zel'dovich, and Vilenkin.[2] This paper arose from an attempt to understand more clearly the general nature (irrespective of specific gauge-theoretic structural models) of the relation between the near and far fields of a string and its internal structure.

From close up, the string may be considered straight and infinite. The exterior geometry is then conical, and linearized Einstein theory[3] yields the relation $\Delta\phi/8\pi=$(inertial mass/unit length) for the angular deficit $\Delta\phi$. The curvature and the force required to hold a test particle at rest both vanish. From the near-field point of view, the effective gravitational mass of the string is zero. The string is nevertheless a repository of positive energy, which must make its presence manifest in the total gravitational mass measured at spatial infinity.

This suggests a schematic picture for the gravitational field of a loop of cosmic string. There is a near zone, whose size is small compared to the loop radius, in which the geometry is locally flat and conical; a far zone, where the field is Schwarzschildean; and, sandwiched between these, a transition zone, of thickness comparable to the loop radius, in which the geometry evolves from one to the other configuration. The relations between the fields in these zones and the internal structure of the string can be neatly formulated for a newborn or momentarily static loop by defining three kinds of "mass": the inertial mass

$$M_{\text{inert}}=\int(-T^0_0)({}^3g)^{1/2}d^3x \ , \tag{1}$$

the "Hiscock mass"[4] (really a measure of angular deficit)

$$M_{\text{His}}=\mu_{\text{His}}\times(\text{length}),\quad \mu_{\text{His}}\equiv\Delta\phi/8\pi \ , \tag{2}$$

and the gravitational or Arnowitt-Deser-Misner (ADM) mass M determined by the asymptotic field at spatial infinity. Linearized theory, which one might reasonably hope to be adequate for grand-unified-theory (GUT) strings, which have

$$\mu\sim(\text{Higgs-boson mass})^2\sim10^{-8} \ , \tag{3}$$

predicts that the "masses" M_{inert}, M_{His}, and M are equal. Concerning the general relationship between them, however, very little is rigorously known. To explore this question is one of the principal objectives of this paper. Since only local and near-field properties are involved in the definitions of M_{inert} and M_{His}, the idealization of an infinite straight string may be used in studying their interconnection. This is the subject of Sec. II, which is concerned with stationary cylindrical distributions and the conditions under which they can be joined to an exterior vacuum having a conical geometry. To deal with the gravitational mass M, the idealization of an infinitely extended source is ineffective, and in Secs. III–VI we therefore turn to a detailed study of the initial-value problem for circular loops of string at a moment of time symmetry. There is an infinite variety of solutions, depending on the amount of incoming gravitational radiation present on the initial time slice, a quantity that is difficult to bring under complete control without access to past lightlike infinity. However, it can be said in broad terms that for small angular deficits our results corroborate the equality

$$M\approx M_{\text{His}}\approx M_{\text{inert}}\quad(\Delta\phi\to0) \ . \tag{4}$$

But M dips below M_{His} by a factor that grows to the order of 2 as $\Delta\phi$ rises towards π. Beyond this, the loop is generally enclosed within a black hole.

Specific properties and relations of this kind lead on to more general issues, of which the briefest mention may suffice here, since they have recently received attention elsewhere. Is the external geometry of a string a unique diagnostic of internal structure—i.e., can a line distribution of stress and energy be uniquely inferred from external properties such as angular deficit?[4,5] Is a distributional description possible at all in general for line sources, stringlike or nonstringlike?[6] In a preliminary reconnaissance of these questions more than a decade ago, one of us concluded: "There exists no simple general prescription, analogous to [the well-known surface-layer formalism], for obtaining the physical characteristics of an arbitrary line source."[7] A careful mathematical analysis by Geroch and Traschen[6] reaffirms this pessimistic conclusion. The difficulty is that Einstein's theory is nonlinear, and a product is not definable for distributions more singular than step functions. It is nevertheless possible to isolate a subclass of "simple" line sources,[7] characterized by the condition

$$(\text{radial stress}) \ll \text{energy density}\ , \tag{5}$$

for which a distributional description is possible in the limit of an infinitely thin source, and a line stress-energy tensor can be defined that is uniquely correlated with the near-field exterior geometry. In particular, for conical geometries, this prescription can be used to assign to a string with any given angular deficit an "equivalent" or "effective" simple line-source structure that happens to reproduce precisely the Hiscock mass:

$$\begin{aligned}\text{effective tension} &= \text{effective mass/unit length}\\ &= \Delta\phi/8\pi\ .\end{aligned} \tag{6}$$

Of course, this effective structure approximates the distribution of stress and energy in the real (finite-thickness) string only when that distribution satisfies the "simplicity" condition (5). Studies of false-vacuum solutions to the gauge-coupled massive scalar-Maxwell-Einstein equations, pioneered by Garfinkle,[8,4] show, however, that gauge strings are generally not simple sources; they have a stress-energy distribution that is strongly model dependent and well approximated by (6) only in the limit of small $\Delta\phi$.

Beyond this loom more difficult dynamical and evolutionary questions. Cosmological scenarios involving cosmic strings generally assume that the smaller loops evaporate by emission of gravitational and other forms of radiation. If, however, an appreciable fraction were to form black holes instead of evaporating completely, this could lead to an unacceptably high γ-ray background due to Hawking radiation from mini black holes. It is thus of great interest to secure an estimate of what this fraction is. Hawking[9] has recently made an initial assault on this problem. He has also shown by an elegant argument[10] that a circular loop collapsing with the speed of light (a reasonable idealization, since the loop is rapidly accelerated by its own tension) must form a black hole if cosmic censorship is valid, with the loss of at most a fraction $1-2^{-1/2}=29.3\%$ of its mass energy in the form of gravitational radiation.

II. INTEGRAL FORMULAS FOR STATIONARY CYLINDRICAL DISTRIBUTIONS AND INFINITE STRAIGHT STRINGS

Internal structures for cosmic strings, "normal"[8] and superconducting,[11] have been worked out by a number of authors on the basis of various field-theoretical models involving one or more gauge-coupled boson or fermion fields. In the case of infinite straight strings the field equations are stationary and become more-or-less tractable, and only this case has so far been considered. This section contains little that is essentially new. Our principal objective is simply to assemble, with concise derivations, some of the more useful integral formulas that relate the exterior geometry and gravitational field of an infinite straight string to its internal distribution of stress-energy. These formulas follow solely from the gravitational field equations; no particular source structure is assumed.

Several of these formula may be derived most conveniently as special cases of a general integral identity, due in essence to Tolman and to Komar,[12] valid for any spacetime admitting a Killing vector. Let V be a spacetime domain bounded by a closed or cylindrical three-space Σ with unit normal n^α: $n^\alpha n_\alpha=\epsilon=+1$ or -1 accordingly as n^α is spacelike or timelike. Suppose that V admits a Killing field ξ^α (timelike, azimuthal, or cylindrical) which generates and is tangent to Σ. Define a scalar ζ ($=t$, ϕ, or z) by the conditions

$$\xi^\alpha\partial_\alpha\zeta=1\ ,\quad n^\alpha\partial_\alpha\zeta=0\ , \tag{7}$$

which are compatible by virtue of $\mathcal{L}_\xi n^\alpha=0$. Then

$$\int_\Sigma \epsilon\xi^a K_a^b(\partial_b\zeta)d\Sigma=\int_V \xi^\alpha R_\alpha^\beta(\partial_\beta\zeta)(-g)^{1/2}d^4x\ . \tag{8}$$

Latin indices refer to three-dimensional intrinsic coordinates of Σ; K_a^b is its extrinsic curvature and our sign conventions follow Misner, Thorne, and Wheeler.[13]

We briefly indicate the simple steps of the proof. From the definition of extrinsic curvature,[13]

$$K_a^b\xi^a\partial_b\zeta=-n_\alpha{}^{|\beta}\xi^\alpha\partial_\beta\zeta=\xi^{\alpha|\beta}n_\alpha\partial_\beta\zeta\ . \tag{9}$$

The Gauss divergence theorem now yields, noting the skew symmetry of $\xi^{\alpha|\beta}$,

$$\int_\Sigma \epsilon\xi^{\alpha|\beta}(\partial_\beta\zeta)n_\alpha d\Sigma=\int_V \xi^{\alpha|\beta}{}_{|\alpha}(\partial_\beta\zeta)(-g)^{1/2}d^4x\ , \tag{10}$$

which reduces to (8) on applying the Ricci commutation relations.

For later reference we digress for a moment to a *bounded* distribution in an asymptotically flat spacetime which admits a timelike Killing vector $\xi_{(t)}^\alpha$. The surface integral in (8) is then easily evaluated in the asymptotic region. Factoring out the t integration and deploying the Einstein field equations, one arrives at the well-known Tolman formula

$$M_{\text{grav}}=\int_{x^0=t=\text{const}}(-T_t^t+T_a^a)(-g)^{1/2}d^3x \tag{11}$$

for the gravitational mass of a bounded stationary distribution. Its simple interpretation in the weak-field limit in terms of Newtonian potential ϕ and gravitational energy Ω is worth recording. Restoring the constants G and c, setting $d\tau={}^{(3)}g^{1/2}d^3x$, $-T_t^t=\rho c^2$, $M_{\text{inert}}=\int\rho\,d\tau$, noting that

$$\Omega=\tfrac{1}{2}G\int\rho\phi\,d\tau=-\int T_a^a d\tau \tag{12}$$

by the Newtonian virial theorem, and that

$$\begin{aligned}(-g)^{1/2}&={}^{(3)}g^{1/2}(-g^{tt})^{-1/2}\\ &={}^{(3)}g^{1/2}(1+\phi/c^2)\end{aligned}$$

we recover from (11) the expected Newtonian result for the total energy through overcompensation by the stress contribution for the red-shift of the volume element:

$$E=M_{\text{grav}}c^2=M_{\text{inert}}c^2+2\Omega-\Omega+O(c^{-2})\ . \tag{13}$$

We turn now to stationary axisymmetric cylindrical distributions with three commuting Killing vectors $\xi^{\alpha}_{(t)}$, $\xi^{\alpha}_{(\phi)}$, $\xi^{\alpha}_{(z)}$. If the distribution is radially bounded, the metric of the surrounding vacuum has the Kasner form

$$(ds^2)_{\rm ext}=d\rho^2+k^2\rho^{2a}d\phi^2+\rho^{2b}dz^2-\rho^{2c}dt^2 \quad (0\le\phi\le 2\pi) \ . \tag{14}$$

The constant k could have been scaled to unity by giving up the standard $(0,2\pi)$ convention for ϕ; a,b,c are constants subject to the Kasner constraints

$$a+b+c=a^2+b^2+c^2=1 \ . \tag{15}$$

The identity (8), applied to the timelike Killing field and a cylindrical slice of unit coordinate length, yields

$$\int_{\Delta z=1}(T^a_a-T^t_t)(-g)^{1/2}d^3x=\tfrac{1}{2}kc \ . \tag{16}$$

Thus, $\frac{1}{2}kc$ may be termed the "Tolman" or "gravitational" mass of the cylinder per unit (coordinate) length. This also measures the force required to hold a test mass stationary in the field (14), given by the absolute derivative

$$-\left.\frac{\delta^2\rho}{\delta s^2}\right|_{\rho=\rm const}=\frac{\partial\ln(-g_{tt})^{1/2}}{\partial\rho}=-K^t_t=\frac{c}{\rho} \ .$$

Applying (8) to $\xi^{\alpha}_{(z)}$ and $\xi^{\alpha}_{(t)}$ gives

$$\int_{\Delta z=1}(T^z_z-T^t_t)(-g)^{1/2}d^3x=\tfrac{1}{4}k\lim_{\rho\to\infty}\rho(K^z_z-K^t_t) =\tfrac{1}{2}k(c-b) \ . \tag{17}$$

From (16), (17), and (15)

$$\int_{\Delta z=1}(T^\rho_\rho+T^\phi_\phi)(-g)^{1/2}d^3x=\tfrac{1}{2}kb \ , \tag{18}$$

$$\int_{\Delta z=1}[2(T^\rho_\rho-T^\phi_\phi)+T^z_z-T^t_t](-g)^{1/2}d^3x=\tfrac{1}{2}k(1-a) \ . \tag{19}$$

In the theory of time-dependent cylindrical fields, expression (19) is sometimes referred to as the "C-energy" per unit length.[14]

Within the general class of cylindrical fields, the *string* as a "topologically trapped region of false vacuum," is characterized, assuming that it is not carrying a current, by invariance under boosts in the (z,t) plane. The exterior vacuum geometry possesses this property if and only if $b=c$, which means

$$\text{either } a=1,\ \ b=c=0 \ \text{ or } \ a=-\tfrac{1}{3},\ \ b=c=\tfrac{2}{3} \ . \tag{20}$$

The first alternative implies that the exterior geometry is locally flat (i.e., the Riemann curvature vanishes), with an angular conical deficit given by

$$\Delta\phi=2\pi(1-k) \ . \tag{21}$$

The second alternative cannot represent a "stringlike" object in an expansive environment because of the anomalous behavior of azimuthal circles, and we follow conventional practice in disregarding it.

The condition necessary and sufficient for a cylindrical vacuum exterior field to be "stringlike" can then be stated in several equivalent forms: (i) The exterior geometry is boost invariant (assuming $b\neq\frac{2}{3}$); (ii) the exterior geometry is conical; (iii) the gravitational or Tolman mass [given by (16)] vanishes and $a\neq 0$; (iv) the "C-energy" [given by (19)] vanishes; (v) the integal (17) vanishes (assuming $b\neq\frac{2}{3}$); (vi) the integral (18) vanishes and $a\neq 0$.

In the static cylindrical models for the interior structure of current free strings with Abelian gauge fields, only the azimuthal components $A_\phi(\rho)$ of the (one or more) vector potentials cannot be gauged to zero. Structures with this property are clearly boost invariant, so that vanishing of (17) follows trivially from the *local* form $T^z_z=T^t_t$. However, $T^\rho_\rho+T^\phi_\phi\neq 0$ in general: the integral condition (vi) specifies when it is possible to join the string to a surrounding vacuum, at least in an asymptotic sense. Its meaning is clarified by the equation

$$\partial_\rho[(-g)^{1/2}(K+3K^\phi_\phi)^{-1}T^\rho_\rho]=-\tfrac{1}{4}(-g)^{1/2}(T^\rho_\rho+T^\phi_\phi) \ , \tag{22}$$

where

$$K=K^a_a=-\tfrac{1}{2}\partial_\rho\ln(-g),\quad K^\phi_\phi=-\tfrac{1}{2}\partial_\rho\ln g_{\phi\phi} \ ,$$

and ρ is proper radial distance from the axis. Equation (22) follows from the cylindrical Einstein field equations in the case of boost invariance. Condition (vi) thus requires the radial stress to approach zero strongly enough at the boundary $\rho=\rho_0$ (finite or infinite) to make

$$\lim_{\rho\to\rho_0}[(-g)^{1/2}(K+3K^\phi_\phi)^{-1}T^\rho_\rho]=0 \ . \tag{23}$$

These results shed light on a paradox encountered in studying the vacuum polarization due to a cosmic string. If the string is idealized as a thin filament in a conical spacetime, vanishing of the local curvature implies that the renormalized vacuum stress tensor T^ν_μ is traceless for a conformal massless field. Conformal scale invariance further requires $T^\nu_\mu\propto\rho^{-4}$. Finally, boost invariance and the conservation identity imply $T^z_z=T^t_t, 3T^\rho_\rho+T^\phi_\phi=0$, leading to the diagonal structure

$$(T^\rho_\rho,T^\phi_\phi,T^z_z,T^t_t)=\gamma\hbar c(\Delta\phi/2\pi)\rho^{-4}(1,-3,1,1) \tag{24}$$

in which the dimensionless parameter γ is characteristic of the field and has been computed explicitly for various fields.[15] The paradox is that the vacuum stress energy (24) appears to imbue the string with a nonvanishing (in fact, negative) gravitational mass according to (16). Thus boost invariance appears to be broken by a boost-invariant effect. The resolution, from (23), is, of course, that (24) diverges so rapidly near the axis that $(-g)^{1/2}(K+3K^\phi_\phi)^{-1}T^\rho_\rho\propto\rho^{-2}$ fails to vanish when $\rho\to 0$, i.e., when one approaches the boundary of the filament from the outside. It is not permissible to ignore the strongly curved interior of the string [where vacuum stresses do *not* have the form (24)] when assessing the integrated effects of vacuum polarization.

The quantity of primary observational interest is the angular deficit of the conical exterior space and its relation to the internal structure. A convenient form of this relation, which holds for any boost-invariant string, is Garfinkle's formula[8] (derived originally in the context of a specific gauge model):

$$\mu_{\rm His} \equiv \Delta\phi/8\pi = \mu_* + \frac{1}{8\pi}\int\int \left[\frac{dV}{d\rho}\right]^2 d\rho\, R\, d\phi \qquad (25)$$

where

$$\mu_* \equiv \int\int (-T^t_t) d\rho\, R\, d\phi \;, \qquad (26)$$

and V,R are defined by (28) below.

A general proof of (25) follows at once from

$$8\pi R T^t_t = R'' + (RV')' + RV'^2 \;, \qquad (27)$$

which is the form of one of the gravitational field equations in the case of boost invariance, when the interior metric is reducible to

$$ds^2 = d\rho^2 + R^2(\rho)d\phi^2 + e^{2V(\rho)}(dz^2 - dt^2) \;. \qquad (28)$$

We obtain the desired result from (27) and

$$\Delta\phi/2\pi = 1 - R'(\infty) = -\int_0^\infty R''(\rho)d\rho,$$

$$V'(\infty) = V'(0) = 0 \;.$$

For weak sources ($G\mu/c^2 \ll 1$), the integrand in (25) is of second order, and μ_* is nearly the inertial mass $\mu_{\rm inert}$ per unit proper length given by

$$\mu_{\rm inert} e^{V(\infty)} = \int\int (-T^t_t) e^V d\rho\, R\, d\phi \qquad (29)$$

so that we recover the linearized result $\Delta\phi = 8\pi\mu$.

III. TIME-SYMMETRIC INITIAL-VALUE PROBLEM FOR A CIRCULAR LOOP OF STRING: GENERAL FORMULATION

The analysis of the previous section for an infinite straight string is able to give a good account of *local* properties, such as the conical angular deficit $\Delta\phi$, and their relation to internal structure, but entirely fails to capture global properties of a finite string, especially the gravitational mass. For a straight string, as we noted, the gravitational mass per unit length vanishes.

The ensuing sections present the results of a first attempt to come to grips with the gravitational properties of finite loops of string. We study a circular loop of string at a moment of time symmetry. This may be considered an idealized or approximate representation of a newborn circular loop just before it begins to collapse, or of an oscillating loop at a moment of maximal expansion. Our considerations are confined to a single spatial three-slice of spacetime. Since we have no access to past lightlike infinity, the clearcut formulation of a condition to exclude incoming gravitational radiation presents a basic difficulty, to be addressed in detail in Sec. IV.

Time-symmetric gravitational fields[16] are in general characterized by invariance of the geometry

$$g_{\mu\nu}(x)dx^\mu dx^\nu = g_{\mu\nu}(x')dx'^\mu dx'^\nu \qquad (30)$$

under the point transformation $x^\mu \to x'^\mu$, where (in suitably adapted coordinates) $x'^0 = -x^0$, $x'^a = x^a$ ($a = 1,2,3$).

On the three-space of time symmetry $x^0=0$, (30) implies

$$\partial_0 g_{ab} = \partial_0 g_{00} = 0, \quad g_{0a} = 0 \quad (x^0 = 0) \;. \qquad (31)$$

The ADM initial-value constraints $G^0_\mu = 8\pi T^0_\mu$ for this hypersurface therefore reduce to

$$^{(3)}R = -16\pi T^0_0, \quad T^0_a = 0 \quad (x^0=0) \;. \qquad (32)$$

We next impose the requirement that the field also be axially symmetric. The three-metric of the time-symmetric slice is then reducible to the standard (Weyl) form

$$ds^2 = e^{2(\nu-\lambda)}(d\rho^2 + dz^2) + \rho^2 e^{-2\lambda} d\phi^2 \quad (x^0=0) \qquad (33)$$

with λ and ν functions of ρ, z. The explicit form of (32) is now

$$\nabla^2\lambda - \tfrac{1}{2}[(\nabla\lambda)^2 + \Delta\nu] = 4\pi e^{2(\nu-\lambda)}(-T^0_0) \qquad (34)$$

with the notation

$$\nabla^2 = \partial^2_\rho + \rho^{-1}\partial_\rho + \partial^2_z, \quad \Delta \equiv \partial^2_\rho + \partial^2_z \;,$$

$$(\nabla\lambda)^2 = (\partial_\rho\lambda)^2 + (\partial_z\lambda)^2 \;.$$

Acceptable solutions of (34) for a bounded source must satisfy the condition of asymptotic flatness,

$$\nu = O(r^{-2}), \quad \lambda \approx -M/r, \quad r \equiv (\rho^2+z^2)^{1/2} \to \infty \;, \qquad (35)$$

which fixes the ADM mass M; and also the condition that the geometry (33) be free of conical singularities on the axis of symmetry $\rho=0$ ("elementary flatness"):

$$\nu = \partial_\rho \nu = 0 \quad (\rho = 0) \;. \qquad (36)$$

Particularizing further to the case of interest to us, we assume that the material source is a circular ring $\rho=a$, $z=0$, momentarily at rest at time $x^0=0$. At this point it is convenient to pass from the coordinates ρ,z to toroidal coordinates σ,ψ defined by

$$\rho = aN^{-2}\sinh\sigma, \quad z = aN^{-2}\sin\psi$$

with

$$N(\sigma,\psi) \equiv (\cosh\sigma - \cos\psi)^{1/2} \quad (0 \le \sigma \le \infty, \; -\pi < \psi \le \pi) \;.$$

so that

$$r = (\rho^2+z^2)^{1/2} = aN^{-1}(\cosh\sigma + \cos\psi)^{1/2} \;. \qquad (37)$$

The metric of the Euclidean background associated with the cylindrical coordinates ρ, z, ϕ then becomes

$$(ds^2)_{\rm Eucl} = d\rho^2 + dz^2 + \rho^2 d\phi^2$$

$$= a^2 N^{-4}(d\sigma^2 + d\psi^2 + \sinh^2\sigma\, d\phi^2) \;. \qquad (38)$$

The geometrical meaning of the coordinates σ,ψ is depicted in Fig. 1. We note that a surface of constant σ is a torus with circular cross sections of radius a cschσ; the central axis of the tube forms a circle of radius $\rho = a$ cothσ in the plane $z=0$. The ring $\rho=a$, $z=0$ corresponds to the limiting torus $\sigma=\infty$.

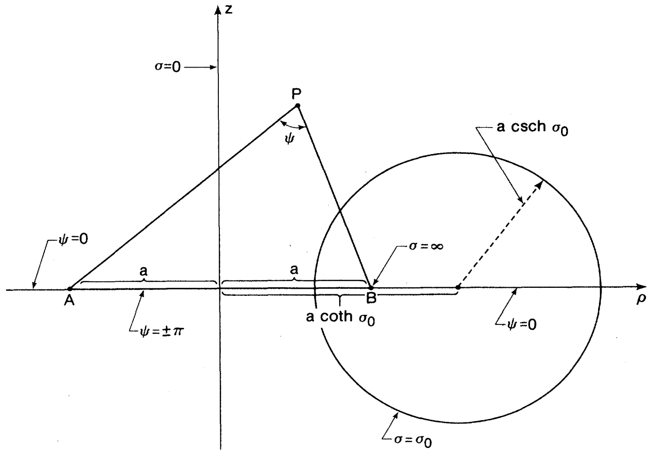

FIG. 1. Toroidal coordinates σ,ψ,ϕ. The figure shows the extension of a half-plane ϕ=const in Euclidean space. A and B are the points at which the equatorial circle $\rho=a$ intersects this plane. For any point $P(\sigma,\psi,\phi)$, $\sigma=\ln(AP/PB)$, and $\psi=\pm$(angle APB), with sign equal to the sign of z.

To ensure that the material source on this ring is "stringlike," we must impose the condition that $\sigma=\infty$ is a conical singularity of the physical metric (33), i.e.,

$$ds^2=a^2N^{-4}e^{-2\lambda}[e^{2\nu}(d\sigma^2+d\psi^2)+\sinh^2\sigma\, d\phi^2] , \quad (39)$$

From the asymptotic form of this metric as $\sigma\to\infty$, one sees readily that $\sigma=\infty$ is indeed a conical singularity with angular deficit

$$\Delta\phi=2\pi-\lim_{\sigma\to\infty} e^{\nu-\sigma}\Big/\int_\sigma^\infty e^{\nu(\sigma',\psi)-\sigma'}d\sigma'$$

provided $\lambda(\sigma,\psi)$ remains regular and bounded as $\sigma\to\infty$, which implies that it approaches a constant value

$$\lambda(\infty,\psi)=\lambda_0 ; \quad (40)$$

and, further, that $\nu(\sigma,\psi)$ has an asymptotically linear form

$$\nu\approx 4\alpha\sigma+\text{const}\approx -4\alpha\ln(p/2a)+\text{const} \quad (\sigma\to\infty) \quad (41)$$

with the definition

$$\alpha\equiv\Delta\phi/8\pi \quad (42)$$

and where $p\equiv[(\rho-a)^2+z^2]^{1/2}$ represents Euclidean distance from the ring.

Using these boundary conditions, it is easy to derive an explicit formal expression for the ADM mass M of the string. Integrate (34) over the vacuum space exterior to a narrow tube $\sigma=\sigma_1$ enclosing the string, using as "volume" element the Euclidean expression $d\tau\equiv d\rho\, dz\, \rho\, d\phi$. Write $\Delta\nu=\nabla^2\nu-\rho^{-1}\partial_\rho\nu$ and observe that the second term integrates to give pure boundary contributions which vanish in the limit $\sigma_1\to\infty$, since the divergent part of ν becomes constant on $\sigma=\sigma_1$ in this limit by virtue of (41). Accordingly,

$$\lim_{\sigma_1\to\infty}\int_{\sigma<\sigma_1}(\Delta\nu)d\tau=16\pi^2 a\alpha . \quad (43)$$

The integral of (34), with the boundary conditions (41) and (35), therefore, give the ADM mass

$$M=2\pi a_{\text{phys}}\alpha e^{\lambda_0}+(8\pi)^{-1}\int(\nabla\lambda)^2 d\tau , \quad (44)$$

where

$$2\pi a_{\text{phys}}=2\pi a e^{-\lambda_0} \quad (45)$$

is the physical circumference of the string loop.

We note, incidentally, that on the singular source $\sigma=\infty$, $\Delta\nu$ is a two-dimensional delta function which must integrate to a value equal and opposite to (43) by virtue of (35); thus,

$$\Delta\nu|_{p\approx 0}=-4\alpha\delta(p)/p . \quad (46)$$

According to (34), this implies that the ring source has energy density

$$-T_0^0=\alpha e^{2(\lambda_0-\nu)}\frac{\delta(p)}{2\pi p}=\frac{\alpha}{1-4\alpha}\frac{\delta(p_{\text{phys}})}{2\pi p_{\text{phys}}} , \quad (47)$$

where p_{phys} measures geodesic distance from the string in terms of the physical metric (39).

We conclude this section with some remarks of a general and heuristic character. It is obvious from (44) that for *weak* fields (i.e., for small values of the angular deficit $8\pi\alpha$, and λ everywhere small), the gravitational mass of the string per unit length is well approximated by α and, according to (47), by its inertial mass per unit length. This validates what is generally assumed in estimating from linearized theory the gravitational radiation emitted by an oscillating loop.

However, when 4α becomes comparable with unity, or if λ becomes of order unity anywhere, the first-order estimate

$$M\approx M_{\text{His}} \quad (4\alpha\ll 1), \quad M_{\text{His}}\equiv(2\pi a_{\text{phys}})\alpha , \quad (48)$$

is modified by the appearance in (44) of the function $\lambda(\rho,z)$, which may be considered, in view of (35), as the "Coulomb potential" of the gravitational field. All material forms of energy are direct sources for λ via (34). Superposed upon this is an indirect coupling to "nonmaterial sources": gravitational-wave packets (see Brill[16] and Sec. V) and conical geometries [see (46)] may be considered to act directly as sources for the "auxiliary potential" $\nu(\rho,z)$—they form islands where $\Delta\nu$ is locally negative; $\Delta\nu$ then reacts secondarily on the field $\psi\equiv e^{-(1/2)\lambda}$ through (34), which in matter-free regions takes the Schrödinger form

$$\nabla^2\psi+(\tfrac{1}{4}\Delta\nu)\psi=0 . \quad (49)$$

It is instructive to compare this situation with the Newtonian analogue of (44). Since there are no gravitational waves and only a single potential ϕ in Newtonian theory, the ring source now affects ϕ only through its inertial mass M_0. Although ϕ is the analogue of λ for weak fields in general, this analogy breaks down near the string, where ϕ (unlike λ) diverges logarithmically. We shall suppose the Newtonian ring replaced by a tube of nonzero thickness, so that the potential ϕ_0 on the tube stays bounded. The Newtonian potential energy is

$$\Omega = -(8\pi G)^{-1}\int (\nabla\phi)^2 d\tau = \tfrac{1}{2}G\int \rho\phi\, d\tau = \tfrac{1}{2}GM_0\phi_0 \ .$$

The total mass energy can thus be expressed in a form that closely resembles (44) (we set $G=c=1$):

$$M = M_0 + \Omega = M_0(1+\phi_0) + (8\pi)^{-1}\int (\nabla\phi)^2 d\tau \ .$$

The Newtonian result $M = M_0 + \Omega < M_0$ leads to the conjecture that the first approximation (48) to the gravitational mass of a string loop may be an overestimate in the general-relativistic case also, at least when the initial hypersurface $x^0=0$ has been swept clear of free gravitational waves. In the following sections we shall examine a wide-ranging class of exact solutions which lend support to this conjecture. At the same time, they indicate that (48) should remain a very tolerable estimate, good to within a factor 2 or so, even for large angular deficits, for the field configurations that could reasonably be expected to develop around a real oscillating loop.

IV. CIRCULAR STRING LOOP: A CLASS OF PARTICULAR SOLUTIONS

The procedure for finding the gravitational field of a circular loop of string at a moment of time symmetry has been formulated as the following boundary-value problem: solve the vacuum equation for the field outside the string,

$$\nabla^2\lambda - \tfrac{1}{2}(\nabla\lambda)^2 = \tfrac{1}{2}\Delta\nu \quad (\sigma \neq \infty) \ , \qquad (50)$$

subject to the boundary conditions (35), (36), (40), and (41).

Since (50) is a single equation for two unknown potentials $\lambda(\rho,z)$ and $\nu(\rho,z)$, the solution of this problem is far from unique. This reflects the circumstance that boundary conditions on one spatial slice cannot exclude the presence there of arbitrary amounts of free gravitational radiation—radiation that might have been emitted by the loop in its past history or simply come in from past null infinity.

There is a conceivable way out of this impasse. That it may be possible to control free gravitational radiation by imposing restrictions on $\nu(\rho,z)$ is suggested by the following result, essentially due to Brill.[16] A space that is everywhere clear of material energy contains gravitational radiation (i.e., has nonvanishing ADM mass) at a moment of time symmetry if and only if $\Delta\nu$ is not identically zero. This result is an easy consequence of (50) or (49), taking the boundary conditions (35) and (36) into account.

At first sight, this suggests that $\Delta\nu=0$ might be quite generally deployable as a condition to supplement (50) and as a prescription for excluding free gravitational waves from a hypersurface of time symmetry, which works at least in the vacuum exterior region even when material sources are present. When the source is a string, however, (43) shows that the condition is not enforceable in quite this form: the region surrounding a string must function as a positive source for $\Delta\nu$, even in the absence of waves, to quench the negative effect (46) of the conical vertex by something akin to isostatic adjustment.

These considerations lead us to the following schematic picture for the behavior of $\Delta\nu$ in a space containing a momentarily static string loop and no free gravitational waves. On the string itself, $\Delta\nu$ is given by (46). In the immediate surroundings, and extending over a distance of the order of the loop radius, is a "near-zone" where the geometry is locally flat and conical, ν is given by (41), and $\Delta\nu=0$. Further out, one passes through a transition zone of comparable thickness in which the geometry evolves from its conical near form to the Schwarzschild-type form characteristic of the far field of a compact source. The principal contribution to the integral (43) must originate in the transition zone, where $\Delta\nu$ is positive. Beyond this, $\Delta\nu$ again vanishes, reflecting absence of gravitational radiation in the "far zone" extending out to infinity.

(While this represents our intuition concerning the behavior of the solution for a real string in a space initially swept as clear as possible of free gravitational waves, it is perhaps arguable that the class of models considered in this section, which are based on the idealization of an infinitely thin transition zone, do not entirely bear out this picture. The point is that the ADM mass of these models approaches its minimum value (zero) in the limit where the transition zone embraces the loop as tightly as possible [see (74) below]. This result is not easy to understand if one holds to the view that a tight-fitting transition zone can be created by focusing a toroidal gravitational shock wave inward upon the loop, since that would require injection of positive energy. This question needs further investigation.)

The detailed specification of initial data needed to sharpen this picture depends on aspects of past history not accessible to our approach. All we can do here is to analyze a class of idealized initial conditions that is hopefully broad enough to encompass the range of configurations that are physically relevant.

The principal goal is to determine how the ADM mass M, defined by the far field, depends on loop radius a_{phys} and angular deficit $\Delta\phi = 8\pi\alpha$, parameters defined by the near field. The transition zone, which is of secondary interest for this purpose, will be idealized as the thin wall of a torodial tube enclosing the string loop. Our hope is that this idealization adequately represents the real physics. It will be assumed that (50) continues to hold (in a distributional sense) within the tube wall, which therefore contains no material energy [cf. (34)]. For moderate angular deficits, the wall might be regarded as a place where an imploding gravitational shock wave and its time-reflected image, an exploding shock, collide at time $x^0=0$. Of course, conditions at $x^0=0$ by no means enforce this interpretation, and it would in any case fail under conditions so extreme that the collision leads to self-focusing singularities in the future and past of $t=0$. (Compare Sec. V.)

Mathematically, this idealization is encoded in the ansatz

$$\nu = \begin{cases} 4\alpha(\sigma-\sigma_0) & (\sigma > \sigma_0), \\ 0 & (\sigma < \sigma_0), \end{cases} \qquad (51)$$

which satisfies $\Delta\nu=0$ except on the tube wall $\sigma=\sigma_0$ and on the string $\sigma=\infty$, and trivially satisfies the boundary conditions (35), (36), and (41).

One may regard (51) as the solution of a two-dimensional electrostatic problem in the ρ,z plane. Since $\sigma=-\ln(BP/AP)$ (see Fig. 1), ν can be interpreted as the potential due to a charge 2α at B enclosed in the earthed circular conducting shell $\sigma=\sigma_0$ (the image charge is at A). Evaluation of the charge density induced on the shell gives

$$\Delta\nu=(4\alpha N^2/a^2)\delta(\sigma-\sigma_0) \quad (0\le\sigma<\infty) . \tag{52}$$

Of course, this value automatically satisfies the constraint (43), since that stems from the same boundary condition.

To obtain λ, it is convenient to recast (50) as an integral equation. Returning to three dimensions, we define

$$V=e^{-\lambda/2}-1 , \tag{53}$$

so that (5) takes the form

$$\nabla^2 V=-\tfrac{1}{4}(1+V)\Delta\nu . \tag{54}$$

With $\Delta\nu$ given by (52), and V regular for $\sigma=\infty$, (54) has the formal solution

$$V(\mathbf{r})=\frac{\alpha}{4\pi a}\int\frac{1+V(\mathbf{r}_0)}{|\mathbf{r}-\mathbf{r}_0|}N_0^2 dS_0 . \tag{55}$$

The integration is over the torus $\sigma=\sigma_0$, we abbreviate $N(\mathbf{r}_0)=(\cosh\sigma_0-\cos\psi_0)^{1/2}=N_0$ and "lengths" and "areas" refer to the Euclidean metric (38), so that

$$dS_0=a^2N_0^{-4}\sinh\sigma_0 d\psi_0 d\phi_0 . \tag{56}$$

The ADM mass M can now be read off from (55) and the asymptotic condition $V\approx\frac{1}{2}M/|\mathbf{r}|$ ($|\mathbf{r}|\to\infty$):

$$M=(\alpha/2\pi a)\int[1+V(\mathbf{r}_0)]N_0^2 dS_0 . \tag{57}$$

Expressed in terms of the Hiscock mass

$$M_{\rm His}=(2\pi a_{\rm phys})\alpha, \quad a_{\rm phys}=a[1+V(\sigma=\infty)]^2 , \tag{58}$$

this becomes

$$M/M_{\rm His}=[1+V(\sigma=\infty)]^{-2}(2\pi a)^{-2} \times\int[1+V(\mathbf{r}_0)]N_0^2 dS_0 . \tag{59}$$

We turn next to the explicit solution of the integral equation (55). The Green's function has the following expansion in toroidal harmonics:[17]

$$\frac{\pi a}{|\mathbf{r}-\mathbf{r}_0|}=N(\mathbf{r})N(\mathbf{r}_0)\sum_{m,n=0}^{\infty}c_{mn}\cos m(\phi-\phi_0)\cos n(\psi-\psi_0) \times P_{n-1/2}^m(\cosh\sigma_<) \times Q_{n-1/2}^m(\cosh\sigma_>) , \tag{60}$$

where $\sigma_<$ and $\sigma_>$ denote the lesser and greater of (σ,σ_0). The numerical coefficients are

$$c_{mn}=(-1)^m\frac{\Gamma(n-m+\frac{1}{2})}{\Gamma(n+m+\frac{1}{2})}\varepsilon_m\varepsilon_n, \quad \varepsilon_n=\begin{cases}1 & (n=0),\\ 2 & (n=1,2,\dots) .\end{cases} \tag{61}$$

The axial and equatorial symmetries of our problem imply that

$$V(\mathbf{r})=V(\sigma,\psi)=V(\sigma,-\psi) . \tag{62}$$

Hence only the $m=0$ term survives integration when (60) is inserted into (55). We obtain the Fourier expansion

$$V(\sigma,\psi)=(\alpha/2\pi)(\sinh\sigma_0) \times N\sum_{N=0}^{\infty}\varepsilon_n a_n P_{n-1/2}(\cosh\sigma_<) \times Q_{n-1/2}(\cosh\sigma_>)\cos n\psi , \tag{63}$$

where

$$a_n=\int_{-\pi}^{\pi}[1+V(\sigma_0,\psi_0)]N_0^{-1}\cos n\psi_0 d\psi_0 . \tag{64}$$

The first part of (64) is a standard integral:

$$\int_{-\pi}^{\pi}N_0^{-1}\cos n\psi_0 d\psi_0=2^{3/2}Q_{n-1/2}(\cosh\sigma_0) . \tag{65}$$

In the second part, if one replaces $V(\sigma_0,\psi_0)$ by its Fourier expansion (63), only the nth term survives integration,

$$\int_{-\pi}^{\pi}V(\sigma_0,\psi_0)N_0^{-1}\cos n\psi_0 d\psi_0=a_n b_n\alpha , \tag{66}$$

where

$$b_n=(\sinh\sigma_0)P_{n-1/2}(\cosh\sigma_0)Q_{n-1/2}(\cosh\sigma_0) . \tag{67}$$

From (64)–(66) one finds explicitly

$$a_n=2^{3/2}Q_{n-1/2}(\cosh\sigma_0)/(1-\alpha b_n) \tag{68}$$

so the solution can finally be written

$$V(\sigma,\psi)=\frac{2^{1/2}}{\pi}(\cosh\sigma-\cos\psi)^{1/2} \times\sum_{n=0}^{\infty}\frac{\varepsilon_n\alpha b_n}{1-\alpha b_n}\frac{P_{n-1/2}(\cosh\sigma_<)}{P_{n-1/2}(\cosh\sigma_0)} \times Q_{n-1/2}(\cosh\sigma_>)\cos n\psi . \tag{69}$$

Equations (53), (69), and (67) explicitly determine λ in the metric (39). We have thus obtained the complete family of solutions for which ν has the special form (51).

It is now straightforward to infer the explicit form of (59). From (65),

$$\lim_{\sigma\to\infty}NQ_{n-1/2}(\cosh\sigma)=2^{-1/2}\pi\delta_{n0} , \tag{70}$$

so

$$1+V(\sigma=\infty)=(1-\alpha b_0)^{-1} . \tag{71}$$

The value of the integral in (59),

$$\int [(1+V(\mathbf{r}_0)]N_0^2 dS_0 = 2\pi a^2 \sinh\sigma_0 \int_{-\pi}^{\pi} [1+V(\mathbf{r}_0)]N_0^{-2} d\psi_0 \ , \quad (72)$$

follows at once by term-by-term integration of (69). The final result is

$$M/M_{\rm His} = (1-\alpha b_0)^2 \times \left\{ 1 + 2\pi^{-2}\sinh\sigma_0 \sum_{n=0}^{\infty} [\alpha b_n/(1-\alpha b_n)] \times [Q_{n-1/2}(\cosh\sigma_0)]^2 \right\} . \quad (73)$$

V. PROPERTIES OF THE SOLUTIONS

The unwieldiness of the solutions (69) has deterred complete exploration of their geometry. However, some broad features can be delineated, and are summarized below and in Table I.

Solutions exist if all denominators $(1-\alpha b_n)$ in (69) are nonzero. Because b_n decreases with n, it is sufficient that $\alpha b_0 < 1$. Now, b_0 is conveniently expressed in terms of complete elliptic integrals, in the form

$$b_0 = (4/\pi)\tanh\tfrac{1}{2}\sigma_0 K(\tanh\tfrac{1}{2}\sigma_0)K(\operatorname{sech}\tfrac{1}{2}\sigma_0)$$

with the asymptotic behavior

$$b_0 \approx \begin{cases} \sigma_0 \ln(8/\sigma_0) & (\sigma_0 \to 0), \\ \sigma_0 + \ln 4 & (\sigma_0 \to \infty) \ , \end{cases}$$

so this constraint limits σ_0 to the range

$$\sigma_0 < \sigma_{\rm crit}(\alpha) \approx \begin{cases} 8\pi/\Delta\phi & (\Delta\phi \to 0) \ , \\ 2.625 & (\Delta\phi = 2\pi) \ . \end{cases} \quad (74)$$

In particular, a solution exists for all angular deficits if the torus $\sigma=\sigma_0$ on which the defect is healed has $\sigma_0 < 2.625$. As a rough guide to the relation between σ_0 and the physical dimensions of the healing torus, we mention that the ratio of its greater and lesser equatorial circumferences (equal to $\coth^2\frac{1}{2}\sigma_0$ in flat space) declines steadily from 16.7 to 8.2 as $\Delta\phi$ varies from 0 to 2π with σ_0 held at 0.5; for $\sigma_0=1$, the range is from 4.7 to 2.25, for $\sigma_0=2.5$, from 1.39 to 1.012. In all cases, the circumference of the string loop is not far from the geometric mean of these equatorial circumferences.

A measure of the degree of external intervention needed to force sudden extinction of the conical defect on the torus $\sigma=\sigma_0$ can be obtained by imagining this torus replaced by a *static* material wall, and deducing the surface stresses S_a^b. [The surface energy density vanishes, because (50) holds distributionally.] A straightforward calculation shows that the stresses are pure shear, and given by

$$S_\psi^\psi = -S_\phi^\phi = (\alpha/4\pi a)[1+V(\sigma_0,\psi)]^{-2}(\cosh\sigma_0 - \cos\psi) \ . \quad (75)$$

The stresses are maximal along the inner equatorial circumference $\psi=\pm\pi$, and the dimensionless number $a_{\rm phys}(S_\psi^\psi)_{\rm max}$ (listed in Table I) is a measure of the extent to which the idealization of instant healing departs from the natural order. It is asymptotic to $\alpha e^{\sigma_0}/8\pi$ for large σ_0. It would seem natural to suppose that the wall actually represents an instantaneous collision of ingoing and outgoing gravitational waves. After the initial situation, which is all we can study, one would have one gravitational shock traveling in toward the core of the string and another expanding away from the string.

The tabulated values of the expression (73) for $M/M_{\rm His}$ show that it declines gradually and steadily from unity as $\Delta\phi$ increases from zero for each fixed σ_0. However, for moderate values of σ_0, the estimate $M/M_{\rm His}\sim 1$ remains good within a factor of 2 even for angular deficits as large as π.

For fixed angular deficit $\Delta\phi$, $M/M_{\rm His}$ declines steadily from unity to zero as σ_0 increases from 0 to its maximum possible value (74). We do not understand the nature of this zero-mass limit, whether it is due to the crudeness of our model or could represent a genuine feature of more realistic models.

The final column of the table lists values of $\mu \equiv M/2\pi a_{\rm phys}$, the gravitational mass per unit proper length. This rises with angular deficit, at first like $\mu \approx \Delta\phi/8\pi$. The rise slackens off, but remains monotonic, for each fixed σ_0 smaller than 1. If $\sigma_0 > 1$, μ reaches a maximum for a value of $\Delta\phi$ depending on σ_0, and descending steadily from 2π as $\sigma_0 - 1$ increases. For reasons expanded upon in Sec. VII, we believe this behavior may signal formation of an apparent horizon at or near the critical value of $\Delta\phi$. Although the complexity of (69) has hindered us from checking this in detail, it is easy to

TABLE I. Properties of circular string loops with "instant healing" on the torus $\sigma=\sigma_0$. For each of three selected values of σ_0, the columns list as functions of angular deficit $\Delta\phi$ (i) the ratio $M/M_{\rm His}$ given by (73); (ii) the gravitational mass μ per unit proper length (given by $\Delta\phi/8\pi$ times the entries in the previous column); and (iii) the dimensionless maximal effective stress $a_{\rm phys}(S_\psi^\psi)_{\psi=\pi}$ given by (75) and (58).

	$\sigma_0=0.5$			$\sigma_0=1$			$\sigma_0=2.5$		
$\Delta\phi/2\pi$	$M/M_{\rm His}$	μ	$a_{\rm phys}S_{\rm max}$	$M/M_{\rm His}$	μ	$a_{\rm phys}S_{\rm max}$	$M/M_{\rm His}$	μ	$a_{\rm phys}S_{\rm max}$
0.2	0.9201	0.046 00	0.006 84	0.8853	0.044 27	0.007 50	0.806 0	0.040 30	0.017 9
0.4	0.8412	0.084 12	0.010 9	0.7718	0.077 18	0.010 8	0.612 1	0.061 21	0.020 1
0.6	0.7635	0.114 5	0.012 8	0.6594	0.098 90	0.011 1	0.418 5	0.062 78	0.013 7
0.8	0.6870	0.137 4	0.013 2	0.5482	0.109 65	0.009 67	0.225 1	0.045 03	0.005 12
1.0	0.6119	0.153 0	0.012 5	0.4385	0.109 62	0.007 28	0.032 02	0.008 006	0.000 125

show that an apparent horizon certainly exists when σ_0 is large and close to the upper limit (74)—more precisely, when

$$1 \ll \sigma_0 \ll (1-\alpha b_0)^{-1} . \tag{76}$$

To show this, we note that for $\sigma_0 \gg 1$ and $\alpha b_0 < 1$ the series (69) is dominated by its first term,

$$V(\sigma,\psi) \approx \frac{2^{1/2}}{\pi} N \frac{\alpha b_0}{1-\alpha b_0} \frac{Q_{-1/2}(\cosh\sigma_0)}{P_{-1/2}(\cosh\sigma_0)} P_{-1/2}(\cosh\sigma) \quad (\sigma < \sigma_0 \gg 1) , \tag{77}$$

because higher terms drop off exponentially with σ_0. At large radii $(\sigma^2+\psi^2 \to 0)$ we then find, recalling (37) and the identity

$$\tfrac{1}{2}\pi Q_{-1/2}(\cosh\sigma)/P_{-1/2}(\cosh\sigma) = K(\tanh\tfrac{1}{2}\sigma)/K(\operatorname{sech}\tfrac{1}{2}\sigma) , \tag{78}$$

where K is the complete elliptic integral, that V has the asymptotic form

$$V \approx \tfrac{1}{2}M/r, \quad ds^2 \approx (1+\tfrac{1}{2}M/r)^4 (dr^2 + r^2 d\Omega^2) \quad (r \gg a) , \tag{79}$$

$$M \approx 2\pi a \alpha b_0 (1-\alpha b_0)^{-1} (\sigma_0 + \ln 4)^{-1} \quad (\sigma_0 \gg 1, \alpha b_0 < 1) . \tag{80}$$

This clearly reveals a "throat" at $r = \frac{1}{2}M$, provided this radius falls within the domain of validity of (79), i.e., provided $M \gg a$. According to (80), this will be so if $(1-\alpha b_0)^{-1} \gg \sigma_0$, thus confirming the presence of a horizon under the condition (76).

However, as σ_0 becomes of order $(1-\alpha b_0)^{-1}$, we have not been able to find evidence of an apparent horizon. At this stage, $M/M_{\rm His}$ is still very small, and of order $\Delta\phi$ for small $\Delta\phi$. Although the model has now become physically ridiculous, as the ratio of the healing radius to the loop radius is of order $e^{-\sigma_0} \approx e^{-1/\Delta\phi}$ which for physically reasonable $\Delta\phi \lesssim 10^{-4}$ is well within the core radius of the matter making up the loop, the result still poses a puzzle we have been unable to resolve. How, apparently by merely changing the amount of gravitational radiation in the initial state, is it possible to reduce the ratio of gravitational mass measured at infinity to the local (Hiscock) mass by such a large factor, especially in the limit of very small $\Delta\phi$? Is this a true violation of our intuitive conviction that the condition of minimal radiation implies a smooth transition region of size about the radius of the loop, or is it, as we suspect, a consequence of the artificial model we have chosen?

VI. CONFORMALLY FLAT TIME-SYMMETRIC CIRCULAR LOOP

The solutions considered in the two preceding sections had the contrived feature that healing of the conical defect of the space around the string happens abruptly on a toroidal surface enclosing the loop. In this section we shall obtain a solution for which the healing is smooth. By its manner of construction and simplicity of form, this has claims to be considered the most natural way to extend the field of a straight string to the case where the shape is circular.

The basic idea of our procedure is easy to explain. For a straight string, as is well known, the conical exterior geometry can be constructed by removing from Euclidean three-space a wedge extending from the string, and then identifying the exposed edges. For a circular loop (supposed imbedded in Euclidean space) we employ a similar method. We remove a pair of spherical caps, mirror images of each other, spanning the loop (Fig. 2). Although the exposed spherical faces have the same intrinsic geometry, their extrinsic curvatures have opposite sign; if we try to glue them together, we cannot avoid interposing a (surface layer) discontinuity. However, it is possible to apply a conformal transformation which (i) reduces to the identity at infinity and (ii) flattens the extrinsic curvatures of the faces, in effect reducing them to a pair of disks spanning the loop, which now can be identified. (Because the conformal transformation provides only a single disposable function, the only extrinsic curvatures that can be equalized in this way are those determined by a single function, which then must necessarily be the mean curvature.[18] Thus, our conformal trick cannot be extended to nonspherical caps and noncircular loops.) The conformal factor is uniquely defined by the boundary conditions (i) and (ii), and the requirement that the conformally flat space be empty: this reduces to the single ADM constraint ${}^{(3)}R = 0$ at a moment of time symmetry.

To implement this program, we fix an arbitrary angular deficit $\Delta\phi = 8\pi\alpha < 2\pi$. Starting with the flat toroidal metric (38), we restrict ψ to the range $-(\pi - \frac{1}{2}\Delta\phi) < \psi < \pi - \frac{1}{2}\Delta\phi$, identify end points of this interval, then apply a conformal factor $(\omega N)^4$. This gives the physical metric

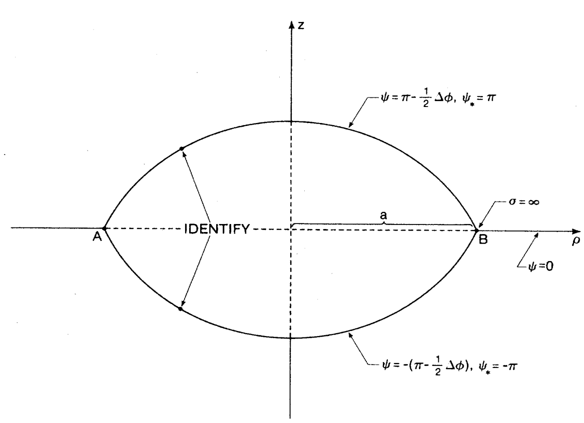

FIG. 2. The conformally flat geometry of the circular string loop, located at $\rho = a$ in this Euclidean map, is constructed by conformally squeezing the two spherical caps $\psi = \pm(\pi - \frac{1}{2}\Delta\phi)$ to become, in effect, equatorial disks spanning the loop, and topologically identifying them.

$$ds^2=(\omega N)^4(ds^2)_{\text{flat}}=\omega^4 ds_*^2 \; , \tag{81}$$

where

$$ds_*^2=a^2(d\sigma^2+d\psi^2+\sinh^2\sigma\, d\phi^2) \tag{82}$$

represents locally the geometry of a three-cylinder whose sections $\psi=$const are pseudospheres with negative curvature ${}^{(2)}R=-2/a^2$.

Excision of the wedge $|\pi-\psi|\le\frac{1}{2}\Delta\phi$ has no effect on the geometry for small ψ, and hence on the asymptotically flat form of the metric, since spatial infinity corresponds to

$$(\sigma^2+\psi^2)^{1/2}\approx 2a/r\to 0 \tag{83}$$

by (37). Also, since it merely excises the segment $|z|\le a\tan\frac{1}{2}\Delta\phi$, it introduces no conical singularity on the axis of symmetry $\sigma=0$. Accordingly, defining

$$\psi_*=\eta\psi, \quad \eta=2\pi/(2\pi-\Delta\phi), \quad -\pi<\psi_*<\pi \; , \tag{84}$$

if $\omega(\sigma,\psi_*)$ is chosen as a smooth function, periodic and even in ψ_*, and such that the function ωN is (i) regular for $\sigma=0$, (ii) asymptotically constant at spatial infinity, and (iii) unity on the string loop $\sigma=\infty$, then (81) will represent an asymptotically flat space that is conical in the immediate vicinity of the loop, with angular deficit $\Delta\phi$ (Fig. 2).

In the conformally transformed geometry (82), the faces $\psi_*=\pm\pi$ that we are gluing together already have vanishing extrinsic curvature. It therefore remains only to adjust the further conformal factor ω so as to evacuate the physical three-geometry (81):

$${}^{(3)}R=\omega^{-4}(R_*-8\omega^{-1}\nabla_*^2\omega)=0 \; , \tag{85}$$

where starred quantities of course refer to the three-metric (82):

$$R_*=-2a^{-2},$$
$$a^2\nabla_*^2\omega=(\sinh\sigma)^{-1}\partial_\sigma(\sinh\sigma\,\partial_\sigma\omega)+\eta^2\partial_{\psi_*}^2\omega \; .$$

Equation (85) is a linear differential equation for ω, of Legendre's form. The general solution periodic and even in ψ_* and such that ωN is bounded as $\sigma\to\infty$ has the form

$$\omega(\sigma,\psi_*)=\sum_{n=0}^{\infty} a_n\varepsilon_n Q_{n\eta-1/2}(\cosh\sigma)\cos n\psi_* \; , \tag{86}$$

with ε_n as in (61). By comparing this with the expansion

$$N^{-1}=(\sqrt{2}/\pi)\sum_{n=0}^{\infty}\varepsilon_n Q_{n-1/2}(\cosh\sigma)\cos n\psi \; , \tag{87}$$

which follows from (65), one sees that the choice

$$\omega(\sigma,\psi_*)=(\sqrt{2}/\pi)\sum_{n=0}^{\infty}\varepsilon_n Q_{n\eta-1/2}(\cosh\sigma)\cos n\psi_* \; , \tag{88}$$

satisfies the boundary conditions (i)–(iii).

Our solution (88) can be reduced to a more convenient integral form by deploying the identity[19]

$$Q_\mu(\cosh\sigma)=2^{-1/2}\int_0^\infty(\cosh\tau-\cosh\sigma)^{-1/2}\times e^{-(\mu+1/2)\tau}d\tau \; . \tag{89}$$

After inserting this in (88), summation of the series is elementary:

$$\omega(\sigma,\psi,\eta)=\frac{1}{\pi}\int_\sigma^\infty\frac{d\tau}{(\cosh\tau-\cosh\sigma)^{1/2}}\frac{\sinh\eta\tau}{\cosh\eta\tau-\cos\eta\psi} \; . \tag{90}$$

Together with (81) and (82), this gives the final form of the metric for a conformally flat string loop with arbitrary angular deficit $\Delta\phi=2\pi(1-\eta^{-1})$.

Because N^{-1} is given by the integral (90) with η replaced by 1, it is easy to verify that the expression

$$I(\eta)\equiv 2^{-1/2}\pi(\omega\eta-N^{-1})|_{\sigma=\psi=0} \tag{91}$$

can be reduced to the integral

$$I(\eta)=\int_0^\infty \operatorname{csch}x(\eta\coth\eta x-\coth x)dx \; . \tag{92}$$

Thus, recalling (83) and (81),

$$\eta\omega N\approx 1+(2a/\pi)I(\eta)(\rho^2+z^2)^{-1/2}\approx 1+\tfrac{1}{2}M/r_{\text{phys}} \quad (r\to\infty) \; ,$$

where

$$r_{\text{phys}}\approx(\omega N)^2(\rho^2+z^2)^{1/2}\approx\eta^{-2}r$$

is the physical asymptotic distance from the origin. This identifies the ADM mass as

$$M=(4a/\pi)\eta^{-2}I(\eta) \; . \tag{93}$$

From the boundary condition $\omega N\to 1$ as $\sigma\to\infty$ and (81) and (82) we find

$$a_{\text{phys}}=a, \quad M_{\text{His}}=2\pi a_{\text{phys}}\alpha=\tfrac{1}{2}\pi a(\eta-1)/\eta \tag{94}$$

so that

$$\frac{M}{M_{\text{His}}}=\frac{8}{\pi^2}\frac{1}{\eta(\eta-1)}I(\eta) \; . \tag{95}$$

When $\eta\to 1$, this ratio tends to unity as it should.

If η is an integer or half-integer, the integral (90) can be evaluated in terms of elementary functions. The elegant way to do this (kindly pointed out to us by Teshima) makes use of the partial fractions expansion

$$\frac{\sinh n\tau}{\sinh\tau(\cosh n\tau-\cos n\psi)}=\frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{\cosh\tau-\cos(\psi+2k\pi/n)}$$

and leads, for integer values $\eta=n$, to

$$\omega=(1/n)\sum_{k=0}^{n-1}[\cosh\sigma-\cos(\psi+2k\pi/n)]^{-1/2} \quad (\eta=n) \; . \tag{96}$$

For half-integer values, Teshima has shown similarly that (96) generalizes to

$$\omega=\frac{4}{\pi m}\sum_{k=0}^{m-1}[\cosh\sigma-\cos(\psi+4k\pi/m)]^{-1/2}\times\arctan\left[\frac{\cosh\frac{1}{2}\sigma+\cos\frac{1}{2}(\psi+4k\pi/m)}{\cosh\frac{1}{2}\sigma-\cos\frac{1}{2}(\psi+4k\pi/m)}\right] \quad (\eta=\tfrac{1}{2}m) . \tag{97}$$

The integral (92) is reducible to elementary form for rational values of η. Here we merely record the result for integer values:

$$\frac{M}{M_{\rm His}}=\frac{4}{\pi n(n-1)}\sum_{k=1}^{n-1}\csc\frac{k\pi}{n} \quad (\eta=n=2,3,\ldots) . \tag{98}$$

VII. CONFORMALLY FLAT LOOP: NATURE OF THE SOLUTIONS

Numerical results derived from the preceding formulas are summarized in Table II.

The first point to note is that the ratio of gravitational mass to Hiscock mass remains quite close to unity for small angular deficits $\Delta\phi$, but declines steadily, reaching $2/\pi$ for $\Delta\phi=\pi$ and falling off as $(4/\pi)\ln\eta/\eta$ as $\eta\equiv(1-\Delta\phi/2\pi)^{-1}\to\infty$. At the same time the gravitational mass per unit length,

$$\mu_{\rm grav}=M/2\pi a_{\rm phys}=(2\pi^2)\eta^{-2}I(\eta) , \tag{99}$$

which has the "conventional" value $\Delta\phi/8\pi$ for small $\Delta\phi$, rises at first, to a maximum of 0.0827 for $\eta=2.55$ (i.e., $\Delta\phi=1.216\pi$), then falls steadily to zero. Since we expect Hiscock mass to be positively correlated with inertial mass (Sec. II), we are driven to conclude that adding more mass to the string leads, after a certain stage, to reduction of its gravitational mass.

Similar behavior has long been familiar in spherical systems with strong gravitational binding. The closest analogy involves a ball of uniform density ρ_0 at a moment of time symmetry.[20] The three-metric is

$$ds^2=[1-2M(r)/r]^{-1}dr^2+r^2d\Omega^2,$$
$$M(r)=\tfrac{4}{3}\pi r^3\rho_0 \quad \text{(interior)} .$$

Setting $r=a\sin\chi$, with boundary at $\chi=\chi_0$, one recognizes the geometry as that of the cap $0\le\chi\le\chi_0$ of the three-sphere

$$ds^2=a^2(d\chi^2+\sin^2\chi\, d\Omega^2), \quad a^2=(\tfrac{8}{3}\pi\rho_0)^{-1} ,$$

immersed in a Schwarzschild exterior solution of appropriate mass. As χ_0 increases from 0 to π, the inertial mass

$$M_{\rm inert}=4\pi a^3\rho_0\int_0^{\chi_0}\sin^2\chi\, d\chi$$

rises steadily to a maximum $2\pi^2a^3\rho_0$. However, the gravitational mass

$$M=M(r=a\sin\chi_0)=\tfrac{4}{3}\pi a^3\rho_0\sin^3\chi_0$$

peaks at $\chi_0=\pi/2$, thereafter falling to zero. The explanation is, of course, that the gravitational energy that can be mined as work in lowering additional layers of density ρ_0 onto the surface comes to exceed their rest-mass energy once the inertial mass exceeds $\sim\rho_0^{-1/2}$, and the gravitational mass begins to decrease.

Since, for $\chi_0>\pi/2$, the circumferential radius r is decreasing outwards at the surface $\chi=\chi_0$, an imbedding diagram for the spherical three-geometry would show a bulbous form, with a minimal two-space, a Schwarzschild

TABLE II. Properties of circular string loop with conformally flat three-geometry. Successive columns list values of (i) $\eta=(1-\Delta\phi/2\pi)^{-1}$; (ii) $M/M_{\rm His}$ given by (95); (iii) gravitational mass μ per unit proper length; (iv) $\delta_{\max}$, given by (100) with A defined as the area of the maximal torus (note that this surface does not exist for $\eta\le2$, and that $\delta_{\max}$ is negative); (v) δ, given by (100) with A defined as the area of the apparent horizon (this exists only for $\eta\ge2$). For $\eta=2.55$ these areas have not been computed.

η	$M/M_{\rm His}$	μ	$-10^2\delta_{\max}$	$10^4\delta$
1	1	0		
1.1	0.9387	0.021 33		
1.2	0.8864	0.036 93		
1.3	0.8412	0.048 53		
1.4	0.8016	0.057 25		
1.5	0.7665	0.063 87		
2.0	0.6366	0.079 58		0
2.25	0.5901	0.081 96	0.066 30	0.5726
2.5	0.5514	0.082 71	0.249 9	1.0054
2.550	0.5444	0.082 725	?	?
2.75	0.5185	0.082 48	0.509 2	1.1300
3.0	0.4901	0.081 68	0.815 4	1.1022
3.5	0.4433	0.079 16	1.504 3	0.9117
4.0	0.4062	0.076 16	2.239 0	0.7181
5.0	0.3505	0.070 10	3.726 2	0.4522
6.0	0.3102	0.064 63	5.164 0	0.3033

"throat," at $r=2M$, and all of the matter inside the bulb, i.e., inside a black hole.

Peaking of the gravitational mass is thus associated with formation of an apparent horizon in the case of spherical distributions. One might conjecture that such a correlation holds quite generally, at least in a loose sense. For the conformally flat loop we find, indeed, that apparent horizons are formed once $\Delta\phi$ reaches π, not far from the mass-peaking value of 1.216π.

On a time-symmetric slice, an apparent horizon always corresponds to a minimal surface: the area of a surface element is stationary under all Lie displacements—timelike, spacelike and lightlike—normal to such a two-surface. For the string geometry (90), closed minimal surfaces, having (necessarily, by a theorem of Gibbons[21]) spherical topology, exist for all values of $\eta \geq 2$ that we have tested (Fig. 3). Their areas A satisfy the Gibbons-Penrose inequality[19]

$$\delta \equiv 1-(A/16\pi M^2)^{1/2} \geq 0 \tag{100}$$

in accordance with a general result of Ludvigsen and Vickers.[22] If cosmic censorship holds, the quantity δ, tabulated in the last column of Table II, gives an upper limit to the fraction of the initial mass M that can be radiated gravitationally as the black hole settles down to its final Schwarzschild form: it never exceeds 0.01%.

All of this is actually trivial for the case $\eta=2$ ($\Delta\phi=\pi$): according to (96) and (37), the metric (81) can now be expressed as

$$ds^2=\tfrac{1}{16}(1+a/r)^4(dr^2+r^2d\Omega^2) .$$

Upon rescaling $r\rightarrow 4r$, this becomes, remarkably, the Schwarzschild metric for a mass $M=\frac{1}{2}a$ on a slice $t=$const. For $\Delta\phi=\pi$, the string loop initially sits on the equator of the horizon $r=a$. For larger angular deficits, the loop is born inside the horizon (Fig. 3).

A curiosity which may be of interest to note is that each geometry with $\Delta\phi > \pi$ also admits a closed *maximal* surface, having toroidal topology and enclosed within the minimal surface. Its area exceeds the minimal area by a few percent and satisfies the sign reverse of inequality (100) (see Table II).

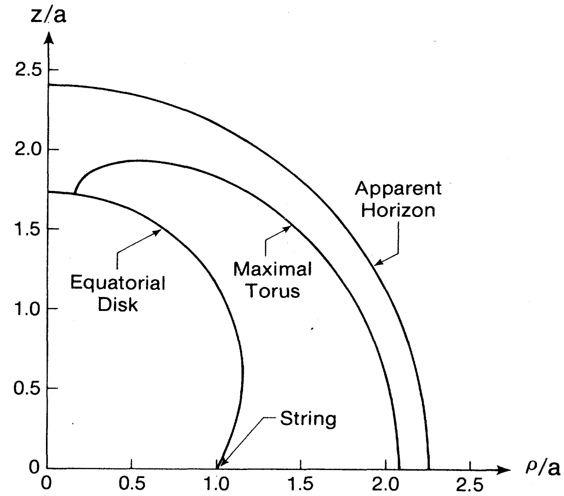

FIG. 3. Euclidean map of an azimuthal section of the conformally flat geometry of a loop (located at $\rho/a=1$) with $\Delta\phi=\frac{4}{3}\pi$. The circular arc $\psi=\frac{1}{3}\pi$ represents what is in reality a section of the equatorial disk $\psi_*=\pm\pi$ in physical space. The physical space contains a maximal surface (a torus) enclosed within a minimal surface with spherical topology—the apparent horizon.

VIII. CONCLUDING REMARKS

In dealing with the gravitational effects of cosmic string loops, it has been customary to take a "bifocal" view. For near-field effects (e.g., gravitational deflection of electromagnetic waves) one treats the string as a conical defect having zero gravitational mass. When treating far-field effects, such as emission of gravitational waves on the basis of covariant linearized theory, one considers the source to be massive. The exact solutions described in the preceding sections validate this dualistic view, and show how it can be integrated into a coherent picture. They confirm that the conventional assumptions of linearized theory are a good approximation for small angular deficits $\Delta\phi$, at the same time bringing out exotic features, such as black-hole formation, that appear when $\Delta\phi$ becomes comparable with unity.

ACKNOWLEDGMENTS

Our warmest thanks go to Bob Teshima, whose numerical work, analytical insight, and sheer enthusiasm for the project were quite indispensable. This work was supported by Canadian Institute for Advanced Research and by Natural Sciences and Engineering Research Council of Canada.

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