On the existence of localized solutions in self-consistent Yang-Mills-matter systems

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(Received 6 August 1986)

Under the assumption of a compact gauge group, localized solutions are absent for Yang-Mills fields coupled to either non-Higgs-scalar fields or (under certain additional conditions) linear spinor fields.

I. INTRODUCTION

It has already been established that for certain external charges one finds solutions of the Yang-Mills equations which screen the external charge so that the total charge vanishes.[1] This phenomenon is called tōtal color screening. The controversy of whether or not the effect is merely a gauge effect[2] was eventually explained in Ref. 3. In our opinion, a more fundamental point may be the objection to the occurrence of external sources in the theory which is expected to describe elementary processes. The old Occam's principle should be observed also in Yang-Mills theory, especially as its classical part has no direct physical applications.

This gives us the incentive to study the existence of localized solutions in self-contained Yang-Mills-matter field theory, where sources appear in a dynamical way. The results obtained below almost exclude the possibility of having finite-energy localized solutions and encourage us to formulate the following.

Conjecture. Finite-energy localized solutions are absent in any "reasonable" self-consistent gauge-matter systems provided that (i) the gauge group is compact, (ii) spinor fields do not interact directly (i.e., they are linear), and (iii) scalar fields are non-Higgs type. The word "reasonable" needs a clarification which is not fully given in this paper. We hope, however, that it will be possible to prove the strong form of the conjecture without any additional assumptions. We have to point out that conditions (i) and (iii) seem to be inevitable. The compactness of a gauge group is necessary to ensure that the kinetic energy of a gauge field is positive definite, which is important for the strategy of our proof. The exclusion of Higgs fields is necessary because, in this case, the localized solutions are known explicitly [Bogomolny-Prasad-Sommerfield (BPS) monopoles].

The order of the rest of this work is as follows. In Sec. II we introduce the models of interest and describe our notation. Section III comprises a partial proof of the conjecture. For scalar fields coupled to Yang-Mills potentials the conjecture has been proven earlier.[4] In the fermion counterpart we make use of a recently found inequality concerning the kinetic energy of fermion fields.

II. STATEMENT OF THE MODEL

In the part concerning scalar fields we have to explain only the term "non-Higgs type." Let ψ denote a scalar field in the fundamental or adjoint representation of a gauge group which is minimally coupled to a gauge potential A^a_μ and let V be a part of the Lagrangian describing self-interaction of a scalar field. We say that $V(\psi)$ is non-Higgs type if

$$\psi^\dagger V'(\psi)\geq 0 \; ; \tag{1}$$

here $V'(s)=(d/ds)V(s)$. This generalizes the standard definition according to which a field is said to be non-Higgs type if $V(s)$ has only one extremum, a minimum at $s=0$, and $V(0)=0$.

The Yang-Mills-Dirac equations read

$$\partial^\mu F^a_{\mu\nu}+gf_{abc}A^{\mu b}F^c_{\mu\nu}=g\bar{\psi}^s{}_A\gamma^\nu{}_{AB}\tfrac{1}{2}T^a_{st}\psi^t_B \; , \tag{2a}$$

$$\gamma^\mu{}_{AB}\partial_\mu\psi^s_B+m\psi^s_A-ig\gamma^\mu{}_{AB}A^a_\mu\tfrac{1}{2}T^a_{st}\psi^t_B=0 \; . \tag{2b}$$

Here, the following notation is used: f_{abc} are the structure constants of a gauge group G, the field-strength tensor $F^a_{\mu\nu}$ is defined by

$$F^a_{\mu\nu}=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+gf_{abc}A^b_\mu A^c_\nu \; , \tag{3}$$

the T's are the gauge group generators. The indices $a,b,c,\ldots=1,\ldots,n$ are the gauge group indices in the adjoint representation, the indices $s,t,u=1,\ldots,N$ are the gauge group indices for the fundamental representation, $A,B,C=1,2,3,4$ are the Dirac-bispinor indices, and $\mu,\nu,\rho=0,1,2,3$ are the Minkowski indices. The indices i,j,k will be used as space indices. The Dirac matrices are, in the spinor representation,

$$\gamma^0=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, \quad \gamma^i=\begin{bmatrix}0 & -\sigma^i\\ \sigma^i & 0\end{bmatrix},$$

where σ's are the Pauli matrices.

III. PROOFS

We begin with a simple observation that to have classical color screening in self-contained systems one needs localized solutions. Thus, we will consider the existence of localized solutions to various Yang-Mills-matter systems.

Let us discuss Yang-Mills fields minimally coupled to non-Higgs scalar fields. The following important result is due to Glassey and Strauss.[4]

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Theorem 1. Let $V(\psi)$ describe the self-interaction of non-Higgs scalar fields in the Lagrangian. Assume that gauge and scalar fields decay at spatial infinity sufficiently fast to ensure the finiteness of the energy-momentum and angular-momentum tensors of the whole Yang-Mills-matter system. Then the localized time-dependent solutions are absent if either (i) $sV'(s)\geq 4V(s)\geq 0$ (for massless scalar fields), or (ii) $sV'(s)\geq 2V(s)\geq 0$ (for massive scalar fields).

Theorem 1 implies also the absence of static finite-energy solutions.[4] The approach of Glassey and Strauss is based on conservation laws of the energy-momentum and angular momentum and therefore requires too strong boundary conditions at spatial infinity (e.g., that the potentials fall off more quickly than $1/r$). In the static case, however, one easily obtains the same results for standard boundary conditions $\psi=O(1/r^{+3/2+\epsilon})$, $A^a_\mu=O(r^{-3/4-\epsilon})$ (as $r\to\infty$) and under weaker conditions on $V(\psi)$, $sV'(s)\geq 0$ (Ref. 5).

Thus the only models where localized solutions could exist are fermion fields coupled to non-Abelian gauge potentials. Below we limit our interest to static Yang-Mills-Dirac fields. A result of Magg[6] states the absence of finite-energy solutions of static Yang-Mills-Dirac equations, but it requires the SU(2) gauge group (there is no obvious way to include other gauge groups), massless spinor fields, and spherical symmetry. Therefore the problem of the existence of localized static solutions of Yang-Mills-Dirac theory is still open. A partial solution to the problem was obtained in Ref. 7. We show that the conjecture is valid for an important sector of Yang-Mills-Dirac theory.

Theorem 2. Assume the constant

$$t=g^2\left[\sum_{\alpha,\mu}\int (A^a_\mu)^3 d^3x\right]^{2/3} \tag{4}$$

is sufficiently small, then localized solutions are absent in Yang-Mills-Dirac static theory both in massive and massless cases and for all compact gauge groups.

Proof. For the Abelian compact group the proof is trivial (Appendix A). It happens also that for all compact semisimple gauge groups the main idea of the proof remains intact. Therefore we assume the SU(2) gauge group and spinor fields in the fundamental representation of SU(2). After some algebra (Appendix B) one obtains from Eq. (2b) the following identity:[7]

$$\sum_\mu\int[(\nabla\phi^M_\mu)^2+m^2(\phi^M_\mu)^2]d^3x =g^2\int\left[\bar{C}^M C^M+\sum_k \bar{D}^M_k D^M_k\right]d^3x\ ; \tag{5}$$

above $M=\pm$ and summation over M is assumed, while the C's and D's are defined by

$$C^M=-A^a_0\frac{M}{2i}\phi^M_a-\frac{1}{2i}A^a_a\phi^M_0-\tfrac{1}{2}\epsilon_{ika}A^a_i\phi^M_k\ ,$$

$$D^M_k=-\frac{M}{2i}\phi^M_0A^k_0-\tfrac{1}{2}\epsilon_{ilk}\phi^M_iA^l_0-\tfrac{1}{2}\epsilon_{iak}A^a_i\phi^M_0 -\frac{1}{2i}A^k_a\phi^M_a+\frac{1}{2i}A^a_a\phi^M_k-\frac{1}{2i}A^a_k\phi^M_a\ .$$

The ϕ's are related to the bispinor ψ by

$$\psi^1\equiv\psi^+,\quad \psi^2\equiv\psi^-,\quad \psi^\pm\equiv\phi^\pm_\mu\sigma^\mu\sigma^2\ .$$

By the use of the Hölder, Minkowski, and Sobolev inequalities one arrives at the following inequality (Appendix C):

$$E\leq ctE\ , \tag{6}$$

where E denotes the left-hand side of (5), c is a constant (which, in principle, can be calculated), and t is defined in the above theorem. Quite analogous inequalities are obtained for other gauge groups with the constant c being dependent only on the group. It is interesting to note that such inequalities are possible only for $n\geq 3$, where n is a number of spacelike coordinates.

Now assume t is so small that $ct<1$; then $E=0$ and nonzero solutions of Eqs. (2b) are absent. This implies (via the Deser no-go theorem[8]) the absence of static solutions of Eq. (2a). This completes the proof.

The rest of this section is devoted to the justification and interpretation of the condition stated in Theorem 2. The condition that t is small ensures that the nonlinear terms in the equations of motion are small; in the limit $t=0$, Eq. (2a) becomes linear (but not necessarily sourceless, since the dependence of the ψ's on the coupling constant g is almost arbitrary). Thus the condition that t is small can be read off as the selection of a sector of Yang-Mills-Dirac theory which is "asymptotically free." Let us point out that the explicit examples of total color screening[1] also belong to this sector. The theorem proven above can be restated as "classical asymptotic freedom is absent." The classical Yang-Mills-Dirac theory does not mimic the asymptotic freedom known in quantum chromodynamics.

The next problem is the justification of such an asymptotic falloff of potentials that ensures $A^2\in L_3$, which is assumed in (4). The L_3 integrability of a function f does not imply even its vanishing at spatial infinity but, on the other hand, it excludes the $(1/r)$ asymptotic falloff. There are two arguments. At first, total color screening solutions are expected to be short range.[1] Second, assume that the total charge is Lorentz invariant. Then the result of Bizon[9] implies that the nonlinearities appearing in the Yang-Mills equations are L_1 integrable. Now the standard procedure (cf. Ref. 10) gives the asymptotic relations

$$A^a_i=\frac{1}{r^2}+\cdots,\quad A^a_0=\frac{T^a}{r}+\cdots\ ;$$

here T^a denotes isospin components of a total charge. If we assume the total charge to vanish then we obtain the required asymptotics of gauge potentials that ensures the L_3 integrability of potentials. Thus we can conclude that the condition stated in the theorem is quite natural because it selects an asymptotically free sector of Yang-Mills-Dirac fields with the Lorentz-invariant total charge. The preceding theorem says that this sector has no static solutions of the Yang-Mills-Dirac equations. The generalization on the case of stationary fields is easy,[7] although it requires faster decay of gauge fields at spatial infinity than previously assumed.

We expect that localized solutions are absent also in the time-dependent case. To prove this, one should show the absence of localized time-dependent solutions of Yang-Mills-Dirac theory, at least in the asymptotically free sector. The absence of static and stationary localized solutions gives a strong argument in favor of the expectation but, on the other hand, the technique used in this work cannot be employed then.

ACKNOWLEDGMENTS

I am indebted to Graham Hall for reading the manuscript. This work was supported in part under Project No. 01.03.

APPENDIX A

For a single fermion field the time component of the current is either nonpositive or non-negative (being dependent on the sign of the coupling constant e):

$$j^0 = e\psi^\dagger\psi \ . \tag{A1}$$

The Gauss-type law

$$\partial_i E_i = j^0 \tag{A2}$$

gives now $q=\int j^0 dV=0$ for short-ranged fields and therefore implies $\psi=0$.

APPENDIX B

Assume the gauge group SU(2) and set

$$\psi \equiv \begin{bmatrix} \psi^+ \\ \psi^- \end{bmatrix} \equiv \begin{bmatrix} \phi_\mu^+ \sigma^\mu \sigma^2 \\ \phi_\mu^- \sigma^\mu \sigma^2 \end{bmatrix} \ . \tag{B1}$$

Then Eq. (2b) reads[7]

$$\mp\partial_0\phi_0^\mp + \partial_i\phi_i^\mp \pm \frac{1}{2i} g A_0^a \phi_a^\mp - \frac{1}{2i} A_a^a \phi_0^\mp - \tfrac{1}{2} g \epsilon_{ika} A_i^a \phi_k^\mp \mp im\phi_0^\pm = 0 \ , \tag{B2}$$

$$\mp\partial_0\phi_k^\mp + \partial_k\phi_0^\mp - i\epsilon_{ikl}\partial_i\phi_l^\mp - \tfrac{1}{2} g\epsilon_{iak} A_i^a \phi_0^\mp - \frac{1}{2i} g A_a^k \phi_a^\mp + \frac{1}{2i} g A_a^a \phi_k^\mp - \frac{1}{2i} g A_k^a \phi_a^\mp \mp im\phi_k^\pm = 0 \ . \tag{B3}$$

Suppose that the Dirac fields are stationary,

$$\partial_0 \phi_\mu^\mp = i\epsilon\phi_\mu^\mp \ ,$$

while the Yang-Mills potentials are static $\partial_0 A_\mu^a = 0$ and satisfy the Coulomb gauge $\partial_i A_i^a = 0$. Multiplication of Eqs. (B2) and (B3) by $(\partial_k \bar{\phi}_k^- \pm im\bar{\phi}_0^\pm - i\epsilon\bar{\phi}_0^\mp)$ and $(\partial_k\bar{\phi}_0^\mp - i\epsilon_{kil}\partial_l\bar{\phi}_l^\mp \mp i\epsilon\bar{\phi}_k^\mp \pm im\bar{\phi}_k^\pm)$, respectively, integration over R^3, and omission of exact differentials yields Eq. (5) (notice that there $\epsilon=0$, i.e., fermion fields are static).

APPENDIX C

We begin with the observation that the right-hand side of Eq. (5) is bilinear both in the fermionic and gauge fields:

$$\sum_M g^2 \int \left[\bar{C}^M C^M + \sum_k \bar{D}_k^M D_k^M \right] d^3x \equiv g^2 \sum_{\substack{\mu,\nu,\alpha \\ \beta,k,l \\ M}} \int \alpha_{\mu\nu\alpha\beta klM} \bar{\phi}_\mu^M \phi_\nu^M A_\alpha^k A_\beta^l d^3x \ . \tag{C1}$$

Here $\alpha_{\mu\alpha\beta\nu klM}$ are constants while the C's and D's were defined below formula (5).

By the use of Hölder and Minkowski inequalities we estimate the right-hand side of (C1) by

$$g^2 c \sum_{\substack{\mu,\nu \\ \alpha,M}} \left[\int (\phi_\mu^M)^6 d^3x \right]^{1/3} \left[\int d^3x (A_\nu^\alpha)^3 \right]^{2/3} \ . \tag{C2}$$

This in turn is estimated by

$$C' g^2 \sum_{\substack{\mu,M \\ \alpha,\nu}} \left[\int (A_\mu^\alpha)^3 d^3x \right]^{2/3} \int |\phi_\nu^M|^2 d^3x \leq C' g^2 \sum_{\alpha,\nu} \left[\int (A_\nu^\alpha)^3 d^3x \right]^{2/3} E \ . \tag{C3}$$

Here

$$E = \sum_{\mu,M} \int [(\nabla\phi_\mu^M)^2 + m^2(\phi_\mu^M)^2] \ .$$

To get (C3) we have to invest the Sobolev inequality

$$\left[\int (\phi)^6 dV \right]^{1/3} \leq 5.478 \int (\nabla\phi)^2 dV \ ; \tag{C4}$$

the above estimation is the sharpest one.[11]

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