An improved presentation of a novel perturbative scheme in field theory

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Bender *et al.* have recently proposed a novel perturbative scheme to analyze self-interacting scalar field theories. Calculations, however, require the use of complicated polynomial interactions for which no general expression has been found. The answers obtained have then to be acted on by a complex differential operator. We propose an alternative approach involving an interaction Lagrangian of only two terms, which presents the whole scheme in a simple way, as well as having some calculational advantage.

A new perturbative scheme has been suggested,[1,2] for calculations in self-interacting scalar field theories with an interaction term of the form ϕ^{2n}. By writing this as $(\phi^2)^{1+\delta}$ it is possible to derive an expansion in powers of δ. The interaction term is

$$L_I=\lambda(\phi^2)^{1+\delta}\mu^{2+(2-n)\delta}\ ,$$

where μ is a mass parameter taking care of the dimensions, n is the number of space-time dimensions and λ is the dimensionless coupling constant. This can be expanded as a power series in δ:

$$L_I=\lambda\mu^2\left[\phi^2+\frac{\delta}{2!}\phi^2\ln(\phi^2\mu^{2-n})+\frac{\delta^2}{3!}\phi^2\ln^2(\phi^2\mu^{2-n})+\cdots\right]. \tag{1}$$

The first term, being only quadratic in ϕ, is included in the free Lagrangian; the rest of the series is the interaction term. Although at first sight progress seems impossible, because of the nonpolynomial interaction terms, it is possible to formulate a diagrammatic expansion using the relation

$$\lim_{\alpha\to 0}\frac{\partial}{\partial\alpha}(\phi^2)^{1+\alpha}=\phi^2\ln\phi^2\ . \tag{2}$$

In Ref. 2 it was shown that to perform a calculation to $O(\delta^k)$, it is possible to write down a polynomial interaction term involving k terms of the form $(\phi^2)^{\alpha_i+1}$ with α_i some parameters which for the moment are taken to be integers. The coefficients of these interaction terms are polynomials in δ and the α_i. One then computes the relevant Feynman diagrams for the quantity in which one is interested, obtaining an answer of $O(\delta^k)$ depending on the parameters α_i. Then, regarding the parameters α_i as continuous one acts on the answer with the differential operator:

$$D=\frac{1}{k}\sum_{m=1}^{k}\sum_{n=1}^{k}\frac{1}{m!}e^{2\pi i m(1-n)/k}\left[\frac{\partial}{\partial\alpha_n}\right]^m \tag{3}$$

finally setting $\alpha_1=\alpha_2=\cdots=\alpha_k=0$. This is patently a very complicated procedure. The interactions have coefficients which are, in general, not simple, and their exact form for arbitrary k is not known.

This can all be avoided by noting the almost trivial fact that the new interaction term

$$L_I'=\lambda\mu^2\left[\frac{\delta}{2!}\phi^2\ln(\phi^2\mu^{2-n})+\frac{\delta^2}{3!}\phi^2\ln^2(\phi^2\mu^{2-n})+\cdots\right] \tag{4}$$

can be written as

$$L_I=\lambda\mu^2[(\phi^2)^{1+\delta}\mu^{\delta(2-n)}-\phi^2]\ . \tag{5}$$

That is, we simply add and subtract the term of zeroth order in δ, adding it to the free Lagrangian, and subtracting it from the interaction term. The important thing to note is that the new interacting term is now of $O(\delta)$. Thus to calculate to $O(\delta^k)$, we need only include diagrams involving up to power k in L_I, i.e., involving *up to k vertices.* To proceed, we first regard δ as an integer, and calculate all the relevant diagrams with up to k vertices. Equation (5) gives rise to two types of vertex, one with $2\delta+1$ external lines, the other with two. Having obtained an answer which is a function of δ, we then regard it as a continuous variable by analytic continuation, and expand the answer to $O(\delta^k)$. It is easily verified that previous calculations can be reproduced.

Note that L_I, Eq. (5), is proportional to λ, so a diagram with k vertices is multiplied by an explicit λ^k. Thus it is possible to use λ to count the number of vertices in the method outlined above. However since $2\lambda\mu^2$ appears as a mass term in the propagator, it is probably best just to count the number of vertices "by hand."

Whether one uses the original approach, or the method outlined here, identical diagrams will appear and have to be calculated. The calculational advantage of our approach, however, lies in the "bookkeeping." In calculating to $O(\delta^k)$, instead of the k parameters α_i, we only have one, namely δ, and expanding an expression as a power series in δ is somewhat simpler than acting with the differential operator D, Eq. (3). This procedure also provides a check on the calculation, since all the diagrams with a fixed number n of vertices, must give an answer which is of $O(\delta^n)$. Thus the terms of order $1,2,\ldots,n-1$ must cancel between these diagrams. This is a nontrivial check, since each individual diagram is of zeroth order in δ.

Since our approach allows a simpler formulation of this expansion in δ, we hope it may be of use in shedding some light upon the possible advantages[1–4] of this new perturbative scheme.

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