Cumulative beam break-up in a periodic linac

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The problem of cumulative beam break-up for a coasting beam in a linac with periodically placed acceleration cavities and transverse focusing units is treated using a general technique. The equations of motion for a series of equally spaced pointlike bunches are reduced to a single Hill's-type differential equation, from which a transfer matrix for a single period is derived. The technique is applied to the special case of a single resonance impedance, yielding a result which is in agreement with that derived elsewhere. The case of two nearly degenerate resonances is investigated using saddle-point integration, in the limit of strong focusing.

I. INTRODUCTION

Gluckstern, Cooper, and Channell[1] (GCC) considered the following model for cumulative transverse beam break-up in a linac.

(i) The linac consists of a series of external betatron-focusing units and cavities placed periodically from $z=0$ to ∞ with period L.

(ii) The beam bunches are pointlike.

(iii) The cavities are the only source of the deflective beam impedance.

(iv) The cavities are of negligible length. This thin-lens approximation is valid if the change in the transverse position of the particle in passing through a cavity is negligible.

Using an elegant technique, GCC solved the model for the special case where the deflective beam impedance consists of only one normal mode.

In this paper we introduce a new method of treating a periodic linac. We shall apply the method specifically to the above model and solve it for the general case of an arbitrary cavity impedance. Here, we treat only the case where the beam is not accelerated; the case with acceleration will be discussed elsewhere.[2]

The paper is organized as follows: in Sec. II we reduce the problem of a periodic linac to a differential equation of Hill's type. Then in Sec. III we solve the corresponding initial-value problem by finding the transfer matrix for a period L. In Sec. IV we show how our result reduces to that of GCC in the special case of a single resonance impedance. In the Appendix we introduce an extended problem and show how our method is related to the eigensolutions of the extended problem.

II. EQUATION OF MOTION AND HILL'S EQUATION

Consider a linac with pointlike bunches moving along the positive z direction according to

$$z=ct-Mc\tau,\quad M=0,1,2,\ldots,\infty\ , \tag{2.1}$$

where M labels different bunches, τ is the bunch separation in seconds, and c is the speed of light.

If we ignore the beam-induced deflective force, the equation of motion of a particle in the Mth bunch is

$$x_M''+K(z)x_M(z)=0,\quad \zeta\geq 0\ , \tag{2.2}$$

where $x'(z)=dx/dz$, and the periodic function $K(z)$, $K(z+L)=K(z)$, describes the effects of the external transverse focusing.

The following notation will be adopted; ν is the betatron tune associated with Eq. (2.2), $\nu\phi$ is the betatron phase, $\mu=2\pi\nu$ is the betatron phase advance per period, β,α,γ are the Twiss parameters, and $\beta_c,\alpha_c,\gamma_c$ are the Twiss parameters evaluated at the cavity. β is in units of meters.

We now include the effects of the transverse cavity wake field induced by the beam. Let us assume the cavities to be located periodically at positions $z=NL$, $N=0,1,2,\ldots$. Then the equation of motion including the wake field force $F_M(z)$ is

$$x_M''(z)+K(z)x_M(z)=F_M(z),\quad z\geq 0\ , \tag{2.3}$$

where F_M is related to the transverse wake function $G(t)$ by

$$F_M(z)=\frac{e^2N_B}{\gamma_0}\sum_{m=0}^{\infty}\sum_{N=0}^{\infty}\delta(z-NL)S_{M-m}x_m(z)\ , \tag{2.4}$$

with N_B being the number of particles per bunch, γ_0 the beam energy in units of rest energy, and

$$S_{M-m}=G((M-m)\tau)\ . \tag{2.5}$$

From the causality condition

$$G(t)=0\quad\text{if}\quad t<0\ , \tag{2.6}$$

we have

$$S_m=0\quad\text{if } m<0\ . \tag{2.7}$$

In terms of the Courant-Snyder variable

$$\xi_M(\phi)=x_M/\sqrt{\beta}\ , \tag{2.8}$$

Eq. (2.3) becomes

$$\ddot{\xi}_M(\phi)+\nu^2\xi_M(\phi) = \frac{e^2N_B\nu\beta_c}{\gamma_0}\sum_{m=0}^{M}\sum_{N=0}^{\infty}\delta(\phi-2\pi N)S_{M-m}\xi_m(\phi)\ , \tag{2.9}$$

where $\dot{\xi}_M(\phi)=d\xi_m(\phi)/d\phi$. This equation defines the model completely. We reduce this equation below to a Hill's-type differential equation.

The variables M and ϕ in Eq. (2.9) can easily be separated. Define

$$\Xi(\phi,\theta)=\sum_{m=0}^{\infty}\xi_m(\phi)e^{im\theta}\ , \tag{2.10}$$

or, equivalently,

$$\xi_m(\phi)=\frac{1}{2\pi}\int_0^{2\pi}d\theta\,\Xi(\phi,\theta)e^{-im\theta}\ , \tag{2.11}$$

also

$$\Delta(\theta)=b\sum_{m=0}^{\infty}S_m e^{im\theta} \tag{2.12}$$

and

$$b=\frac{e^2N_B\nu\beta_c}{\gamma_0}\ . \tag{2.13}$$

Then, from the above equations and Eq. (2.7),

$$\ddot{\Xi}(\phi,\theta)+\nu^2\Xi(\phi,\theta)=\Delta(\theta)\sum_{N=0}^{\infty}\delta(\phi-2\pi N)\Xi(\phi,\theta)\ . \tag{2.14}$$

This is a differential equation of Hill's type with period 2π. The focusing function including the effect of the beam-induced force is

$$\bar{K}(\phi,\theta)=\nu^2-\Delta(\theta)\sum_{N=0}^{\infty}\delta(\phi-2\pi N)\ . \tag{2.15}$$

This is a complex function since Δ is complex.

Equation (2.14) is equivalent to Eq. (2.9); once the solution of Eq. (2.14) is found, we can obtain the solution of Eq. (2.9) by using (2.11).

III. INITIAL-VALUE PROBLEM

The initial-value problem of Eq. (2.14) can be solved completely; that is, given $\Xi(0,\theta)$ and $\dot{\Xi}(0,\theta)$, one can find $\Xi(\phi,\theta)$ explicitly for $\phi>0$. (We adopt a shorthand notation $x_-=x-\epsilon$, with ϵ being a small positive number.) However, we shall only find the expression relating $\Xi_N(\theta)$ and $\dot{\Xi}_N(\theta)$ to $\Xi_0(\theta)$ and $\dot{\Xi}_0(\theta)$, where $\Xi_N(\theta)$ is the value of $\Xi(\phi,\theta)$ at the Nth cavity:

$$\Xi_N(\theta)=\Xi(2\pi N,\theta),\quad \dot{\Xi}_N(\theta)=\dot{\Xi}(2\pi N_-,\theta)\ . \tag{3.1}$$

The transfer matrix $T(\theta)$ over a period which relates $(\Xi_{N+1},\dot{\Xi}_{N+1})$ to $(\Xi_N,\dot{\Xi}_N)$ can be calculated from Eq. (2.14) by an elementary method. It is

$$T(\theta)=\begin{bmatrix}\cos\mu+\dfrac{\Delta(\theta)}{\nu}\sin\mu & \dfrac{1}{\nu}\sin\mu\\ -\nu\sin\mu+\Delta(\theta)\cos\mu & \cos\mu\end{bmatrix}\ . \tag{3.2}$$

Note that $\det(T)=1$; hence, T can be written in a standard form

$$T(\theta)=\begin{bmatrix}\cos\bar{\mu}+\bar{\alpha}\sin\bar{\mu} & \bar{\beta}\sin\bar{\mu}\\ -\bar{\gamma}\sin\bar{\mu} & \cos\bar{\mu}-\bar{\alpha}\sin\bar{\mu}\end{bmatrix} \tag{3.3}$$

with $\bar{\beta}\bar{\gamma}-\bar{\alpha}^2=1$. $\bar{\mu}$, $\bar{\alpha}$, $\bar{\beta}$, and $\bar{\gamma}$ are all complex functions of θ; they are given in terms of μ and $\Delta(\theta)$ by

$$\cos\bar{\mu}=\cos\mu+\frac{\Delta}{2\nu}\sin\mu\ , \tag{3.4}$$

$$\bar{\beta}=\sin\mu/(\nu\sin\bar{\mu})\ , \tag{3.5}$$

$$\bar{\alpha}=\Delta\sin\mu/(2\nu\sin\bar{\mu})\ , \tag{3.6}$$

$$\bar{\gamma}=\nu(\sin\mu-\Delta\cos\mu/\nu)/\sin\bar{\mu}\ . \tag{3.7}$$

Note that $\bar{\beta}$ is dimensionless.

We shall call $\bar{\nu}(\theta)=\bar{\mu}(\theta)/2\pi$ the coherent betatron tune of the mode θ; the reason for this is explained in the Appendix.

Using

$$\begin{bmatrix}\Xi_N\\ \dot{\Xi}_N\end{bmatrix}=T^N\begin{bmatrix}\Xi_0\\ \dot{\Xi}_0\end{bmatrix}\ , \tag{3.8}$$

where

$$T^N=\begin{bmatrix}\cos N\bar{\mu}+\bar{\alpha}\sin N\bar{\mu} & \bar{\beta}\sin N\bar{\mu}\\ -\bar{\gamma}\sin N\bar{\mu} & \cos N\bar{\mu}-\bar{\alpha}\sin N\bar{\mu}\end{bmatrix}\ , \tag{3.9}$$

we have

$$\Xi_N=\left[\cos N\bar{\mu}+\frac{\sin\mu}{2\nu}\Delta C^1_{N-1}(\cos\bar{\mu})\right]\Xi_0 +\frac{\sin\mu}{\nu}C^1_{N-1}(\cos\bar{\mu})\dot{\Xi}_0\ , \tag{3.10}$$

$$\dot{\Xi}_N=-(\nu\sin\mu-\Delta\cos\mu)C^1_{N-1}(\cos\bar{\mu})\Xi_0 +\left[\cos N\bar{\mu}-\frac{\sin\mu}{2\nu}\Delta C^1_{N-1}(\cos\bar{\mu})\right]\dot{\Xi}_0\ , \tag{3.11}$$

where we have introduced the Gegenbauer polynominal $C^1_{N-1}(\cos x)=\sin Nx/\sin x$.

Let us define, in analogy to (3.1),

$$\xi_{M,N}=\xi_M(2\pi N),\quad \dot{\xi}_{M,N}=\dot{\xi}_M(2\pi N)\ . \tag{3.12}$$

Then, from Eqs. (3.10), (3.11) and (2.10), (2.11),

$$\xi_{M,N}=\frac{1}{2\pi}\sum_{m=0}^{\infty}\int_0^{2\pi}d\theta\,e^{i(m-M)\theta}\left[\left[\cos N\bar{\mu}+\frac{\sin\mu}{2\nu}\Delta C^1_{N-1}(\cos\bar{\mu})\right]\xi_{m,0}+\frac{\sin\mu}{\nu}C^1_{N-1}(\cos\bar{\mu})\dot{\xi}_{m,0}\right]\ , \tag{3.13}$$

$$\dot{\xi}_{M,N}=\frac{1}{2\pi}\sum_{m=0}^{\infty}\int_0^{2\pi}d\theta\, e^{i(m-M)\theta}\left[-(\nu\sin\mu-\Delta\cos\bar{\mu})C_{N-1}^{1}(\cos\bar{\mu})\xi_{m,0}+\left[\cos N\bar{\mu}-\frac{\sin\mu}{2\nu}\Delta C_{N-1}^{1}(\cos\bar{\mu})\right]\dot{\xi}_{m,0}\right] . \tag{3.14}$$

Note from Eq. (2.12) that $\Delta(\theta)$ has only Fourier components corresponding to a non-negative Fourier-conjugate index m. The terms within the square brackets in Eqs. (3.13) and (3.14) can be expanded in power series in Δ; hence those terms cannot have components with negative index. As a consequence, the integrals in these equations vanish if $m>M$. We conclude that we can make the replacement

$$\sum_{m=0}^{\infty}\rightarrow\sum_{m=0}^{M} \tag{3.15}$$

in Eqs. (3.13) and (3.14).

By utilization of Eq. (2.8), Eqs. (3.13) and (3.14) with (3.15) can be transformed into equations relating $x_{M,N}$ and $x'_{M,N}$ to $x_{M,0}$ and $x'_{M,0}$, where

$$x_{M,N}=x_M(NL),\quad x'_{M,N}=x'_M(NL_-) . \tag{3.16}$$

The result is

$$x_{M,N}=\frac{1}{2\pi}\sum_{m=0}^{M}\int_0^{2\pi}d\theta\, e^{i(m-M)\theta}\left\{x_{m,0}\left[\cos N\bar{\mu}+\sin\mu\left[\alpha_c+\frac{\Delta}{2\nu}\right]C_{N-1}^{1}(\cos\bar{\mu})\right]+x'_{m,0}\beta_c\sin\mu C_{N-1}^{1}(\cos\bar{\mu})\right\} , \tag{3.17}$$

$$x'_{M,N}=\frac{1}{2\pi}\sum_{m=0}^{M}\int_0^{2\pi}d\theta\, e^{i(m-M)\theta}\left\{x_{m,0}\frac{1}{\beta_c}\left[-(1+\alpha_c^2)\sin\mu+\frac{\Delta}{\nu}(\cos\mu-\alpha_c\sin\mu)\right]C_{N-1}^{1}(\cos\bar{\mu})\right.$$
$$\left.+x'_{m,0}\left[\cos N\bar{\mu}-\sin\mu\left[\alpha_c+\frac{\Delta}{2\nu}\right]C_{N-1}^{1}(\cos\bar{\mu})\right]\right\} . \tag{3.18}$$

Let us close this section by taking the above two equations in the limit of no external focusing, $K(\phi)=0$. This limit amounts to setting $\mu\rightarrow 0$, $\alpha\rightarrow 0$, and $\beta\mu\rightarrow L$. We obtain

$$x_{M,N}=\frac{1}{2\pi}\sum_{m=0}^{M}\int_0^{2\pi}d\theta\, e^{i(m-M)\theta}\{x_{m,0}[\cos N\bar{\mu}+\pi\Delta C_{N-1}^{1}(\cos\bar{\mu})]+x'_{m,0}LC_{N-1}^{1}(\cos\bar{\mu})\} \tag{3.19}$$

and

$$x'_{M,N}=\frac{1}{2\pi}\sum_{m=0}^{M}\int_0^{2\pi}d\theta\, e^{i(m-M)\theta}\left[x_{m,0}\frac{2\pi}{L}\Delta C_{N-1}^{1}(\cos\bar{\mu})+x'_{m,0}[\cos N\bar{\mu}-\pi\Delta C_{N-1}^{1}(\cos\bar{\mu})]\right] . \tag{3.20}$$

Also, Eqs. (3.4) and (2.14) become

$$\cos\bar{\mu}(\theta)=1+\pi\Delta(\theta) \tag{3.21}$$

and

$$\Delta(\theta)=\frac{e^2N_B}{\gamma_0}\frac{L}{2\pi}\sum_{m=0}^{\infty}S_m e^{im\theta} . \tag{3.22}$$

IV. SINGLE-CAVITY MODE

We consider here the case where the wake field of the cavity, $G(t)$, consists of a single mode

$$S_m=\frac{1}{2m_pc}\frac{R}{Q}e^{-m\omega\tau/2Q}\sin(m\omega\tau) , \tag{4.1}$$

where m_p is the particle mass, R the shunt impedance, and ω the mode angular frequency, and show that Eq. (3.17) reduces to the result of GCC.

$\Delta(\theta)$ is in this case, from Eqs. (2.12), (2.13), and (4.1),

$$\Delta(\theta)=-a\frac{\sin\omega\tau}{\cos\omega\tau-\cos\left[\theta+i\frac{1}{2Q}\omega\tau\right]} \tag{4.2a}$$

$$=-a\frac{\sin\omega\tau}{\cos\omega\tau-\frac{1}{2}(e^{-\omega t/2Q}e^{i\theta}+e^{\omega\tau/2Q}e^{-i\theta})} , \tag{4.2b}$$

with

$$a=\frac{e^2N_B}{4m_pc\gamma_0}\nu\beta_c\frac{R}{Q} . \tag{4.3}$$

Let us change the variable of integration in Eq. (3.17) to

$$\zeta=\rho e^{i\theta} , \tag{4.4}$$

with $\rho=e^{-\omega\tau/2Q}<1$. Equations (3.17) and (4.2b) become

$$x_{M,N}=\frac{1}{2\pi i}\sum_{m=0}^{M}\rho^{M-m}\int_{C_\rho}d\zeta\,\zeta^{m-M-1}P_{M,N}(\zeta) \tag{4.5}$$

and

$$\Delta = -a\frac{\sin\omega\tau}{\cos\omega\tau - \frac{1}{2}(\zeta+\zeta^{-1})} , \tag{4.6}$$

where $P_{M,N}$ represents the quantities in curly brackets in Eq. (3.17), and C_ρ is the circular contour with radius ρ going clockwise in the complex ζ plane. (See Fig. 1.)

Note that Δ has two singularities at $\zeta = e^{+i\omega\tau}$ and $\zeta = e^{-i\omega\tau}$ which lie on the unit circle in the complex ζ plane. Therefore, the integrand of Eq. (4.5) has these two singularities plus a regular singularity at $\zeta=0$.

Let us change the contour of integration from C_ρ to $C_{1-\epsilon}$, where $C_{1-\epsilon}$ is also depicted in Fig. 1, and then change the variable $\zeta \to \theta$, where

$$\zeta = e^{i\theta} .$$

We obtain

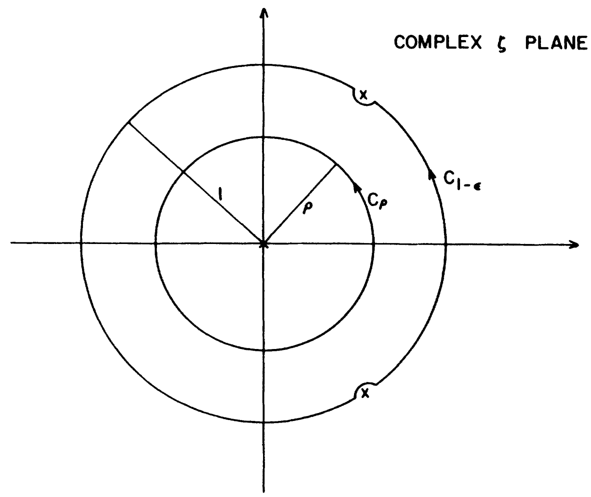

FIG. 1. Integration contours for single resonance beam break-up calculation.

$$x_{M,N} = \frac{1}{2\pi}\sum_{m=0}^{M} e^{-(m-M)\omega\tau/2Q}\int_0^{2\pi} d\theta\, e^{i(m-M)\theta}\left\{x_{m,0}\left[\cos N\mu_1 + \sin\mu\left[\alpha_c + \frac{\Delta_1}{2\nu}\right]C^1_{N-1}(\cos\mu_1)\right] + x'_{m,0}\beta_c \sin\mu C^1_{N-1}(\cos\mu_1)\right\} , \tag{4.7}$$

where

$$\Delta_1(\theta) = -a\frac{\sin\omega\tau}{\cos\omega\tau - \cos\theta} \tag{4.8}$$

and

$$\cos\mu_1 = \cos\mu + \frac{\Delta_1}{2\nu}\sin\mu . \tag{4.9}$$

This is the result of GCC. The integral of Eq. (4.7) has singularities at $\theta = \omega\tau$ and $2\pi - \omega\tau$. The contour of integration should be taken above these singularities.

V. TWO-CAVITY MODES

To demonstrate the usefulness of our method, the case of a cavity with two nearby resonances is treated here using saddle-point integration, in the limit of strong focusing. Of interest are effects resulting from the proximity of the resonances to each other. The technique of GCC is specific to a single-cavity mode and cannot address a problem of this nature.

When the deflecting wake fields inside the accelerating cavities are modeled by a sum of two resonances, the function $\Delta(\theta)$ of (2.12) can be written

$$\Delta(\theta) = \sum_{l=1}^{2} \frac{a_l \sin(\omega_l\tau)}{\cos\left[\theta + \frac{i\omega_l\tau}{2Q_l}\right] - \cos(\omega_l\tau)} , \tag{5.1}$$

where

$$a_l = \frac{e^2 N_B}{4m_p c\gamma_0}\nu\beta_c\frac{R_l}{Q_l} . \tag{5.2}$$

The notation should be clear from the discussion of Sec. IV.

The strong focusing limit will be taken to mean that the coherent betatron tune $\bar{\nu}(\theta)$, over the region of integration, is not significantly different from the tune ν without wake fields. Therefore, Eq. (3.4) can be expanded about $\bar{\nu} = \nu$ to yield

$$\bar{\mu} - \mu \approx -\frac{\Delta(\theta)}{2\nu} , \tag{5.3}$$

and the absolute value of the quantity on the right-hand side is small compared to unity.

Here we will consider the case where the leading bunch is offset by an amount $x_{0,0} = x_0$, with all trailing bunches on axis. The initial angles $x'_{m,0}$ are assumed to vanish. Supposing further that $\alpha_c = 0$ (e.g., smooth focusing), the particle trajectories, from (3.17), are described by

$$\frac{x_{M,N}}{x_0} \approx \frac{1}{4\pi}\int_0^{2\pi} d\theta\, e^{-iM\theta}\left(e^{iN(\mu-\Delta/2\nu)} + e^{-iN(\mu-\Delta/2\nu)}\right) . \tag{5.4}$$

Because saddle points are expected to occur near points where the denominators in (5.1) vanish (i.e., near $\theta + i\omega_l\tau/2Q_l = \pm\omega_l\tau$), the denominators in (5.1) can be expanded about their respective resonance points to yield

$$\Delta(\theta) \approx \sum_{l=1}^{2} \pm\frac{a_l}{\theta + i\omega_l\tau/2Q_l \pm \omega_l\tau} . \tag{5.5}$$

Consider the first term in (5.4). The exponent $f(\theta)$ leading to break up is

$$f(\theta)=-iM\theta+iN\mu+i\frac{N}{2\nu}\sum_{l=1}^{2}\frac{a_l}{\theta+\alpha_l}\ , \tag{5.6}$$

where

$$\alpha_l=i\frac{\omega_l\tau}{2Q_l}-\omega_l\tau\ . \tag{5.7}$$

We can greatly simplify our task, and retain most of the physics, by assuming the resonances to be of equal strength, i.e., by assuming $a_1=a_2=a$. In this case, the exponent (5.6) can be written

$$f(\bar\theta)=-iM\bar\theta+iM\bar\alpha+iN\mu+i\frac{Na}{\nu}\frac{\bar\theta}{\bar\theta^2-\delta^2}\ , \tag{5.8}$$

where we have introduced the shifted variable

$$\bar\theta=\theta+\bar\alpha\ , \tag{5.9}$$

with

$$\bar\alpha=(\alpha_1+\alpha_2)/2 \tag{5.10}$$

and

$$\delta=(\alpha_1-\alpha_2)/2\ . \tag{5.11}$$

Saddle points $\bar\theta_s$ are located by setting the first derivative of (5.8) to zero:

$$\bar\theta_s^2=-\frac{Na}{2M\nu}+\delta^2\pm\left[\left[\frac{Na}{2M\nu}\right]^2-4\delta^2\frac{Na}{2M\nu}\right]^{1/2}\ . \tag{5.12}$$

It is instructive to treat two separate cases, depending on the size of the complex frequency split δ. The first case, when $4|\delta^2| \ll Na/2M\nu$, after expanding the square root, gives

$$\bar\theta_s^2\approx\begin{cases}-\delta^2\ ,\\ -\dfrac{Na}{M\nu}+3\delta^2\ .\end{cases} \tag{5.13}$$

Notice the location of the saddle points $\bar\theta_s=\pm i\delta$ in relation to the two poles of $f(\theta)$ at $\pm\delta$. The path of steepest descent passes directly between the poles, meaning that one will pick up an unwanted residue if this path is used. Therefore, only the second saddle point is relevant to the calculation.

The value of the exponent at the relevant saddle point is given by

$$f(\bar\theta_s)=iN\mu+iM\bar\alpha+2E-M^2\delta^2/E\ , \tag{5.14}$$

where

$$E=(MNa/\nu)^{1/2}\ . \tag{5.15}$$

The result of the saddle-point calculation is therefore given by

$$\frac{x_{M,N}}{x_0}\approx\frac{1}{4\sqrt{\pi}M}\sqrt{E}\,e^{iN\mu+iM\bar\alpha+2E-M^2\delta^2/E}+\text{c.c.} \tag{5.16}$$

The complex-conjugate term arises from the second term in (5.4) using the alternate sign in (5.5). This result shows how the growth is suppressed as two degenerate modes are separated.

Consider now the case where $4|\delta^2| \gg Na/2M\nu$. Expanding the square root in (5.12) gives, to leading order,

$$\bar\theta_s\approx\pm(\delta\pm iE_2/M)\ , \tag{5.17}$$

where

$$E_2=(MNa/2\nu)^{1/2}\ . \tag{5.18}$$

Only the saddle points at $\pm\delta+iE_2/M$ give exponents with a positive real part. The exponents corresponding to this are given by

$$f(\bar\theta_s)=iN\mu-iM(\pm\delta-\bar\alpha)+2E_2\pm i\frac{Na}{4\nu\delta}\ , \tag{5.19}$$

yielding the result

$$\frac{x_{M,N}}{x_0}\approx\frac{1}{4\sqrt{\pi}M}\sqrt{E_2}\left\{\exp\left[iM\alpha_1+2E_2+iN\left[\mu+\frac{a}{4\nu\delta}\right]\right]+\exp\left[iM\alpha_2+2E_2+iN\left[\mu-\frac{a}{4\nu\delta}\right]\right]\right\}+\text{c.c.} \tag{5.20}$$

This result shows a "tune shift" inversely proportional to the complex frequency split δ, and opposite in sign for the two modes.

Considering the number of approximations which have been made up to this point, it is appropriate to go back and investigate the domain of validity for the results (5.16) and (5.20). First, a more thorough calculation would yield an asymptotic series in powers of $1/E$ for (5.16), or $1/E_2$ for (5.20). The leading correction corresponds to multiplying our results by $(1-\frac{3}{16}E^{-1})$ or $(1-\frac{3}{16}E_2^{-1})$, respectively. Clearly, a condition for the validity of our results is that the quantities E and E_2 be larger than, or on the order of, unity. Note, however, that there will be additional corrections, which could be comparable to $1/E$, related to the approximations regarding the size of the complex frequency split δ.

Second, Eq. (5.5) was written under the assumption that the denominators appear to be small when $\theta=\theta_s$. Writing (5.5) using the notation of (5.9)–(5.11), we have

$$\Delta(\bar\theta)=\frac{a}{\bar\theta+\delta}+\frac{a}{\bar\theta-\delta}\ . \tag{5.21}$$

Inserting the relevant value $\bar\theta_s$ from (5.13), we see that the denominators will be small if $E \ll M$. Similarly, insertion of the saddle points (5.17) leads to the conclusion that the denominator of one or the other of the terms in (5.21) will be small if $E_2 \ll M$.

Third, implicit in the result (5.20) is the assumption that the distance between the two saddle points (5.17), i.e., 2δ, is large in comparison to the saddle width. This width can be approximated by $\Delta\theta \approx |[2/f''(\theta_s)]|^{1/2} \approx \sqrt{E_2}/M$.

Recall that the assumption leading to (5.17) was that $2|\delta| \gg \sqrt{Na/2M\nu} = E_2/M$. Therefore, the two saddle points (5.17) will be far enough apart provided that $2|\delta|/\Delta\theta \gg 1$, which implies that $\sqrt{E_2}$ must be greater than or on the order of unity.

Finally, the condition for which the strong-focusing approximation (5.3) is valid is that $\Delta(\bar{\theta}_s)/(2\nu)$ be small compared to one. This in turn leads to the conclusion that both E and E_2 must be small compared to N.

VI. CONCLUSIONS

A new technique of solving the cumulative beam break-up problem has been presented, which for coasting beams yields the results (3.17) and (3.18), and can be easily extended to include the effects of acceleration.[2] It was shown that these expressions are in agreement with the results of GCC, but are valid for a general impedance. The power of the technique was demonstrated for the case of two overlapping resonances, which was treated approximately using saddle-point integration.

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APPENDIX: EXTENDED PROBLEM

Equation (2.9) is a homogeneous linear equation of two independent variables ϕ and M. It is well known (e.g., from quantum mechanics) that the initial-value problem in ϕ of such an equation is closely related to the eigenvalue problem of the other variable M. We find here the physical meaning of θ and $\bar{\mu}(\theta)$ introduced in Secs. II and III by relating them to the eigenvalue problem associated with an extension of Eq. (2.9).

We solved earlier the initial-value problem associated with the equation

$$\ddot{\xi}_M(\phi)+\nu^2\xi_M(\phi)=b\sum_{m=0}^{M}\sum_{N=0}^{\infty}\delta(\phi-2\pi N)S_{M-m}\xi_m(\phi)\,,$$
$$\phi\geq 0,\ M=0,1,2,\ldots,\infty\,. \tag{A1}$$

Assuming $\xi_{M,0}$ and $\dot{\xi}_{M,0}$ to be known for $M=0,1,2,\ldots$, we were able to find expressions for $\xi_{M,N}$ and $\dot{\xi}_{M,N}$ with $N=1,2,\ldots$.

In order to find an interpretation for θ and $\bar{\mu}(\theta)$, let us extend the range of M and m to be from $-\infty$ to $+\infty$ instead of from 0 to ∞. We thus consider

$$\ddot{\xi}_M(\phi)+\nu^2\xi_M(\phi)=b\sum_{m=-\infty}^{M}\sum_{N=0}^{\infty}\delta(\phi-2\pi N)S_{M-m}\xi_m(\phi)\,,$$
$$\phi\geq 0,\ M=-\infty,\ldots,-1,0,1,\ldots,\infty\,. \tag{A2}$$

The solution of the initial-value problem of Eq. (A2) can be reduced to that of Eq. (A1) by setting

$$\xi_{M,0}=\dot{\xi}_{M,0}=0,\quad M<0\,. \tag{A3}$$

We now discuss Eq. (A2). First, we concentrate on the indices M and m. S_{M-m} on the right-hand side can be regarded as an element of an ∞-dimensional matrix $\hat{S}$. This matrix can be diagonalized easily since what is involved is a convolution sum. The eigenvector of $\hat{S}$ is

$$\xi_m(\phi)=\frac{1}{2\pi}\Xi(\phi,\theta)e^{-im\theta}\,,$$
$$m=-\infty,\ldots,-1,0,1,\ldots,\infty\,, \tag{A4}$$
$$0\leq\theta\leq 2\pi\,,$$

where θ parametrizes different eigensolutions; the factor $1/2\pi$ in front of this equation is arbitrary.

We have yet to find the ϕ dependence of Ξ so that $\xi_m(\phi)$ satisfies Eq. (A2). Upon substitution of Eq. (A4) in Eq. (A2), we obtain

$$\ddot{\Xi}(\phi,\theta)+\nu^2\Xi(\phi,\theta)=\Delta(\theta)\sum_{N=0}^{\infty}\delta(\phi-2\pi N)\Xi(\phi,\theta)\,, \tag{A5}$$

where $\Delta(\theta)$ is defined by Eqs. (2.12) and (2.13), and Eq. (2.7) has been used. Recall that we have obtained this equation in Sec. II [see Eq. (2.14)]; hence, the transfer matrix of this transfer equation is the same as before: namely, Eq. (3.3).

It is now evident that $\bar{\mu}(\theta)$ as given by Eq. (3.4) is the betatron phase advance per period if the beam in the extended problem is in the θth mode, or, if the phases of the bunches are arranged coherently according to Eq. (A4).

To solve the initial-value problem of (A2), we expand $\xi_M(\phi)$ in terms of the eigensolutions of Eq. (A2):

$$\xi_m(\phi)=\frac{1}{2\pi}\int_0^{2\pi}d\theta\,\Xi(\phi,\theta)e^{-im\theta}\,, \tag{A6}$$

or

$$\Xi(\phi,\theta)=\sum_{m=-\infty}^{\infty}\xi_m(\phi)e^{im\theta}\,. \tag{A7}$$

We find that $\Xi(\phi,\theta)$ satisfies Eq. (A5).

A method similar to that of Sec. III gives the solution of Eq. (A2) as

$$\xi_{M,N}=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int_0^{2\pi}d\theta\,e^{i(m-M)\theta}\left[\left[\cos N\bar{\mu}+\frac{\sin\mu}{2\nu}\Delta C_{N-1}^{1}(\cos\bar{\mu})\right]\xi_{m,0}+\frac{\sin\mu}{\nu}C_{N-1}^{1}(\cos\bar{\mu})\dot{\xi}_{m,0}\right]. \tag{A8}$$

The solution to Eq. (A1) is obtained from this equation by using Eqs. (A3). The result is Eq. (3.13).

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