Analytical and semiclassical aspects of matter-enhanced neutrino oscillations

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Matter-enhanced neutrino oscillations for the case of two neutrino flavors are studied and three different approximation methods for calculating the unaveraged oscillation probability are presented. Two of these methods provide simple analytical expressions, valid for small or large vacuum mixing angles, which are very simple to evaluate numerically. The third method provides an expression uniformly valid over the entire range of mixing angles. In their respective regions of validity, the approximate calculations using these methods are found to be in very good agreement with the exact calculations.

I. INTRODUCTION

If neutrinos have nonzero masses, their mass eigenstates do not need to be the same as the weak-interaction charged-current eigenstates. In such a case different flavors of neutrinos would oscillate back and forth as they evolve in time. Neutrino oscillations in a vacuum were first conjectured by Pontecorvo[1,2] and by Maki, Nakagawa, and Sakata.[3] After it was experimentally determined[4] that the measured solar-neutrino capture rate on Earth is about one-third of the standard solar-model prediction,[5] neutrino oscillations were proposed as a possible mechanism to resolve the discrepancy between experiment and theory. By considering a slab of constant electron density equal to the resonant density, Wolfenstein demonstrated that even a small mixing angle can result in complete conversion of electron neutrinos into muon neutrinos due to the modifications caused by the coherent forward scattering of the neutrinos in electronic matter.[6] Although the possibility of a matter resonance was noted by Barger, Whisnant, Pakvasa, and Phillips[7] subsequent to Wolfenstein's result, it was generally assumed that such a resonance conversion is too restrictive to be applicable to the solar-neutrino problem.

More recently Mikheyev and Smirnov demonstrated that the resonance condition can be satisfied in a medium with varying electron density.[8] Bethe rederived this result and showed that the Mikheyev-Smirnov-Wolfenstein (MSW) mechanism has an adiabatic solution for the near-exponential electron density distribution of the Sun.[9] Detailed aspects of this mechanism have been investigated by various authors.[10–20] In principle, it is possible to investigate matter-enhanced neutrino oscillations numerically. However, the simultaneous existence of a macroscopic (stellar radius) scale and a microscopic (quantum mixing) scale leads to a time-consuming numerical integration.

As was stressed by Bethe[9] and Messiah,[15] the main contribution of Mikheyev and Smirnov is to recognize the importance of adiabatic level crossing. Level crossing and other salient features of the underlying physics of the matter-enhanced neutrino oscillations can be understood analytically. Already considerable work was done in this direction. Barger, Phillips, and Whisnant derived an approximate analytic solution for the adiabatic propagation to calculate solar-neutrino capture rates of ^{37}Cl and ^{71}Ga detectors.[11] Haxton discussed an extension of the Landau-Zener level-crossing approximation for neutrino propagation in matter.[16]

In this paper we point out a formal analogy between supersymmetric quantum mechanics[21] and the neutrino oscillations for two flavors. We exploit this analogy by applying analogs of recently introduced supersymmetry-inspired primitive[22] and uniform[23] semiclassical approximations to neutrino oscillations. We demonstrate that the adiabatic condition and the condition of validity of the semiclassical approximations are the same. We also obtain another approximation valid for small mixing angles. These approximations have different regions of validity as we discuss in the following sections.

In Sec. II we review salient features of neutrino oscillations in matter and summarize exact and approximate results given in the literature. A "hidden supersymmetry" of the neutrino oscillations for two flavors is outlined in the Appendix. In Sec. III we present a semiclassical analysis of matter-enhanced neutrino oscillations using the primitive Wentzel-Kramers-Brillouin (WKB) approximation and establish its equivalence to the adiabatic approximation. Another approximation, based on the logarithmic perturbation theory, appropriate for small mixing angles, is presented in Sec. IV. As expected, the results given in that section reduce to the nonadiabatic results obtained by previous authors in the limit of very small mixing angles. In Sec. V another WKB approximation, which is uniformly valid for all vacuum mixing angles, is elucidated. Section VI contains a discussion of our results and conclusions.

II. NEUTRINO OSCILLATIONS IN MATTER, EXACT RESULTS

For two flavors of neutrinos ν_e and ν_μ traveling through matter, the mass matrix $\mathcal{M}$ in the flavor basis is[9]

$$\mathcal{M}=\tfrac{1}{2}(m_1^2+m_2^2+A)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+\frac{1}{2}\begin{bmatrix}A-(\delta m^2)\cos2\theta & (\delta m^2)\sin2\theta\\ (\delta m^2)\sin2\theta & -A+(\delta m^2)\cos2\theta\end{bmatrix}, \tag{2.1}$$

where m_1 and m_2 are mass eigenvalues, θ is the vacuum mixing angle,

$$\delta m^2 = m_2^2 - m_1^2 , \tag{2.2a}$$

and A is the Wolfenstein correction[6] to the effective mass:

$$A = 2\sqrt{2} G_F N_e E . \tag{2.2b}$$

In Eqs. (2.2), E is the neutrino energy, G_F is the Fermi constant, and N_e is the number of electrons per unit volume. We find it convenient to represent the electron density as a dimensionless quantity

$$\zeta = \frac{A}{m_2^2 - m_1^2} . \tag{2.3}$$

We assume that ζ is a function of the distance from the center of the Sun. Hence, in the ultrarelativistic limit ($R \sim ct$), we can take ζ to be a function of time. After discarding an overall phase, the Schrödinger-type equation which follows from Eq. (2.1) takes the form[14]

$$i\hbar \frac{\partial}{\partial t} \begin{bmatrix} \Psi_e(t) \\ \Psi_\mu(t) \end{bmatrix} = \tfrac{1}{2}\Delta \begin{bmatrix} \zeta - \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\zeta + \cos 2\theta \end{bmatrix} \begin{bmatrix} \Psi_e(t) \\ \Psi_\mu(t) \end{bmatrix} , \tag{2.4}$$

where we defined

$$\Delta = \frac{\delta m^2}{2E} = \frac{m_2^2 - m_1^2}{2E} . \tag{2.5}$$

In Eq. (2.4), $\Psi_e(t)$ and $\Psi_\mu(t)$ are the wave functions for the electron and muon neutrinos, respectively. Since in this paper we wish to analyze Eq. (2.4) semiclassically, we write Planck's constant $\hbar$ explicitly in the equations above.

The instantaneous eigenvalues of the mass matrix, Eq. (2.1), are[9]

$$m_\pm^2 = \tfrac{1}{2}(m_1^2 + m_2^2 + A) \pm 2E \left[\phi^2(t) + \left[\frac{\Delta \sin 2\theta}{2} \right]^2 \right]^{1/2} , \tag{2.6}$$

where we introduced the function

$$\phi(t) = \frac{\Delta}{2}[\zeta(t) - \cos 2\theta] . \tag{2.7}$$

The splitting between the mass eigenstates has a minimum as a function of t if $\zeta > 0$ ($m_2^2 > m_1^2$). This minimum occurs when $\phi(t) = 0$ or $\zeta(t) = \cos 2\theta$, which corresponds to the resonance investigated by Mikheyev and Smirnov.

Equation (2.4) represents two coupled, first-order differential equations. Differentiating Eq. (2.4) once with respect to time, one can obtain two decoupled, second-order differential equations for Ψ_e and Ψ_μ:

$$-\hbar^2 \frac{\partial^2 \Psi_e}{\partial t^2} - [\phi^2(t) + i\hbar\phi'(t)]\Psi_e = \left[\frac{\Delta \sin 2\theta}{2} \right]^2 \Psi_e \tag{2.8a}$$

and

$$-\hbar^2 \frac{\partial^2 \Psi_\mu}{\partial t^2} - [\phi^2(t) - i\hbar\phi'(t)]\Psi_\mu = \left[\frac{\Delta \sin 2\theta}{2} \right]^2 \Psi_\mu , \tag{2.8b}$$

where a prime denotes derivative with respect to time. Equations (2.8a) and (2.8b) are formally equivalent to one-dimensional, time-independent Schrödinger equations with "potentials" $-(\phi^2 \pm i\hbar\phi')$ and "energy" $(\Delta \sin 2\theta/2)^2$. This connection was first noted by Nötzold.[24]

There is also a formal analogy between Eqs. (2.8) and supersymmetric quantum mechanics.[21] If these "potentials" had opposite (positive) overall sign, Eqs. (2.8a) and (2.8b) would be supersymmetric partners of each other with the same "energy" spectra and the function $\phi(t)$ would be the superpotential. Although Eqs. (2.8) do not represent the standard supersymmetric quantum-mechanical system, the problem of the two flavors of neutrinos propagating through matter still has a "hidden" supersymmetry, which we discuss in the Appendix.

As we mentioned earlier, Eqs. (2.8) are analogous to Schrödinger equations describing the motion of a particle above a complex potential barrier with real part $-\phi^2$ and imaginary part $\mp i\hbar\phi'$ ($-$ for the electron-neutrino flux loss and $+$ for the muon-neutrino flux gain). The maximum of the real part of the barrier ($\phi = 0$) corresponds to the resonance density. Of course for the solar-neutrino problem the initial conditions are different than the boundary conditions of the standard time-independent Schrödinger equation. We take (up to a constant phase)

$$\Psi_e(t=0) = 1 \tag{2.9a}$$

and

$$\Psi_\mu(t=0) = 0 , \tag{2.9b}$$

or equivalently

$$\left. \frac{\partial \Psi_e}{\partial t} \right|_{t=0} = -\frac{i}{\hbar}\phi(t=0) . \tag{2.9c}$$

Using the equivalence between the time-independent Schrödinger equation and Eqs. (2.8) one can write down exact analytic expressions for the neutrino conversion probability in matter for a number of electron densities. Nötzold studied[24] the density $V_0 \tanh(r/r_0)$, which corresponds to a Rosen-Morse barrier.[23] For the Sun an exponential density is a very good approximation. Pizzochero,[25] Toshev,[26] and Petcov[27] studied the solutions of Eqs. (2.8) for an exponential density, which corresponds to a Morse barrier.[23] Toshev also studied[28] the situation for a linear density, which corresponds to a parabolic barrier. In all these cases, the conversion probability was obtained in terms of hypergeometric functions. Unfortunately, those exact solutions are of limited value since the numerical evaluation of hypergeometric functions of complicated arguments is not much easier than numerical integration of Eq. (2.4).

III. SEMICLASSICAL ANALYSIS OF MATTER-ENHANCED NEUTRINO OSCILLATIONS

In this section we apply the WKB approximation to Eq. (2.8a). Given a differential equation of the form

$$\frac{d^2\Psi}{dt^2}+f(t)\Psi=0\ , \tag{3.1}$$

the condition of validity for the WKB approximation is[29]

$$\frac{1}{2}\left|\frac{f'}{\sqrt{f}}\right| \ll |f|\ . \tag{3.2}$$

Since Eq. (2.8a) is written in a form which resembles supersymmetric quantum mechanics, the approximation we use is very similar to the recently introduced supersymmetry-inspired primitive WKB approximation.[22] (Our superpotential $i\phi$ is pure imaginary and *independent* of $\hbar$, whereas in the usual formulation of the supersymmetric quantum mechanics the superpotential is pure real and has a complicated $\hbar$ dependence.) For Eq. (2.8a), the function $f(t)$ of Eq. (3.1) takes the form

$$f(t)=\frac{\Delta}{2\hbar^2}\left[\frac{\Delta}{2}(\zeta^2-2\zeta\cos 2\theta+1)+i\hbar\zeta'\right]\ . \tag{3.3}$$

The turning points of the real part of $f(t)$ [the solution of $\mathrm{Re} f(t)=0$] are given by

$$\zeta_{1,2}=e^{\pm i2\theta}\ . \tag{3.4}$$

We immediately see that, for small θ, since the turning points begin coalescing, the WKB approximation will fail. We can also derive a more precise condition of validity for the WKB approximation. Using Eq. (3.2) we get

$$\frac{2\hbar}{\Delta}\frac{|\zeta'(\zeta-\cos 2\theta)|}{|\zeta^2-2\zeta\cos\theta+1|^{3/2}} \ll 1\ . \tag{3.5}$$

At the resonance, ζ goes from $\cos 2\theta-\sin 2\theta$ to $\cos 2\theta+\sin 2\theta$ as was shown by Bethe.[9] Setting $\zeta=\cos 2\theta\pm\sin 2\theta$, Eq. (3.5) takes the form (with $\hbar=1, c=1$)

$$2^{-3/2}\frac{4E}{\delta m^2\sin 2\theta} \ll \tan 2\theta\left|\frac{1}{N_e}\frac{dN_e}{dr}\right|^{-1}\ . \tag{3.6}$$

The width of the resonance region is[9]

$$\delta r=2\left|\frac{1}{N_e}\frac{dN_e}{dr}\right|^{-1}\tan 2\theta \tag{3.7a}$$

and the neutrino-oscillation distance at the resonance is[9]

$$L_{\text{res}}=\frac{4\pi E}{\delta m^2\sin 2\theta}\ . \tag{3.7b}$$

Inserting Eqs. (3.7a) and (3.7b) into (3.6), we find the condition of validity for the WKB approximation to be

$$\frac{\sqrt{2}}{\pi}L_{\text{res}} \ll \delta r\ , \tag{3.8}$$

which is essentially the adiabatic condition ($L_{\text{res}} \ll \delta r$).

To find the WKB solution to Eq. (2.8a), we make the substitution

$$\Psi_e=\exp\left[\frac{i}{\hbar}S\right] \tag{3.9}$$

and expand the argument of the exponential in powers of $\hbar$:

$$S=S_0+\hbar S_1+O(\hbar^2)\ . \tag{3.10}$$

Inserting Eqs. (3.9) and (3.10) into (2.8a) and equating the same powers of $\hbar$, we get differential equations for S_0 and S_1:

$$S_0'^2=\phi^2+\left[\frac{\Delta\sin 2\theta}{2}\right]^2=\hbar^2\,\mathrm{Re} f(t)\ , \tag{3.11}$$

$$iS_0''-2S_1'S_0'+i\phi'=0\ . \tag{3.12}$$

Equation (3.12) has two sets of solutions:

$$S_0=\pm\int\left[\phi^2+\left[\frac{\Delta\sin 2\theta}{2}\right]^2\right]^{1/2}dt \tag{3.13a}$$

and

$$S_1=\pm\frac{i}{2}\ln\left\{\phi+\left[\phi^2+\left[\frac{\Delta\sin 2\theta}{2}\right]^2\right]^{1/2}\right\}+\frac{i}{2}\ln S_0'\ . \tag{3.13b}$$

The general solution Ψ_e, which satisfies the initial conditions of Eqs. (2.9), then takes the form

$$\Psi_e(t)=\tfrac{1}{2}T_-(0)T_-(t)\exp\left\{+\frac{i}{\hbar}\int_0^t\left[\phi^2(t)+\left[\frac{\Delta\sin 2\theta}{2}\right]^2\right]^{1/2}dt\right\}$$
$$+\tfrac{1}{2}T_+(0)T_+(t)\exp\left\{-\frac{i}{\hbar}\int_0^t\left[\phi^2(t)+\left[\frac{\Delta\sin 2\theta}{2}\right]^2\right]^{1/2}dt\right\}\ , \tag{3.14a}$$

where we have defined

$$T_\pm(t)\equiv\left[1\pm\frac{\phi}{\left[\phi^2+\left[\frac{\Delta\sin 2\theta}{2}\right]^2\right]^{1/2}}\right]^{1/2}\ . \tag{3.14b}$$

Using Eq. (3.14a) we can calculate the probability of an electron neutrino created at $t=0$, to remain as an electron neutrino once it leaves matter (e.g., the Sun). In vacuum we have $\phi = -\Delta\cos 2\theta/2$. Inserting this value into Eq. (3.14b), we find that in vacuum $T_\pm$ take the values

$$T_\pm(t)\,|_{\text{vacuum}} = (1 \mp \cos 2\theta)^{1/2} \; . \tag{3.15}$$

Substituting Eq. (3.15) into (3.14a), one obtains the probability

$$P(\nu_e \to \nu_e) = \frac{1}{2}\left[\left[1 - \frac{\phi\cos 2\theta}{[\phi^2 + (\Delta\sin 2\theta/2)^2]^{1/2}}\right] + \frac{1}{2}\frac{\Delta\sin^2 2\theta}{[\phi^2+(\Delta\sin 2\theta/2)^2]^{1/2}}\cos\left\{\frac{2}{\hbar}\int_0^t\left[\phi^2(t) + \left[\frac{\Delta\sin 2\theta}{2}\right]^2\right]^{1/2} dt\right\}\right] \; . \tag{3.16}$$

Hence if we begin with 100% electron neutrinos in matter before the resonance region, the average probability of them remaining as electron neutrinos after they leave matter is

$$\langle P(\nu_e \to \nu_e)\rangle = \tfrac{1}{2}[1 - \mathcal{R}_0 \cos 2\theta/(1+\mathcal{R}_0^2)^{1/2}] \; , \tag{3.17a}$$

where

$$\mathcal{R}_0 = 2\phi/\Delta\sin 2\theta \; . \tag{3.17b}$$

Equation (3.17a) was previously obtained by Barger, Phillips, and Whisnant in the context of the adiabatic approximation [cf. Eq. (13) of Ref. 11].

In Fig. 1 we plot Eq. (3.14a) for a number of angles. We see that it is a good approximation for larger angles, but it breaks down for smaller angles as stated earlier.

IV. MATTER-ENHANCED NEUTRINO OSCILLATIONS FOR SMALL VACUUM MIXING ANGLES

In the previous section we have shown that the primitive WKB approximation for matter-enhanced neutrino oscillations is equivalent to the adiabatic approximation and, hence, its region of validity is restricted to large [i.e., those satisfying the inequality in Eq. (3.6)] mixing angles. In this section we introduce another approximation, which is valid for small mixing angles.

We choose

$$g \equiv (1 - \cos 2\theta) \tag{4.1}$$

as our expansion parameter. Using Eq. (4.1), Eq. (2.8a) takes the form

$$-\hbar^2 \frac{\partial^2 \Psi_e}{\partial t^2} - \left[\Phi^2 + i\hbar\Phi' + g\frac{\Delta^2}{2}\zeta\right]\Psi_e = 0 \; , \tag{4.2}$$

where we have defined

$$\Phi \equiv \phi(\theta = 0) = \frac{\Delta}{2}(\zeta - 1) \; . \tag{4.3}$$

We solve Eq. (4.2) using the logarithmic perturbation theory,[30] i.e., by expanding $\ln\Psi_e$ in a power series in g. Note that the WKB approximation studied in the previous section can also be thought of as a logarithmic perturbation, where the expansion parameter is $\hbar$.

We proceed, as in the last section, by making the substitution

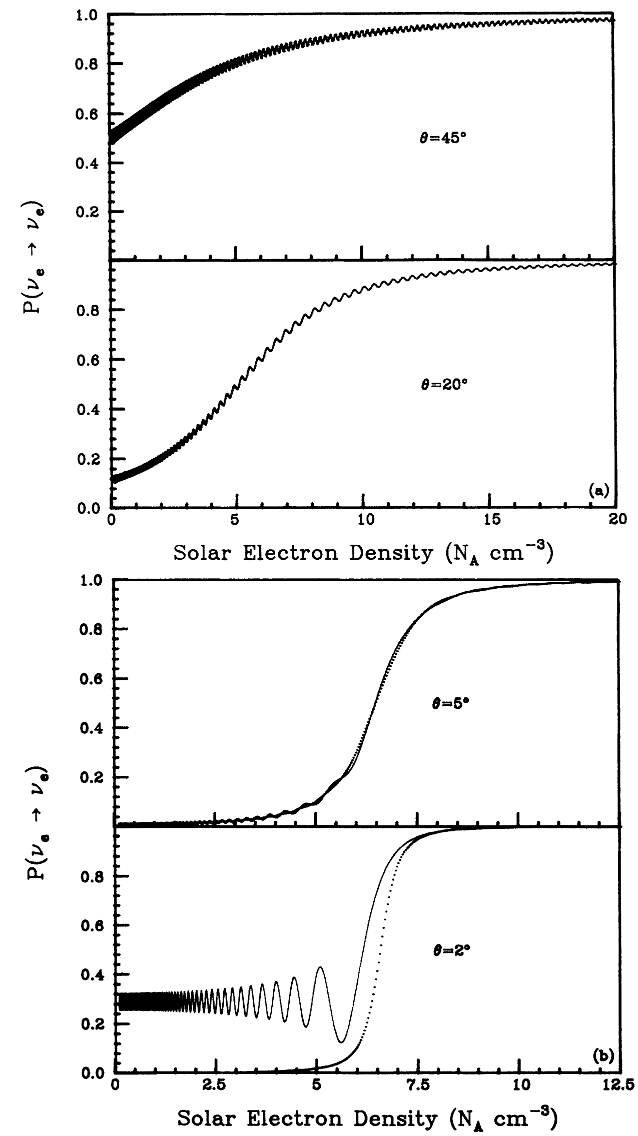

FIG. 1. Probabilities for an electron neutrino to remain an electron neutrino for several different mixing angles. The solid line represents the exact values and the dotted line shows the results of the simple WKB approximation given by Eq. (3.14a). For definiteness, we have chosen an exponential solar-electron distribution $\zeta = \zeta_0 e^{-r/r_0}$ with $\zeta_0 = 137 N_A$ cm^{-3}, $r_0 = 0.092 \times R_{\text{Sun}} = 6.4032 \times 10^8$ m, and the ratio of neutrino mass difference to neutrino energy to be $\delta m^2/E = 10^{-12}$ eV. (a) Results for mixing angles of 45° and 20°. (b) Results for mixing angles of 5° and 2°.

$$\Psi_e = e^S \ . \tag{4.4}$$

We then expand the derivative of the argument of the exponential in powers of g:

$$S' = c_0 + g c_1 + g^2 c_2 + \cdots \ . \tag{4.5}$$

Inserting Eqs. (4.4) and (4.5) into (4.2) and equating the same powers of g, we get differential equations for $c_0, c_1, c_2, \ldots$:

$$-\hbar^2(c_0' + c_0^2) = (\Phi^2 + i\hbar\Phi') \ , \tag{4.6a}$$

$$-\hbar^2(c_1' + 2c_0 c_1) = \frac{\Delta^2}{2}\zeta \ , \tag{4.6b}$$

$$-\hbar^2(c_2' + c_1^2 + 2c_0 c_2) = 0 \ , \tag{4.6c}$$

and so on. The initial conditions, Eqs. (2.9a) and (2.9c), can be rewritten as

$$S(t=0) = 0 \tag{4.7a}$$

and

$$S'(t=0) = -\frac{i}{\hbar}[\Phi(t=0) + \tfrac{1}{2} g\Delta] \ . \tag{4.7b}$$

The initial condition of Eq. (4.7a) can be satisfied by choosing the integration constant appropriately as one calculates S from S'. To satisfy Eq. (4.7b) we choose

$$c_0(t=0) = -\frac{i}{\hbar}\Phi(t=0) \ , \tag{4.8a}$$

$$c_1(t=0) = -\frac{i}{2\hbar}\Delta \ , \tag{4.8b}$$

$$c_2(t=0) = 0, \ldots \ . \tag{4.8c}$$

Using these initial conditions, Eqs. (4.6) can be easily solved to yield

$$c_0(t) = -\frac{i}{\hbar}\Phi(t) \ , \tag{4.9a}$$

$$c_1(t) = -\frac{\Delta^2}{2\hbar^2} e^{+iQ(t)} \int_0^t dt' e^{-iQ(t')} - \frac{i\Delta}{2\hbar} \ , \tag{4.9b}$$

and

$$c_2(t) = -e^{+iQ(t)} \int_0^t dt' c_1^2(t') e^{-iQ(t')} \ , \tag{4.9c}$$

where we have defined

$$Q(t) = \frac{2}{\hbar}\int_0^t \Phi(t') dt' \ . \tag{4.10}$$

The solution of Eq. (4.2) is given as

$$\Psi_e(t) = \exp\left[\int_0^t dt' c_0(t') + g\int_0^t dt' c_1(t') + g^2 \int_0^t dt' c_2(t') + \cdots \right] , \tag{4.11}$$

which can be calculated to any desired accuracy by including the appropriate number of terms.

For very small mixing angles, where we can neglect contributions which are second order in g, Eq. (4.11) takes a particularly simple form:

$$\Psi_e(T) = \exp\left[-\frac{iQ(T)}{2} - g\frac{i\Delta T}{2\hbar} - g\frac{\Delta^2}{2\hbar^2}\int_0^T dt\, e^{+iQ(t)}\int_0^t dt' e^{-iQ(t')}\right] + O(g^2) \tag{4.12}$$

from which one can calculate the probability of an electron neutrino created at $t=0$, to remain as an electron neutrino at $t=T$:

$$P(\nu_e \to \nu_e) = \exp\left[-g\frac{\Delta^2}{2\hbar^2}\left|\int_0^T dt\, e^{iQ(t)}\right|^2\right] + O(g^2) \ . \tag{4.13}$$

Results obtained using Eq. (4.13) for $P(\nu_e \to \nu_e)$ for several mixing angles are plotted in Fig. 2, where they are compared to the exact solution. As expected, the agreement is very good for smaller mixing angles, but the approximation fails for larger angles.

As was emphasized by Rosen and Gelb,[10] for extremely small mixing angles ($g \ll 1$), the Mikheyev-Smirnov-Wolfenstein (MSW) enhancement region is very small. Consequently, for such small values of θ (or of g), the integral in Eq. (4.13) can be evaluated using the stationary phase approximation.[31] From Eq. (4.10), we obtain the stationary point t_R, as a solution to the equation

$$Q'(t_R) = \frac{\Delta}{\hbar}[\zeta(t_R) - 1] = 0 \ , \tag{4.14}$$

which, as expected, is the point at which the MSW enhancement occurs for very small angles [in general, $\zeta(t_R) = \cos 2\theta$ as was discussed in Sec. II]. For a monotonically decreasing electron density there is only one such stationary point and so we can approximate the integral in Eq. (4.13) as

$$\int_0^T dt\, e^{iQ} \approx \left[\frac{2\pi\hbar}{\Delta|\zeta'(t_R)|}\right]^{1/2} e^{i\pi/4} e^{iQ(t_R)} \ . \tag{4.15}$$

Substituting Eq. (4.15) into (4.13) we obtain

$$P(\nu_e \to \nu_e) \approx \exp\left[-\frac{g\pi\Delta}{|\zeta'(t_R)|}\right] + O(g^2) \ . \tag{4.16}$$

Since for small angles $g = 2\theta^2$ and $\zeta(t_R) = 1$, we can rewrite the above result as

$$P(\nu_e \to \nu_e) \approx \exp\left[-\frac{2\pi\theta^2\Delta}{|\zeta'(t_R)/\zeta(t_R)|}\right] + O(g^2) \ , \tag{4.17}$$

which is the result obtained in the Landau-Zener approximation [compare Eq. (4.17) with Eq. (9) of Ref. 14 and Eq. (20a) of Ref. 16 for small mixing angles]. Of course, Eq. (4.11) is valid for larger values of θ, where Eq. (4.17) fails.

V. UNIFORM SEMICLASSICAL ANALYSIS OF MATTER-ENHANCED NEUTRINO OSCILLATIONS

The primitive WKB and small-angle approximations outlined in the previous sections work well in their respective regions of applicability, but there might exist a range of angles between these two regions where neither approximation is appropriate. An approximation which is uniformly valid for all mixing angles would be free of such a limitation. The uniform semiclassical approach allows the WKB method to be extended to the region of small angles despite the fact that the turning points of the problem are very near the real axis, thus providing an approximation that is valid over the entire angular range. As is also the case with the other approximations presented in this paper, this method is not restricted to any particular choice of density distribution.

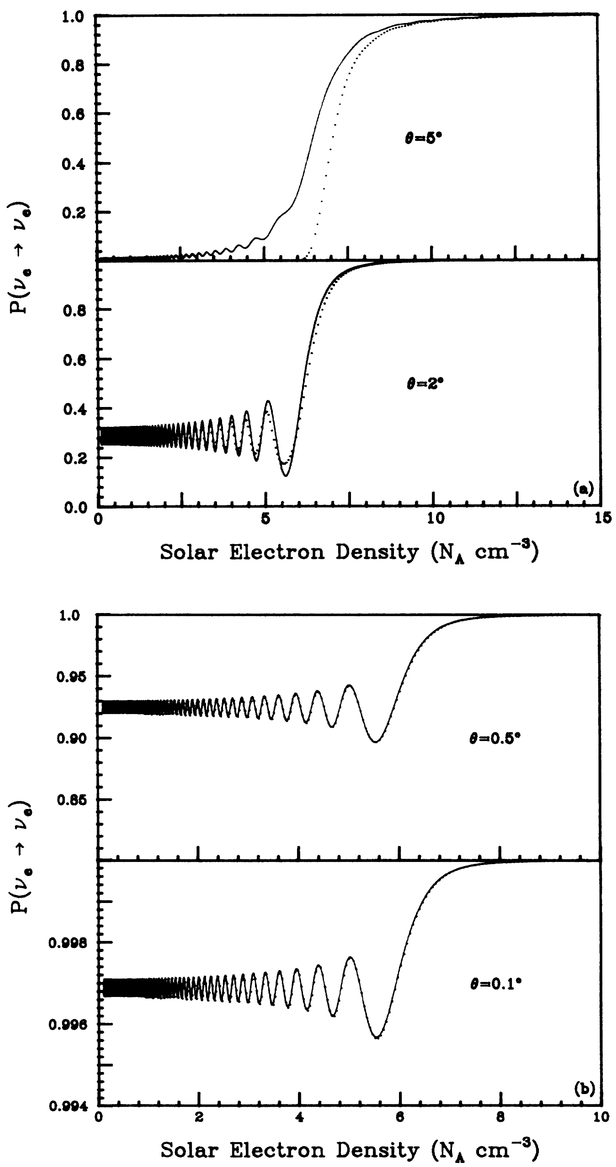

FIG. 2. Probabilities for an electron neutrino to remain an electron neutrino for several different mixing angles. The solid line represents the exact values and the dotted line shows the results of the small-angle approximation given by Eq. (4.13). We have used the same parameters as were used in Fig. 1. (a) Results for mixing angles of 5° and 2°. (b) Results for mixing angles of 0.5° and 0.1°.

In this section we apply the uniform semiclassical approximation to the problem of matter-enhanced neutrino oscillations. We take advantage of the "hidden" supersymmetry in the equation governing the time development of the neutrino probability by using the supersymmetry-inspired uniform approximation.[23] This method has the advantage of separating the purely real $\phi^2(t)$ term from the purely imaginary $\phi'(t)$ term in Eqs. (2.8a) and (2.8b). The supersymmetric uniform approximation outlined in Ref. 23 was applied there to the problem of bound-state wave functions in a potential well. With small modifications, the method can be applied to the case of above barrier scattering which is formally equivalent to the matter-enhanced neutrino oscillation problem considered here.

As we described in Sec. II, the propagation of the electron neutrino in matter is described by the equation

$$-\hbar^2\frac{\partial^2\Psi_e}{\partial t^2}-[\phi^2(t)+i\hbar\phi'(t)]\Psi_e(t)=\Lambda\Psi_e(t)\ , \tag{5.1}$$

where we have defined

$$\Lambda\equiv\left[\frac{\Delta\sin 2\theta}{2}\right]^2 .$$

We wish to solve Eq. (5.1) by using the approximation

$$\Psi_e(t)\sim K(t)U(S(t))\ , \tag{5.2}$$

where the function $U(S(t))$ is a known solution of the mapping equation

$$-\hbar^2\frac{\partial^2 U(S)}{\partial S^2}-(S^2+i\eta\hbar)U(S)=\Omega U(S)\ . \tag{5.3}$$

In Eq. (5.3), the value of η is chosen to be ± 1 depending on whether ϕ' is positive or negative. If we ignore the explicitly $\hbar$-dependent terms in the effective "potentials" of Eqs. (5.1) and (5.3) we see that the turning point topologies of the "local momenta" are identical as long as $\Omega>0$. This equivalence of turning point topology (here the topology is that of complex-conjugate pairs) is critical in developing the correct uniform approximation to the solution of Eq. (5.1). Note that if the mapping "potential" were chosen to be a constant (instead of S^2), the primitive WKB approximation of Sec. III would be obtained.

Substituting the assumed form of Eq. (5.2) for $\Psi_e(t)$ into Eq. (5.1), utilizing the fact that $U(S)$ satisfies Eq. (5.3), and making the choice

$$K(t)=[S'(t)]^{-1/2} \tag{5.4}$$

simplifies the problem to that of solving

$$\hbar^2\frac{K''}{K}-(S')^2(\Omega+i\eta\hbar+S^2)+[\Lambda+\phi^2(t)]+i\hbar\phi'(t)=0\ . \tag{5.5}$$

Now, in the spirit of the WKB method, we expand the function $S(t)$ in a power series of $\hbar$ and keep only the two lowest-order terms

$$S(t) \approx S_0(t) + \hbar S_1(t) \ . \tag{5.6}$$

Substituting Eq. (5.6) into (5.5) and separating powers of $\hbar$ leads to the two equations

$$[\Lambda + \phi^2(t)] = (\Omega + S_0^2)(S_0')^2 \tag{5.7}$$

and

$$i\phi'(t) = 2(\Omega + S_0^2) S_0' S_1' + (S_0')^2 (i\eta + 2S_0 S_1) \ , \tag{5.8}$$

where only terms up to first order in $\hbar$ have been retained. Equation (5.7) can be immediately solved to give

$$\int^{S_0(t)} \sqrt{\Omega + \sigma^2}\, d\sigma = \int^{t} \sqrt{\Lambda + \phi^2(y)}\, dy \ . \tag{5.9}$$

Choosing the zeros of $\Omega + S_0^2(t)$ to correspond to those of $\Lambda + \phi^2(t)$ produces a wave function that is uniformly valid at the turning points and away from them. In particular, we choose the turning point t_0 such that $\phi(t_0) = +i\sqrt{\Lambda}$ corresponds to $S_0(t_0) = +i\sqrt{\Omega}$ and $\phi(t_0^*) = -i\sqrt{\Lambda}$ corresponds to $S_0(t_0^*) = +i\sqrt{\Omega}$ where t_0 has a positive imaginary part. With these choices, Ω is determined by

$$\int_{t_0}^{t_0^*} \sqrt{\Lambda + \phi^2(y)}\, dy = \frac{-i\Omega\pi}{2} \tag{5.10}$$

and $S_0(t)$ by

$$\int_{t_0}^{t} \sqrt{\Lambda + \phi^2(y)}\, dy = \int_{i\sqrt{\Omega}}^{S_0(t)} \sqrt{\Omega + \sigma^2}\, d\sigma \ . \tag{5.11}$$

Note also that, to lowest order in $\hbar$, Eq. (5.4) becomes

$$K(t) \equiv [S_0'(t)]^{-1/2} = \left[\frac{\Omega + S_0^2(t)}{\Lambda + \phi^2(t)} \right]^{1/4} . \tag{5.12}$$

Now Eq. (5.8) can be solved for $S_1(t)$ with the result

$$S_1(t) = \frac{i}{2[\Omega + S_0^2(t)]^{1/2}} \left[\ln \frac{\phi(t) + [\phi^2(t) + \Lambda]^{1/2}}{i\sqrt{\Lambda}} - \eta \ln \frac{i\sqrt{\Omega}}{S_0(t) + [S_0^2(t) + \Omega]^{1/2}} \right] . \tag{5.13}$$

From Eqs. (5.10), (5.11), and (5.13) we see that if t is real, then $S_0(t)$ is also real and $S_1(t)$ is imaginary.

The solutions of the mapping equation are the parabolic cylinder functions

$$D_\nu(\pm(1+i)S(t))$$

with $\nu = -(1 - \eta + i\Omega/2)$. Therefore, we can write the neutrino probability amplitude as

$$\Psi_e(t) = K(t)[A D_\nu(y(t)) + B D_\nu(-y(t))] \ , \tag{5.14}$$

where $y(t) = (1+i)S(t)$. The constants A and B are to be determined by the initial conditions. With the initial conditions of Eqs. (2.9), the approximate solution is found to be

$$\Psi_e(t) = \frac{K(t)}{K(0)} \frac{\Gamma(-\nu)}{\sqrt{2\pi}} \Bigg[-D_\nu(y(t)) D_\nu'(-y(0)) - D_\nu(-y(t)) D_\nu'(y(0)) + \left[-i\phi(0) - \frac{K'(0)}{K(0)} \right] [D_\nu(-y(t)) D_\nu(y(0)) - D_\nu(y(t)) D_\nu(-y(0))] \Bigg] , \tag{5.15}$$

where $\Gamma(-\nu)$ is the gamma function and a prime denotes a derivative with respect to the argument of the parabolic cylinder functions. For large enough t [when $y(t)$ gets large], it is possible to replace the parabolic cylinder functions appearing in Eq. (5.15) with their asymptotic forms.

We should contrast our result with that obtained by Haxton.[13,16] Equation (5.15) is formally very similar to Haxton's result,[16] derived using the Landau-Zener approximation, where the matter density is replaced by a linear function of t that has the correct magnitude and first derivative at the crossing point. Our result is derived for an arbitrary matter density, provided that the turning point topology discussed earlier is realized. For a monotonically decreasing density that would be the case. Indeed, for the special case of a linear matter density, Eq. (5.15) reduces to the result given in Ref. 16.

Figure 3 compares the results of Eq. (5.15) with the exact results for several mixing angles. Note that the uniform method provides an excellent approximation to the exact result over the entire range of mixing angles, in contrast with the primitive WKB and the small-angle approximations.

VI. CONCLUSIONS

In this paper we studied matter-enhanced neutrino oscillations for the case of two neutrino flavors and presented three different approximation methods for calculating unaveraged oscillation probabilities. The first approximation is a WKB approximation which exploits a formal analogy between supersymmetric quantum mechanics and neutrino oscillations for two flavors. We showed that the condition of validity for this approximation is the same as the adiabatic condition, so that it is applicable for large [as defined by Eq. (3.6)] vacuum mixing angles. The second approximation is achieved within the context of logarithmic perturbation theory by taking the

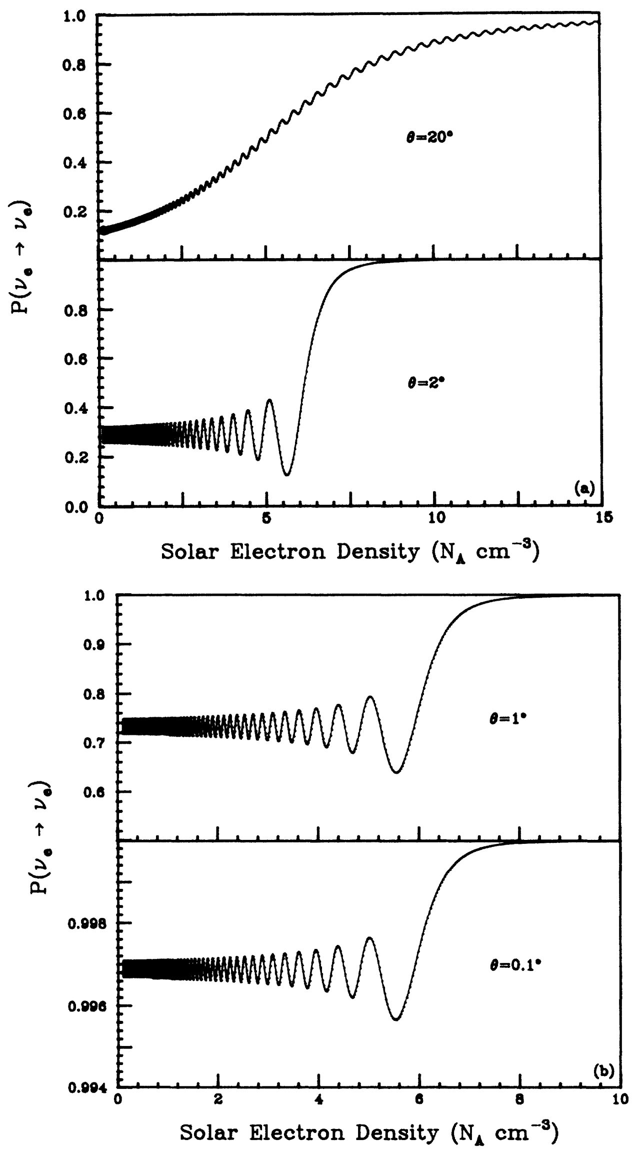

FIG. 3. Probabilities for an electron neutrino to remain an electron neutrino for several different mixing angles. The solid line represents the exact values and the dotted line shows the results of the uniform semiclassical approximation given by Eq. (5.15). We have used the same parameters as were used in Fig. 1. (a) Results for mixing angles of 20° and 2°. (b) Results for mixing angles of 1° and 0.1°.

vacuum mixing angle as the expansion parameter; hence, it is designed to be valid for small vacuum mixing angles. Both of these approximations are numerically much simpler and faster than the exact numerical integration of the equations describing neutrino oscillations. The third approximation is another WKB approximation uniformized to be valid for all mixing angles. In their respective regions of validity, the approximate calculations using these methods are found to be in very good agreement with the exact calculations.

Our approximations give the unaveraged neutrino oscillation probability in all three cases. Upon averaging they yield previously obtained results. However, since we have simple expressions for the unaveraged oscillation probability, we expect that they will be especially suitable to use when the oscillation length is comparable to the distance traveled by neutrinos in matter. (For neutrinos traveling through the Earth that might be the case.) In such cases our expressions could provide simple estimates of the expected neutrino capture rate.

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APPENDIX: SUPERSYMMETRY OF NEUTRINO OSCILLATIONS WITH TWO FLAVORS

To demonstrate the "hidden" supersymmetry of the problem of two flavors of neutrinos propagating through matter, we introduce the operators

$$B_{\pm}=i\hbar\frac{\partial}{\partial t}\pm\phi(t) \ . \tag{A1}$$

Using Eq. (A1), Eqs. (2.5) can be written as

$$H_e\Psi_e\equiv B_+B_-\Psi_e=\left[\frac{\Delta\sin 2\theta}{2}\right]^2\Psi_e \tag{A2a}$$

and

$$H_\mu\Psi_\mu\equiv B_-B_+\Psi_\mu=\left[\frac{\Delta\sin 2\theta}{2}\right]^2\Psi_\mu \ . \tag{A2b}$$

Introducing

$$H=\begin{bmatrix} H_e & 0 \\ 0 & H_\mu \end{bmatrix}=\begin{bmatrix} B_+B_- & 0 \\ 0 & B_-B_+ \end{bmatrix} , \tag{A3}$$

Eqs. (A2a) and (A2b) can be written together as

$$H\begin{bmatrix} \Psi_e \\ \Psi_\mu \end{bmatrix}=\left[\frac{\Delta\sin 2\theta}{2}\right]^2\begin{bmatrix} \Psi_e \\ \Psi_\mu \end{bmatrix} . \tag{A4}$$

By further defining the operators

$$Q_1=\sqrt{2}\begin{bmatrix} 0 & B_+ \\ B_- & 0 \end{bmatrix} \tag{A5a}$$

and

$$Q_2=i\sqrt{2}\begin{bmatrix} 0 & B_+ \\ -B_- & 0 \end{bmatrix} , \tag{A5b}$$

one can show that the following commutation and anticommutation relations are satisfied:

$$\{Q_i,Q_j\}=2\delta_{ij}H; \quad i,j=1,2 \ , \tag{A6a}$$

$$[H,Q_i]=0; \quad i,j=1,2 \ . \tag{A6b}$$

Equations (A6) are the commutation and anticommutation relations of the superalgebra SU(1/1), realized here in a non-Hermitian representation.

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