Motion of test particles and light rays around massive conducting cosmic string

Marek Demiański*
Center for Theoretical Physics, Department of Physics, University of Texas at Austin, Austin, Texas 78712
(Received 20 July 1987)

An exact solution of the coupled Einstein-Maxwell equations describing an infinite, straight, massive, conducting cosmic string is presented. Equations of motion of test particles and light rays are derived. Radial motion, motion in a plane perpendicular to the string, and motion along a line parallel to the string are discussed. When the string is represented by a line singularity and its linear mass density is not too large and it is carrying electric current then radially moving test particles very close to the string encounter a repulsive force.

Incorporation of gauge theories of elementary particles into the standard framework of the big-bang cosmological model opened up several interesting possibilities. It turned out that since the big bang the Universe could have undergone a number of phase transitions. These phase transitions can have important cosmological consequences. Zel'dovich, Kobzarev, and Okun[1] and independently Kibble[1] pointed out that phase transitions can produce vacuum domain structures such as domain walls, strings, and monopoles. Recently cosmic strings have attracted a lot of attention (see an excellent review by Vilenkin[3]). It was shown that cosmic strings could bend light rays and produce double images of quasars. In the early Universe strings might produce density perturbations leading to formation of galaxies. Witten[4] has shown that under certain conditions strings could behave like superconducting wires carrying a current of up to 10^{20} A.

Here we would like to investigate motion of neutral test particles and light rays in a space-time of an infinite, straight, massive, superconducting string. Several years ago Witten[5] obtained an exact solution of the coupled Einstein-Maxwell equations which describes a space-time with an infinite, straight current-carrying wire. It is very easy to generalize this solution so that it describes an infinite, straight, massive, conducting cosmic string. If the string is oriented along the z axis then the metric can be written in the form

$$ds^2=\left[\frac{\rho}{\rho_0}\right]^{2a^2}\left[\left[\frac{\rho}{\rho_0}\right]^{a}+A\left[\frac{\rho}{\rho_0}\right]^{-a}\right]^2(dt^2-d\rho^2)$$
$$-\rho^2\left[\left[\frac{\rho}{\rho_0}\right]^{a}+A\left[\frac{\rho}{\rho_0}\right]^{-a}\right]^2\gamma^2d\phi^2$$
$$-\left[\left[\frac{\rho}{\rho_0}\right]^{a}+A\left[\frac{\rho}{\rho_0}\right]^{-a}\right]^{-2}dz^2\,, \qquad (1)$$

where $\gamma=1-4G\mu_F$, μ_F is the linear "field" energy density of the string, $a\approx 2G\mu_M$, μ_M is the linear mass density, $A=I^2G/a^2\gamma^2$, I is the current flowing on the string, ρ_0 is a constant, and G is the gravitational constant. We use units in which the velocity of light c is set to be equal to 1.

When both the current and the linear mass density are zero the solution reduces to the well-known solution representing a cosmic string. The Witten solution is recovered when the linear "field" energy density vanishes. When $I\neq 0$ it is not possible to go to the limit $a\to 0$ unless one allows the current to grow indefinitely so that I/a is finite.

When $I\neq 0$ there is a nonzero magnetic field and the only nonvanishing component of the Maxwell tensor is

$$F_{13}=-\frac{2I}{\rho\gamma}\left[\left[\frac{\rho}{\rho_0}\right]^{a}+A\left[\frac{\rho}{\rho_0}\right]^{-a}\right]^{-2}. \qquad (2)$$

To investigate the asymptotic structure of this space-time for large and small ρ let us introduce the radial geodesic distance $\hat\rho$ related to ρ by

$$\frac{\hat\rho}{\rho_0}=\frac{1}{a^2+a+1}\left[\frac{\rho}{\rho_0}\right]^{a^2+a+1}+\frac{A}{a^2-a+1}\left[\frac{\rho}{\rho_0}\right]^{a^2-a+1}, \qquad (3)$$

where we have normalized the scale so that $\hat\rho=0$ when $\rho=0$. For $\rho\to\infty$ we have

$$\frac{\hat\rho}{\rho_0}\approx\frac{1}{a^2+a+1}\left[\frac{\rho}{\rho_0}\right]^{a^2+a+1}$$

and asymptotically the metric assumes the Kasner form

$$ds^2=\left[\frac{\hat\rho}{\rho_0}\right]^{2p_1}d\tilde t^2-d\tilde\rho^2-\rho_0^2\left[\frac{\hat\rho}{\rho_0}\right]^{2q_1}\tilde\gamma^2d\phi^2$$
$$-\left[\frac{\hat\rho}{\rho_0}\right]^{2r_1}d\tilde z^2\,, \qquad (4)$$

where

$$\tilde t=(a^2+a+1)^{a(a+1)/(a^2+a+1)}t\,,$$
$$\tilde z=(a^2+a+1)^{-a/(a^2+a+1)}z\,,$$
$$\tilde\gamma=(a^2+a+1)^{(1+a)/(a^2+a+1)}\gamma\,,$$
$$p_1=\frac{a(a+1)}{a^2+a+1},\quad q_1=\frac{1+a}{a^2+a+1}\,,$$
$$r_1=\frac{-a}{a^2+a+1}$$

so $p_1+q_1+r_1=1=p_1^2+q_1^2+r_1^2$.

When $\rho\to 0$,

$$\frac{\hat{\rho}}{\rho_0}\approx\frac{A}{a^2-a+1}\left[\frac{\rho}{\rho_0}\right]^{a^2-a+1}$$

and asymptotically the metric assumes the Kasner form

$$ds^2=\left[\frac{\hat{\rho}}{\rho_0}\right]^{2p_2}d\hat{t}^{\,2}-d\hat{\rho}^{\,2}-\rho_0^2\left[\frac{\hat{\rho}}{\rho_0}\right]^{2q_2}\hat{\gamma}^{\,2}d\phi^2-\left[\frac{\hat{\rho}}{\rho_0}\right]^{2r_2}d\hat{z}^{\,2}\,, \tag{5}$$

where

$$\hat{t}=A\left[\frac{a^2-a+1}{A}\right]^{a(a-1)/(a^2-a+1)}t\,,$$

$$\hat{\gamma}=A\left[\frac{a^2-a+1}{A}\right]^{(1-a)/(a^2-a+1)}\gamma\,,$$

$$\hat{z}=A^{-1}\left[\frac{a^2-a+1}{A}\right]^{a/(a^2-a+1)}z$$

$$p_2=\frac{a(a-1)}{a^2-a+1},\quad q_2=\frac{1-a}{a^2-a+1}\,,$$

$$r_2=\frac{a}{a^2-a+1}$$

so $p_2+q_2+r_2=1=p_2^2+q_2^2+r_2^2$. In both asymptotic regions the nonzero components of the Riemann tensor behave like $\hat{\rho}^{-2}$.

It is apparent that the space-time described by the metric (1) admits three commuting Killing vectors $\partial/\partial t$, $\partial/\partial\phi$, and $\partial/\partial z$ and therefore the equations of geodesic motion will possess three first integrals related to the conservation laws of energy, z component of momentum, and z component of angular momentum. To derive the equations of motion of electrically neutral test particles and light rays we use the Hamilton-Jacobi method and we assume that the generating function S has the form

$$S=-Et+R(\rho)+p_z z+J_z\phi\,. \tag{6}$$

From the relativistic Hamilton-Jacobi equation

$$S_{,a}S_{,b}g^{ab}=m^2\,, \tag{7}$$

we obtain

$$R(\rho)=\int[E^2-U(\rho)]^{1/2}d\rho\,, \tag{8}$$

where

$$U(\rho)=m^2\left[\frac{\rho}{\rho_0}\right]^{2a^2}\left\{\left[\left[\frac{\rho}{\rho_0}\right]^a+A\left[\frac{\rho}{\rho_0}\right]^{-a}\right]^2+\left[\left[\frac{\rho}{\rho_0}\right]^a+A\left[\frac{\rho}{\rho_0}\right]^{-a}\right]^4\left[\frac{p_z}{m}\right]^2+\left[\frac{J_z}{\rho_0 m\gamma}\right]^2\left[\frac{\rho}{\rho_0}\right]^{-2}\right\}. \tag{9}$$

Equations of motion of the test particles are

$$\frac{d\rho}{dt}=E^{-1}[E^2-U(\rho)]^{1/2}\,, \tag{10a}$$

$$\frac{d\rho}{d\phi}=\frac{\rho_0^2\gamma^2}{J_z}\left[\frac{\rho}{\rho_0}\right]^{2(1-a^2)}[E^2-U(\rho)]^{1/2}\,, \tag{10b}$$

$$\frac{d\rho}{dz}=p_z^{-1}\left[\left[\frac{\rho}{\rho_0}\right]^a+A\left[\frac{\rho}{\rho_0}\right]^{-a}\right]^{-4}\left[\frac{\rho}{\rho_0}\right]^{-2a^2}\times[E^2-U(\rho)]^{1/2}\,. \tag{10c}$$

To obtain equations of motion of light rays we take the limit $m\to 0$, $J_z/E\to l$, and $p_z/E\to k$ and we get

$$\frac{d\rho}{dt}=[1-\tilde{U}(\rho)]^{1/2}\,, \tag{11a}$$

$$\frac{d\rho}{d\phi}=\frac{\rho_0^2\gamma^2}{l}\left[\frac{\rho}{\rho_0}\right]^{2(1-a^2)}[1-\tilde{U}(\rho)]^{1/2}\,, \tag{11b}$$

$$\frac{d\rho}{dz}=k^{-1}\left[\frac{\rho}{\rho_0}\right]^{-2a^2}\left[\left[\frac{\rho}{\rho_0}\right]^a+A\left[\frac{\rho}{\rho_0}\right]^{-a}\right]^{-4}\times[1-\tilde{U}(\rho)]^{1/2}\,, \tag{11c}$$

where

$$\tilde{U}(\rho)=k^2\left[\frac{\rho}{\rho_0}\right]^{2a^2}\left[\left[\frac{\rho}{\rho_0}\right]^a+A\left[\frac{\rho}{\rho_0}\right]^{-a}\right]^4+\left[\frac{l}{\rho_0\gamma}\right]^2\left[\frac{\rho}{\rho_0}\right]^{2(a^2-1)}. \tag{12}$$

At first let us consider the radial motion of test particles and photons. Radial motion of test particles is governed by the equation

$$\frac{d\rho}{dt}=\left\{1-\left[\frac{m}{E}\right]^2\left[\frac{\rho}{\rho_0}\right]^{2a^2}\left[\left[\frac{\rho}{\rho_0}\right]^a+A\left[\frac{\rho}{\rho_0}\right]^{-a}\right]^2\right\}^{1/2}. \tag{13}$$

The test particle can move only in the region where

$$\left[\frac{\rho}{\rho_0}\right]^{a^2}\left[\left[\frac{\rho}{\rho_0}\right]^{a}+A\left[\frac{\rho}{\rho_0}\right]^{-a}\right]\le\frac{E}{m}\ . \qquad (14)$$

It is interesting to notice that when $A\neq 0$ and $0<a<1$ radially moving test particles encounter a potential barrier at small and at large ρ. Radially moving test particles will therefore oscillate between the turning points $\rho_{\min}$ and $\rho_{\max}$. The period of oscillations is given by

$$T=\int_{\rho_{\min}}^{\rho_{\max}}\frac{d\rho}{\left\{1-\left[\frac{m}{E}\right]^2\left[\frac{\rho}{\rho_0}\right]^{2a^2}\left[\left[\frac{\rho}{\rho_0}\right]^{a}+A\left[\frac{\rho}{\rho_0}\right]^{-a}\right]^2\right\}^{1/2}}\ . \qquad (15)$$

A test particle placed at the minimum of the potential at

$$\left[\frac{\rho}{\rho_0}\right]=\left[A\frac{1-a}{1+a}\right]^{1/2a}$$

will stay there forever.

When $A=0$ radial motion is possible in the region $0\le\rho\le\rho_{\max}=(E/m)^{1/a(a+1)}$ and all test particles will be trapped by the string.

Photons move in the radial direction along the null lines $\rho\pm t=\text{const}$. One should notice however that ρ is not the proper distance and radially moving photons do feel the curvature of the space-time generated by the current and the matter density of the string. Photons moving radially outward are not trapped by the string.

The effective potential of a test particle moving in the plane perpendicular to the string assumes the form

$$U(\rho)=m^2\left[\frac{\rho}{\rho_0}\right]^{2a^2}\left\{\left[\left[\frac{\rho}{\rho_0}\right]^{a}+A\left[\frac{\rho}{\rho_0}\right]^{-a}\right]^2+\left[\frac{J_z}{\rho_0 m\gamma}\right]^2\left[\frac{\rho}{\rho_0}\right]^{-2}\right\}. \qquad (16)$$

When $0<a<1$ the potential rises at small and at large ρ and the extremal points of the effective potential appear at ρ such that

$$a\left[\frac{\rho}{\rho_0}\right]^2\left[(1+a)\left[\frac{\rho}{\rho_0}\right]^{2a}+2Aa-A^2(1-a)\left[\frac{\rho}{\rho_0}\right]^{-2a}\right]=(1-a^2)\left[\frac{J_z}{\rho_0 m\gamma}\right]^2. \qquad (17)$$

Depending on the values of a, A, and $J_z/\rho_0 m\gamma$ this equation can possess several real positive roots corresponding to local maxima and minima of the potential. Therefore stable circular motion along the string should be possible. In general, the trajectory of a test particle is not a closed curve and it is confined between $\rho_{\max}$ and $\rho_{\min}$.

When $a\le 1$ the effective potential is regular at $\rho=0$ and it rises to infinity with $\rho\to\infty$. In this case all test particles moving in the plane perpendicular to the string fall at the symmetry axis.

Photons moving in a plane perpendicular to the string obey the following equations:

$$\frac{d\rho}{dt}=[1-\hat{U}(\rho)]^{1/2}\ , \qquad (18a)$$

$$\frac{d\rho}{d\phi}=\frac{\rho_0^2\gamma^2}{l}\left[\frac{\rho}{\rho_0}\right]^{2(1-a^2)}[1-\hat{U}(\rho)]^{1/2}\ , \qquad (18b)$$

where

$$\hat{U}(\rho)=\left[\frac{l}{\rho_0\gamma}\right]^2\left[\frac{\rho}{\rho_0}\right]^{2(a^2-1)}. \qquad (19)$$

When $0\le a<1$ and $l\neq 0$ the standard centrifugal barrier appears and photons cannot reach the symmetry axis.

To analyze bending of light by the string let us compute the total change in the angular coordinate $\Delta\phi$ of a photon incoming from infinity with the impact parameter l and escaping to infinity. It is given by

$$\Delta\phi=2\int_{\rho_{\min}}^{\infty}\frac{l}{\rho_0^2\gamma^2}\left[\frac{\rho}{\rho_0}\right]^{2(a^2-1)}\times\frac{d\rho}{\left[1-\frac{l^2}{\rho_0^2\gamma^2}\left[\frac{\rho}{\rho_0}\right]^{2(a^2-1)}\right]^{1/2}}\ , \qquad (20)$$

where $\rho_{\min}=\rho_0(\rho_0\gamma/l)^{1/(a^2-1)}$. Let us introduce a new variable $x=(\rho/\rho_0)^{-1}$, then

$$\Delta\phi=2\frac{l}{\rho_0\gamma^2}\int_0^{x_0}\frac{dx}{x^{2a^2}\left[1-\left[\frac{l}{\rho_0\gamma}\right]^2x^{2(1-a^2)}\right]^{1/2}}\ . \qquad (21)$$

When $a\ge 1/\sqrt{2}$, $\Delta\phi$ is infinite and photons coming from infinity spiral around the string infinitely many times before escaping again to infinity. When $0\le a<1/\sqrt{2}$, $\Delta\phi$ is finite, and for $a=0$ we have $\Delta\phi=\pi/\gamma$.

Finally let us investigate motion of particles and light rays along a line parallel to the symmetry axis. In this case $J_z=0$ and $p_z\neq 0$. A test particle can move along a line parallel to the symmetry axis if it is placed at the extremum of the potential. Position of extrema are determined from the equation

$$\left[\frac{\rho}{\rho_0}\right]^a \left\{1+a+(2+a)\left[\frac{p_z}{m}\right]^2 \left[\left[\frac{\rho}{\rho_0}\right]^a + A\left[\frac{\rho}{\rho_0}\right]^{-a}\right]^2\right\}$$

$$- A\left[\frac{\rho}{\rho_0}\right]^{-a} \left\{1-a+(2-a)\left[\frac{p_z}{m}\right]^2 \left[\left[\frac{\rho}{\rho_0}\right]^a + A\left[\frac{\rho}{\rho_0}\right]^{-a}\right]^2\right\}=0 . \quad (22)$$

When $0<a<1$ and $A\neq 0$ this equation possesses at least one real positive root and therefore in this case test particles can move along a line parallel to the symmetry axis. Energy of the test particle is then given by the relation

$$\left[\frac{E}{m}\right]^2 = \left[\frac{\rho}{\rho_0}\right]^{2a^2} \left[\left[\frac{\rho}{\rho_0}\right]^a + A\left[\frac{\rho}{\rho_0}\right]^{-a}\right]^2 \times\left\{1+\left[\frac{p_z}{m}\right]^2 \left[\left[\frac{\rho}{\rho_0}\right]^a + A\left[\frac{\rho}{\rho_0}\right]^{-a}\right]^2\right\}, \quad (23)$$

and its velocity is

$$\frac{dz}{dt} = \frac{p_z}{E}\left[\frac{\rho}{\rho_0}\right]^{2a^2} \left[\left[\frac{\rho}{\rho_0}\right]^a + A\left[\frac{\rho}{\rho_0}\right]^{-a}\right]^4 . \quad (24)$$

When $0<a<2$ also photons can propagate along a line parallel to the symmetry axis which is at a distance

$$\frac{\rho}{\rho_0} = \left[A\frac{2-a}{2+a}\right]^{1/2a} .$$

The momentum of these photons should be $k=\frac{1}{16}A^{-(1+a/2)}(2-a)^{1-a/2}(2+a)^{1+a/2}$ and as expected $dz/dt=1$.

Detailed analysis of the motion of test particles and light rays in the general case and discussion of possible astrophysical applications will be given elsewhere.

I would like to thank Professor John A. Wheeler, Professor Philip Candelas, Center for Theoretical Physics, and Center for Relativity of the University of Texas at Austin for hospitality. This work was supported in part by the Polish Interdisciplinary Research Project No. CPBP01.03.

*Permanent address: Institute for Theoretical Physics, University of Warsaw, Warsaw, Poland, and International Center for Relativistic Astrophysics, University of Rome, Rome, Italy.

[1]Ya. B. Zel'dovich, I. Yu. Kobzarev, and L. B. Okun, Zh. Eksp. Teor. Fiz. **67**, 3 (1974) [Sov. Phys. JETP **40**, 1 (1975)].

[2]T. W. B. Kibble, J. Phys. A **9**, 1387 (1976).

[3]A. Vilenkin, Phys. Rep. **121**, 263 (1985).

[4]E. Witten, Nucl. Phys. **B249**, 557 (1985).

[5]L. Witten, in *Gravitation: An Introduction to Current Research*, edited by L. Witten (Wiley, New York, 1962).