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Nonperturbative treatment of the functional Schrödinger equation in QCD

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Use of the functional Schrödinger equation in QCD is complicated by the need to maintain gauge invariance at the same time one is dealing with nonperturbative effects (at least at long distances). Moreover, asymptotic freedom must be recovered at short distances. In this paper we show how to set up and solve a set of Ward identities which ensure gauge invariance of the wave functional, which can then be parametrized simply and used with various nonperturbative algorithms (e.g., variational). Parametrizations are found both for the vacuum wave functional and for $J^{PC}=0^{\pm+}$ quantum soliton states of QCD. A nonperturbative algorithm, based on the observation that a wave functional is really a partition function in the presence of a special kind of source, is set up and applied; this algorithm is based on dressed-loop expansions of partition functions. One recovers this way some earlier results found from the study of Schwinger-Dyson equations, plus some new results concerning the 0^{-+} soliton, whose wave functional involves the Chern-Simons secondary class function.

I. INTRODUCTION

The functional Schrödinger equation (FSE) approach to QCD has been advocated by numerous authors,[1] but its practical implementation is difficult. For example, simple Gaussian wave functionals, which work as expected for scalar field theories,[2] do not allow one to incorporate either asymptotic freedom or local non-Abelian gauge invariance.

This paper has two purposes: The first is to discuss how to ensure gauge invariance and how to do nonperturbative calculations; the second is to discuss applications. We show how gauge invariance (or alternatively, Gauss's law) can be imposed on a simple parametrization of the wave functional, both for the vacuum state and for soliton states, while retaining the power to deal with nonperturbative problems. (That is, order-by-order perturbation theory will certainly respect Gauss's law, but is useless for QCD at long distances.) Once we have discovered an appropriate gauge-invariant wave functional, which will, in general, contain an infinite number of functions (ultimately to be related through the FSE), we need some nonperturbative techniques for finding these functions. At short distances the nonperturbative techniques must yield the usual asymptotically free results of renormalization-group-improved (RGI) perturbation theory. A simple Gaussian trial wave functional, while certainly capable of uncovering nonperturbative results,[2] is neither gauge invariant nor does it show asymptotic freedom. In this paper we show that the infinitely many functions which necessarily appear in the wave functional ψ are related by Ward identities similar to those obeyed in ghost-free gauges by the Green's functions of QCD, and we show how to solve these Ward identities and write ψ in a manifestly gauge-invariant form. The answer is not, as one might naively expect, that ψ depends only on field strengths (which are not gauge invariant in QCD anyway). Other gauge-invariant quantities, depending on the gauge potential, are available, such as Wilson loops or constructions of the type

$$\int (dg)\psi\{ A_i^g(\mathbf{x})\} \ , \tag{1.1}$$

where the $A_i(\mathbf{x})$ constitute the space components of the gauge potential at zero time, A_i^g is A_i gauge transformed by the gauge transformation $g(\mathbf{x})$, and ψ is some sort of possibly noninvariant trial wave function. Our result, given in Sec. III, uses the so-called gauge technique[3] to restrain the functions in ψ to a manifestly gauge-invariant set, at the same time reproducing perturbation theory (and thus, asymptotic freedom) at short distances. The gauge technique of Cornwall and Hou[3] introduces massless poles into various vertex functions, but these are longitudinally coupled (i.e., pure-gauge terms) and never enter the S matrix, as discussed in some detail in Ref. 4 and by Cornwall and Hou.[3] The massless poles are akin to similar poles associated with composite Goldstone particles in dynamical symmetry breaking but, of course, there is no symmetry breaking in QCD. These poles can be recovered by doing the integrations over all gauge transformations in (1.1), as the reader can easily check for QED, with a simple Gaussian for $\psi\{A_i\}$.

It is, of course, useless to try to deal practically with an infinite number of functions in ψ, just as one cannot treat the infinitely many Green's functions occurring in the Schwinger-Dyson equations. Fortunately, the gauge technique is well suited to a truncation in which only one independent function remains, yet ψ is still gauge invariant. We will show that ψ for the vacuum has the form

$$\psi=e^{-S} \ , \tag{1.2}$$

$$S=\sum \left[\tfrac{1}{2} A_i \Omega_{ij} A_j + \frac{1}{3!} A_i A_j A_k \Omega_{ijk} + \cdots \right] , \tag{1.3}$$

and that the Ward-identity constraints on the $\Omega_{ij}, \dots$ necessary for gauge invariance have one solution where the Ω functions with three or more indices are entirely expressed in terms of the two-point (and conserved) function Ω_{ij}. [In (1.3), the sum is over space coordinate and group matrix indices; the Ω functions are translationally invariant functions of the coordinates of the spacelike matrix of gauge potentials $A_i(\mathbf{x})$. See Sec. III for details.] As discussed in Ref. 4, where a similar treatment is given for the Schwinger-Dyson equations, this simple truncation yields one-loop asymptotic freedom in the ultraviolet, and the omitted degrees of freedom are unimportant in the infrared. This kind of truncation can be systematically improved but we will not deal with that issue here.

What nonperturbative means are available for determining the Ω functions (or, after truncation, the single function Ω_{ij})? The two obvious ones are direct substitution in the FSE and a variational approach. In this paper we introduce a third algorithm very similar in spirit to dressed-loop expansions of a partition function in the presence of a source.[2,5] The reason for the similarity is that ψ is, in fact, a partition function, in the presence of a source linearly related to the argument $A_i(\mathbf{x})$ of ψ. In Sec. II we develop the dressed-loop expansion for ψ in the context of a scalar field theory, ending up with an approximate dressed-loop equation for the two-point Ω function. This equation is cognate to the one-dressed-loop propagator equation of Ref. 2 for scalar field theory, or in a gauge theory to the similar development of Ref. 4. In essence, the equation for the two-point Ω function is the on-shell version of dressed-loop Schwinger-Dyson equations for propagators. It should be noted here that in a gauge theory the dressed-loop expansion requires vertices to be dressed as well as propagators, as done in Ref. 5 but not in Ref. 2. The gauge technique automatically furnishes an appropriate dressed vertex, obeying the needed Ward identity.

The final step in these formal developments is to discuss the corresponding problems for solitons. Of course, classical QCD has only instantons as solitons, but nonperturbative quantum effects can lead to quantum solitons, which have no classical counterpart, of a quite different character from instantons. [As an example, the Bardeen-Cooper-Schrieffer (BCS) electron-phonon Lagrangian of superconductivity has no classical solitons, but the Ginzburg-Landau Lagrangian, which describes nonperturbative effects, does.] A soliton is described by a wave functional ψ for which $A_i(\mathbf{x})$ has an expectation value, just as for solitons in scalar field theories.[2] This expectation value, termed $\overline{A}_i(\mathbf{x})$, must be gauge invariant (since ψ is) and will obey some equation involving the Ω functions. This notion of a gauge-invariant expectation value of A_i is unfamiliar, but it is inherent in any discussion of gauge-theory solitons. The point is that in the soliton wave functional $\psi\{A_i,\overline{A}_i\}$ only the coordinates A_i undergo gauge transformations, and under them ψ is invariant; parameters of ψ, such as $\overline{A}_i$, are c numbers under gauge changes of the A_i. We show, in Sec. III, how to express these ideas in a more familiar form by writing $\overline{A}_i$ in terms of a classical potential $\mathcal{A}_i$ which transforms as usual, and in terms of a special unitary matrix which is a function of $\mathcal{A}_i$. The combination which forms $\overline{A}_i$ is automatically gauge invariant.[6] Gauge-invariant soliton wave functionals are discussed for $J^{PC}=0^{\mp +}$ glueball states corresponding to massive solitons which can be exhibited as static spherically symmetric objects. The 0^{-+} soliton wave functional involves the Chern-Simons secondary class function. Other soliton states which QCD may have (e.g., vortices, which are responsible for confinement[7]) will not be discussed here.

Let us briefly discuss applications of these formal techniques. In many cases the applications yield results which have already been published and to avoid an excessively lengthy paper, details of the FSE approach will not be given.

We have already said that the vacuum wave functional ψ of (1.2) and (1.3) can be expressed, in a certain approximation, solely in terms of the two-point function Ω_{ij} which is conserved and can be written in momentum space as

$$\Omega_{ij}(\mathbf{k})=\left[\delta_{ij}-\frac{k_ik_j}{\mathbf{k}^2}\right]\Omega(\mathbf{k}) . \tag{1.4}$$

In perturbation theory $\Omega(\mathbf{k})=|\mathbf{k}|$. The nonperturbative one-dressed-loop equation for Ω, discussed in Secs. II and III, conveys the same information as the one-dressed-loop Schwinger-Dyson equation for the propagator. This was shown in Ref. 2 for ϕ^4 theory. It turns out also to be true for gauge theory, provided one uses the modified and gauge-invariant propagator of Ref. 4. Now in ϕ^4 theory with no Lagrangian mass term, the vacuum expectation value $\langle\phi^2\rangle$ will induce a mass through the seagull graph, which can be seen either from the FSE or from the dressed-loop expansion of the partition function.[2] Similarly in QCD, a condensate $\langle G_{\mu\nu}^2\rangle$, which necessarily implies a condensate of (among other things) $\langle A_\mu A_\nu\rangle$, will also drive a gluon mass, in a sense we make specific below. Of course, it is not only the seagull graph which is responsible; that by itself would not be gauge invariant. Either the FSE technique used here or the Schwinger-Dyson techniques of Ref. 4 ensure a gauge-invariant description of this process of gluon mass generation.

It should be emphasized that generation of a gluon mass is not inconsistent with confinement; indeed, part of this process is the induction of long-range pure-gauge potentials (associated with the vortices mentioned earlier[7]) which are responsible for confinement. This mechanism has been known for a long time.[7,8] Moreover, lattice-gauge-theory simulations[9,10] show evidence of a gluon mass of 600–800 MeV, consistent with the estimates of Ref. 4. Of course, the gluon mass is not directly measurable, any more than a confined quark mass; it is simply a parameter which appears in various theoretical calculations the way a mass usually would. In the present case the mass m appears in the function $\Omega(\mathbf{k})$ of (1.4), where it serves as an infrared cutoff. As we will see in Secs. II and III, Ω has the significance of the on-shell energy of a particle, thus, for a massive particle one would have

$$\Omega(\mathbf{k})=(\mathbf{k}^2+m^2)^{1/2} \tag{1.5}$$

instead of the free kinetic energy $|\mathbf{k}|$. QCD is gauge invariant even in the presence of such a mass term by virtue of the long-range pure-gauge terms discussed before and which are seen, for example, in the kinematic tensor of (1.4). The precise solution of the one-dressed-loop FSE equation for Ω is well approximated by (1.5) which will be adequate for our purposes here. Detailed discussion of this equation and its solution will be given elsewhere.

Given that $\Omega(\mathbf{k})$ has the form (1.5), one can discuss solitons of the theory which owe their existence to the mass term and which are strongly coupled. The soliton picture is quite different from that found in the familiar weak-coupling semiclassical approach,[11] but, except for some brief remarks at the end of this section, that is all we will discuss here. If we use (1.5) in the semiclassical expression for the soliton energy as a function of the soliton potential $\overline{A}_i(\mathbf{x})$, we find an effective action whose 0^{++} soliton solution has already been thoroughly discussed,[12,13] and we will not repeat that work here. This soliton is important for chiral-symmetry breakdown[13] since it has three-dimensional fermion zero modes.[14] In itself it has no topological character and, thus, is unstable to decay when coupled to quarks. However, it can couple in interesting ways to topological solitons.[15]

There is also a 0^{-+} soliton (glueball) for which we can construct an approximate trial wave function by multiplying the 0^{++} functional by the Chern-Simons secondary class integral W given in (3.28) below. This is really a quantum-mechanical soliton, and owes its existence to fluctuations around a potential minimum; it has no classical interpretation as an object sitting at the bottom of a well, any more than the first excited state of a harmonic oscillator does. Just like this first excited state (which is a good analog to the 0^{-+} soliton, in having odd parity), the 0^{-+} soliton is heavier than the 0^{++} soliton, as we very crudely estimate in Sec. III. We show that the 0^{++} and the 0^{-+} states mix with each other under changes of the θ angle of the vacuum.[15]

Past the semiclassical approximation, things get very complicated. Higher-order dressed-loop corrections correspond, in part, to what is called soliton entropy (e.g., Ref. 16) in a four-dimensional Euclidean context. For the case of particlelike solitons such as discussed here, described by thick world lines in the four-dimensional context, Stone and Thomas[17] have shown how to map a partition function of such world lines onto a scalar field theory, with an effective field $S(x)$ for the 0^{++} world lines, and a field $P(x)$ for the 0^{-+} world lines. If the entropy in the partition function is large enough, the world lines condense and S gets a vacuum expectation value. Presumably there is some way of describing all this from the FSE but it is unknown to the author. However, it is of some interest that an earlier phenomenological description[18] of scalar and pseudoscalar glueballs invoked all the features mentioned above: S-P mixing under charge of θ; $\langle S\rangle\neq 0$; $M_P > M_S$, etc.

There seems to be no reason to believe that the soliton glueballs occurring in the FSE are any different from conventional glueballs,[19,20] but we have no evidence that they are the same either. Work is in progress on this and other difficult points of strongly coupled soliton theory.

II. SCHRÖDINGER WAVE FUNCTIONALS

Before turning to gauge theories, we briefly discuss the FSE for scalar field theories such as Φ^4. This will help us to understand how to develop various nonperturbative techniques for solving the FSE. These techniques are closely related to the dressed-loop expansion[2] of the effective action, as was long ago recognized for the Gaussian approximation to ψ. However, as we noted earlier, this approximation is neither gauge invariant nor asymptotically free, so some further developments are needed. Among the most important of these is an algorithm, usually called the gauge technique,[3] for ensuring that three- and four-point vertices obey certain Ward identities. A closely related algorithm, which we need not discuss in detail here, extends the dressed-loop expansion of the effective action to include fully dressed vertices,[5] a step which is essential in maintaining gauge invariance. (Presumably the same results could be found using a background gauge.[21])

We begin by exploring the relation between the wave functional ψ and the partition function $Z(J)$ for the theory in question, in the presence of a source J which turns out to be linearly related to the coordinates of ψ. Our only concern here will be to find ψ for the ground state or vacuum; later we will see how to address the question of low-lying soliton states.

A. Scalar field theory

Consider the vacuum wave functional $\psi\{\phi(\mathbf{x})\}$ of a scalar field theory. The Hamiltonian H (with $H\psi=E\psi$) is

$$H=\int d^3x\left[-\frac{1}{2}\left[\frac{\delta}{\delta\phi}\right]^2+\tfrac{1}{2}(\nabla\phi)^2+\frac{m^2}{2}\phi^2+V(\phi)\right]. \tag{2.1}$$

Assuming that $\psi\{0\}\neq 0$, we write

$$\psi\{\phi\}\psi\{0\}=\lim_{\beta\to+\infty}e^{\beta E}\int_0^{\phi}(d\Phi)\exp\left[-\int\mathcal{L}_E\right]. \tag{2.2}$$

The functional integral goes over all fields $\Phi(\mathbf{x},\tau)$ with $\Phi(\mathbf{x},0)=0$ and $\Phi(\mathbf{x},\beta)=\phi(\mathbf{x})$ and $\mathcal{L}_E$ is the Euclidean Lagrangian

$$\mathcal{L}_E=\tfrac{1}{2}(\partial_\mu\Phi)^2+\tfrac{1}{2}m^2\Phi^2+V(\Phi)\equiv\mathcal{L}_0+V(\Phi). \tag{2.3}$$

The unadorned integral sign stands for $\int_0^\beta d\tau\int d^3x$. The right-hand side (RHS) of (2.2) is formally given by

$$\lim_{\beta\to\infty} e^{\beta E}\exp\left[-\int V(\delta/\delta J)\right]Z_0(J)\Big|_{J=0}, \tag{2.4}$$

where

$$Z_0(J)=\int_0^{\phi}(d\phi)\exp\left[-\int\mathcal{L}_0+\int J\phi\right] \tag{2.5}$$

$$=\exp\left[-\beta E_0-\tfrac{1}{2}\int d^3x\,d^3y\,\phi(\mathbf{x})\Omega^0\phi(\mathbf{y}) + \int J(x)\hat{\phi}(x) + \tfrac{1}{2}\int\int J(x)\Delta_0(x,y)J(y)\right]. \tag{2.6}$$

The following notation is used in (2.6): Ω^0 is an operator whose matrix elements are

$$\langle\mathbf{x}\,|\,\Omega^0\,|\,\mathbf{y}\rangle=\langle\mathbf{x}\,|\,(-\nabla^2+m^2)^{1/2}\,|\,\mathbf{y}\rangle =\int\frac{d^3k}{(2\pi)^3}(\mathbf{k}^2+m^2)^{1/2}e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}. \tag{2.7}$$

Also,

$$\hat{\phi}_0(x)\equiv\hat{\phi}_0(\tau,\mathbf{x})=\frac{\sinh\Omega^0\tau}{\sinh\Omega^0\beta}\phi(\mathbf{x}) \tag{2.8}$$

and

$$\Delta_0(x,y)=\left\langle\mathbf{x}\left|\frac{1}{\Omega^0\sinh\beta\Omega^0}[\sinh\Omega^0\tau\sinh\Omega^0(\beta-\tau')\theta(\tau'-\tau)+\sinh\Omega^0\tau'\sinh\Omega^0(\beta-\tau)\theta(\tau-\tau')]\right|\mathbf{y}\right\rangle. \tag{2.9}$$

Δ_0 is the inverse of the operator $-(\Box-m^2)$ or $-(\partial_\tau^2-\Omega^{0^2})$; it vanishes at $\tau=0,\beta$. In (2.6), $E_0=\frac{1}{2}\mathrm{Tr}\Omega^0$ is the free-field zero-point energy. Note that the term quadratic in ϕ in (2.6) can be written as

$$\tfrac{1}{2}\int d^3x\,d^3y\,\phi(\mathbf{x})\Omega^0\phi(\mathbf{y})=\lim_{\beta\to\infty}\tfrac{1}{2}\int(\partial_\tau\hat{\phi}_0)^2+(\Omega^0\hat{\phi}_0)^2 \equiv I_0(\hat{\phi}_0), \tag{2.10}$$

where I_0 is the free-field action. Also note that for purposes of calculating the energy E we may replace Δ_0 by the usual Euclidean propagator

$$\Delta_0(x,y)\to\left\langle\mathbf{x}\left|\frac{1}{2\Omega^0}e^{-\Omega^0|\tau-\tau'|}\right|\mathbf{y}\right\rangle =\frac{1}{(2\pi)^4}\int d^4k\frac{e^{ik\cdot(x-y)}}{k^2+m^2}. \tag{2.11}$$

This is the formal $\beta\to\infty$ limit of (2.9), less a term which does not contribute in the graphs for E. However, the substitution (2.11) *cannot* be made when calculating the ϕ-dependent terms in (2.4). A suitable form for (2.9) in the limit $\beta\to\infty$ is, for this purpose,

$$\Delta^0\to\left\langle\mathbf{x}\left|\frac{1}{2\Omega^0}(e^{-\Omega^0|\tau-\tau'|}-e^{-\Omega^0(\tau+\tau')})\right|\mathbf{y}\right\rangle, \tag{2.12}$$

which still obeys the boundary conditions of vanishing at τ or $\tau'=0,\infty$.

We now make the elementary remark that $Z_0(J)$ in (2.5) can be written as a functional integral with $\phi(\mathbf{x})$-independent boundary conditions:

$$Z_0(J)=\int(d\Phi)\exp\left[-I_0(\Phi)-I_0(\hat{\phi}_0)+\int J(\Phi+\hat{\phi}_0)\right]. \tag{2.13}$$

Here we used (2.10) to rewrite the second term in the exponential of (2.6). Using (2.4) and (2.13) in (2.2), we find

$$e^{-\beta E}\psi\{\phi(\mathbf{x})\}=N\int(d\Phi)\exp\left[-I_0(\Phi)-\int V(\Phi+\hat{\phi}_0)-I_0(\hat{\phi}_0)\right], \tag{2.14}$$

where N is a normalization constant, and the limit $\beta\to\infty$ is understood. The formal shift of variables $\Phi\to\hat{\phi}_0$ yields ψ as (up to overall factors) a partition function $Z(J)$ in the presence of the source $J=-\Delta_0^{-1}\hat{\phi}_0$. This source seems to vanish if one uses (2.8), but the factors of Δ_0^{-1} in J are always compensated by factors of Δ_0 from the Feynman rules. It is, in fact, just as simple to use the form (2.14) without shifting variables so that ψ is a partition function in the presence of sources that excite composite operators.

Either through the above remarks, or by direct substitution, one sees that ψ has the generic form

$$\psi=e^{-S},\quad S=\sum_N\frac{1}{N!}\mathrm{Tr}\Omega_N\phi_1\cdots\phi_N, \tag{2.15}$$

where Tr indicates an integral over the spatial variables $\mathbf{x}_1,\ldots,\mathbf{x}_N$ (and any necessary other index sum), and the Ω_N are translationally invariant operators in the $\mathbf{x}_i$, with $\phi_i\equiv\phi(\mathbf{x}_i)$. The time-honored ways to determine the Ω_N are by direct substitution in the FSE, or variationally. Another way, which exploits the fact that ψ is a partition function, is to use dressed-loop expansions which are usually done for the effective action, the Legendre transform of $\ln Z(J)$. All of these ways are closely related even in the face of necessary approximations which keep the problem tractable, and all are capable of dealing with nonperturbative phenomena.

Let us take up the dressed-loop expansion. The idea is to resum the perturbative series for ψ arising from (2.14) so that it depends not on free quantities such as Ω_0 but only on their fully dressed counterparts. In perturbation theory, the zeroth-order or free-field values for the Ω_N of (2.15) are $\Omega_2=\Omega^0$, all other $\Omega_N=0$. From now on we use the simple notation Ω in place of Ω_2 to indicate the coefficient of the quadratic term in S, and our task is to resum the graphs contributing to S so that as much as possible they depend on Ω, not Ω^0. We write the potential for our field theory as

$$V(\Phi)=\frac{g}{3!}\Phi^3+\frac{\lambda}{4!}\Phi^4 \tag{2.16}$$

and show, in Fig. 1, some of the Feynman graphs contributing to S. Solid lines indicate free propagators as in (2.12) and heavy dots indicate the field $\hat{\phi}_0$ of (2.8). Begin with the Born term of Fig. 1(a); it is easily found to contribute to S a term

$$\frac{g}{3!}\,\mathrm{Tr}(\Omega_1^0+\Omega_2^0+\Omega_3^0)^{-1}\phi_1\phi_2\phi_3\ , \tag{2.17}$$

where, in momentum space, $\Omega_i^0=(\mathbf{k}_i^2+m^2)^{1/2}$. Among the possible corrections to the graphs are those which are one-particle reducible, such as Fig. 1(e). Straightforward calculation shows that these have the effect of replacing the Ω_i^0 in (2.17) by Ω_i, where

$$\Omega_i\simeq\Omega_j^0+\frac{\Pi_0}{2\Omega_i^0}+\cdots\ , \tag{2.18}$$

where Π_i is the *on-shell* proper self-energy. On shell means that in the proper self-energy $\Pi(\omega_i,\mathbf{k}_i)$ we must replace ω_i by Ω_i^0. We will soon see that (2.18) is the perturbative expansion of the equation which determines the fully dressed quality Ω. Since the factor $(\sum\Omega_i^0)^{-1}$ in (2.17) comes from doing the τ integrations over the fields $\hat{\phi}_0$ of (2.8), and since the Ω_i^0 get replaced by Ω_i, it is clear that one part of the resummation process is the replacement of the field $\hat{\phi}_0$, depending on Ω^0, by the field $\hat{\phi}$:

$$\hat{\phi}(\hat{x},\tau)=\frac{\sinh\Omega\tau}{\sinh\Omega\beta}\phi(\hat{x})\ . \tag{2.19}$$

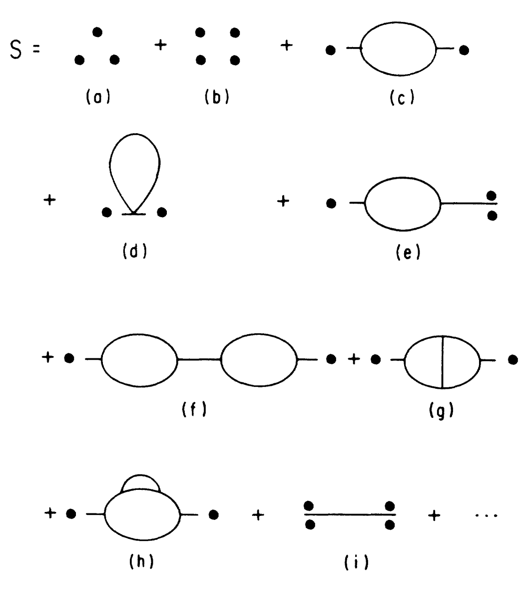

FIG. 1. Some Feynman graphs contributing to the connected wave functional. Solid lines indicate free propagators and heavy dots represent $\hat{\phi}_0$.

With this substitution, one-particle-reducible graphs such as Figs. 1(e) and 1(f) that have self-energy corrections on them can be dropped. Please note that this does not remove all one-particle-reducible graphs; for example, Fig. 1(i) survives as a skeleton graph.

Now turn to the one-particle-irreducible (1PI) graphs, e.g., Figs. 1(c), 1(g), and 1(h). The general structure of the sum of such graphs is

$$\tfrac{1}{4}g^2\langle T[\Phi^2(x)\Phi^2(0)]\rangle_{1\mathrm{PI}}=\Pi(x)_{1\mathrm{PI}}\ , \tag{2.20}$$

where $\Pi_{1\mathrm{PI}}$ is the one-particle-irreducible proper self-energy, that is, the proper self-energy which emerges by leaving out all graphs involving $\langle\Phi\rangle$. (We leave it to the reader to complete the discussion by including such graphs.) Taken by themselves, Figs. 1(c), 1(g), and 1(h) would involve a convolution of Π with two powers of $\hat{\phi}_0$, but the one-particle-reducible graphs such as Fig. 1(f) intervene to change $\hat{\phi}_0$ to $\hat{\phi}$. A similar discussion holds for graphs in Π which involve the $\lambda\Phi^4$ coupling, such as Fig. 1(d).

Based on these findings, we are led to conjecture that the following equation for Ω essentially captures the important nonperturbative phenomenon which can be found in a dressed-loop expansion:

$$\tfrac{1}{2}\,\mathrm{Tr}\phi\Omega\phi=I_0(\hat{\phi})+\tfrac{1}{2}\int\hat{\phi}\Pi\hat{\phi}\ , \tag{2.21}$$

where the fully dressed proper self-energy Π is constructed by replacing the frequency by Ω. Equation (2.21) simply equates the quadratic term in S of (2.15) to the sum of the free-field contribution [see (2.14)] and the resummed contributions of graphs such as shown in Fig. 1.

Even if there were, in principle, no corrections to (2.21)—and there doubtless are—it is necessarily approximate in the absence of exact knowledge of the functional dependence of Π on Ω. We have investigated (2.21) with the simple approximation of replacing Ω^0 by Ω in the free propagator (2.12), and then using this propagator in the standard skeleton graphs for Π (with bare vertices). To one-dressed-loop order, this yields the same result as direct solution of the FSE (and slightly improved results over a Gaussian variational wave functional). At two-dressed-loop order some differences appear, which we can highlight by studying the anharmonic oscillator in quantum mechanics, which is a Φ^4 theory in zero space dimensions. Except for renormalization effects, one can easily convert the results given below to true Φ^4 field theories by letting Ω, which is a number in quantum mechanics, become an operator and symmetrizing various denominators.

Consider the Hamiltonian

$$H=\tfrac{1}{2}p^2+\tfrac{1}{2}\omega^2x^2+\frac{\lambda}{4}x^4 \tag{2.22}$$

and corresponding ground-state wave function

$$\psi=e^{-S},\quad S=\tfrac{1}{2}\Omega x^2+\frac{1}{4!}\Omega_4x^4+\frac{1}{6!}\Omega_6x^6+\cdots \tag{2.23}$$

obeying $H\psi=E\psi$. Table I shows a comparison of the three methods we have referred to, to $O(\lambda^2)$; the last column gives E for $\omega=0$, the strong-coupling limit, where of course perturbation theory fails. It is only a

TABLE I. Comparison of methods and results for the anharmonic oscillator. The exact numerical result for $\omega=0$ is shown in the upper right-hand corner.

Technique used	Ω equations	Ω_4 equations	Ω_6 equations	Energy E for $\omega=0$ ($E=0.420\lambda^{1/3}$)
Perturbation theory	$\Omega=\omega+\frac{3\lambda}{8\omega^2}-\frac{21\lambda^2}{64\omega^5}+\cdots$	$\Omega_4=\frac{3\lambda}{2\omega}-\frac{33}{32}\frac{\lambda^2}{\omega^4}+\cdots$	$\Omega_6=-\frac{15\lambda^2}{4\omega^3}+\cdots$	
Gaussian variational	Minimize $\frac{\Omega}{4}+\frac{\omega^2}{4\Omega}+\frac{3\lambda}{16\Omega^2}$			$\frac{3}{4}(\frac{3}{16}\lambda)^{1/3}=0.429\lambda^{1/3}$
Direct substitution in Schrödinger equation	$\Omega^2=\omega^2+\frac{1}{2}\Omega_4+\cdots$	$\Omega_4=\frac{3\lambda}{2\Omega}+\frac{\Omega_6}{8\Omega}+\cdots$	$\Omega_6=-\frac{5}{3}\frac{\Omega_4^3}{\Omega}+\cdots$	(a) $\Omega_N=0,\ N\geq 6$: $(\frac{3}{32}\lambda)^{1/3}=0.454\lambda^{1/3}$ (b) $\Omega_N=0,\ N\geq 8$: $(\frac{9}{136}\lambda)^{1/3}=0.404\lambda^{1/3}$
Equation (2.21)	$\frac{1}{2}\Omega=\frac{1}{4\Omega}(\Omega^2+\omega^2)+\frac{3\lambda}{16\Omega^2}-\frac{3\lambda^2}{128\Omega^5}+\cdots$			$\left[\frac{3+\sqrt{3}}{64}\right]^{1/3}\lambda^{1/3}=0.420\lambda^{1/3}$

coincidence that the dressed-loop expansion, Eq. (2.21), gives the correct numerical result to three figures. Note that, to $O(\lambda)$, direct solution of the Schrödinger equation and Eq. (2.21) give the same answers. Had we included a gx^3 coupling as well, this remark would also hold to $O(g^2)$, and it also holds for the Φ^3-Φ^4 field theory with $V(\Phi)$ given in (2.16). The strong-coupling results are reasonably insensitive to the order of truncation, as shown in the third row of Table I.

B. Gauge theory

The main problem we face here is, of course, maintaining gauge invariance. In the canonical formalism[15] of gauge theory there is no momentum conjugate to the time component A_0, so we may set $A_0=0$ and the wave functional ψ depends on $A_i(\mathbf{x})$. Here we introduce the notation

$$A_\mu=\sum_a \frac{1}{2i}\lambda_a A_\mu^a \tag{2.24}$$

connecting the anti-Hermitian potential matrix A_μ and the components A_μ^a; the matrix generators λ_a are normalized to

$$\mathrm{Tr}\lambda_a\lambda_b=2\delta_{ab} \ . \tag{2.25}$$

The functional Hamiltonian H is

$$H=\int d^3x \sum_a \left[-\frac{1}{2}\left[\frac{\delta}{\delta A_i^a}\right]^2+\tfrac{1}{4}(G_{ij}^a)^2\right] \tag{2.26}$$

with

$$G_{ij}^a=\partial_i A_j^a-\partial_j A_i^a+g\epsilon_{abc}A_j^b A_j^c \tag{2.27}$$

and H is invariant under spatial gauge transformations (expressed more compactly in matrix notation):

$$A_i\to VA_iV^{-1}+g^{-1}V\partial_iV^{-1} \ ,$$
$$\Pi_i\to V\Pi_iV^{-1},\quad \Pi_i\equiv -i\frac{\delta}{\delta A_i} \ . \tag{2.28}$$

Here $V(\mathbf{x})$ is a unitary representation of the gauge group. For the time being we will concern ourselves only with "small" gauge transformations, with generators

$$D_i^{ab}\Pi_i^b,\quad D_i^{ab}=\delta^{ab}\partial_i+g\epsilon^{acb}A_i^c \ . \tag{2.29}$$

Questions of "large" gauge transformations with nonzero winding numbers will be discussed later in connection with the P soliton. Since $[\mathbf{D}\cdot\mathbf{\Pi},H]=0$ the eigenfunctions of H form representations of the gauge group and we will only be interested in singlet representations:

$$D_i^{ab}\left[-i\frac{\delta}{\delta A_i^b}\right]\psi=0 \ . \tag{2.30}$$

This is, of course, Gauss's law on physical states.

We can construct a formally gauge-invariant ψ_v for the vacuum by writing $\ln\psi_v$ as an infinite (functional) power series in $A_i^a(\mathbf{x})$, with coefficients related by Ward identities:

$$\psi_v\{A_i^a\}=\exp\left[-\left[\frac{1}{2!}\sum A_i^a\Omega_{ij}A_j^a+\frac{g}{3!}\sum A_i^aA_j^bA_k^c\Omega_{ijk}^{abc}+\cdots\right]\right] \ . \tag{2.31}$$

Here the Ω's are operators in the spatial variables $\mathbf{x}_i,\mathbf{x}_j,\ldots$ and the sum is over group indices, spatial indices, and the $\mathbf{x}_i$. The Ω's depend only on differences

$\mathbf{x}_i - \mathbf{x}_j$, by translational invariance. Under the infinitesimal gauge transformation corresponding to (3.24),

$$\delta A_i^a = \partial_i \omega^a + g\epsilon^{abc} A_i^c \omega^b \ . \tag{2.32}$$

ψ_v of (2.31) is invariant, provided that the bilinear coefficient $\Omega_{ij}(\mathbf{x}_i - \mathbf{x}_j)$ is transverse,

$$\Omega_{ij} = (\delta_{ij} - \partial_i \partial_j \nabla^{-2})\Omega \ , \tag{2.33}$$

and that all the rest of the coefficients obey Ward identities of the simple type found in ghost-free gauges, e.g.,

$$\frac{\partial}{\partial x_{1i}} \Omega_{ijk}^{abc}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \epsilon^{abc} \Omega_{jk}(\mathbf{x}_2 - \mathbf{x}_3)[\delta(\mathbf{x}_1 - \mathbf{x}_2) - \delta(\mathbf{x}_1 - \mathbf{x}_3)] \ . \tag{2.34}$$

Similarly, $\partial_i \Omega_{ijkl}^{abcd}$ is related to sums of Ω_{jkl}^{bcd}, etc. These Ward identities simply project out the part of A_i which is gauge independent [cf. Eq. (1.1)].

Just as for the spinless field theories discussed earlier, the Ω coefficients can be determined from direct substitution in the FSE, by variational means, or through an analog of (2.21) which resums terms in a partition function. All of these are capable of generating nonperturbative effects, such as spontaneous generation of a gauge-invariant gluon mass,[4] but certain minimum requirements must be satisfied. As is well known, the requirements of gauge invariance can only be met if *all* graphs with the same number of loops are kept in a given process. This applies to graphs containing dressed loops as well as the usual perturbative expansion. For example, consider the usual gluon proper self-energy $\Pi_{\mu\nu}$. This is not gauge invariant (although it is certainly possible to construct a gauge-invariant proper self-energy by adding certain terms coming from the vertex and other graphs; see Ref. 4), but gauge invariance requires that $q^\mu \Pi_{\mu\nu}(q) = 0$. To one-dressed-loop order the *two* graphs shown in Fig. 2 must be present in a ghost-free gauge (and a ghost graph as well in gauges with ghosts) in order that this be true, and moreover the fully dressed vertex shown by the open circle must obey its own Ward identity. Unfortunately, a Gaussian trial wave functional, with only the first term in the exponent of (2.31) saved, yields an equation for Ω_{ij} which is equivalent to saving only Fig. 2(b). This is clear from the discussion of Ref. 2 concerning scalar field theories. Not only does a Gaussian ψ fail to implement gauge invariance, it also does not show the effects of asymptotic freedom, because Fig. 2(b) is momentum independent.

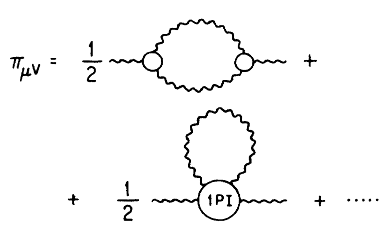

FIG. 2. One-dressed-loop Feynman graphs for $\Pi_{\mu\nu}$ (in a ghost-free gauge) corresponding to the minimal terms in ψ necessary to implement gauge invariance and asymptotic freedom.

We will show below how to construct three- and higher-point Ω's in (2.31) so that they are guaranteed to satisfy Ward identities such as (2.34). The construction will involve only one unknown operator, the two-point function Ω defined in (2.33). Direct substitution in the FSE will then yield an equation for Ω plus new identically conserved terms in the higher-point Ω functions. These last terms, which can be straightforwardly included, are not important for a qualitative understanding of nonperturbative effects, since they vanish in the zero-momentum limit and are of higher order in g^2 than the other terms saved. Dropping them corresponds to the usual implementation of the gauge technique. It is evident that working at the one-dressed-loop level of Fig. 2 requires us to understand the Ward identities through the four-point Ω function, which is of $O(g^2)$. Keeping terms of formal order g^4 and higher corresponds to looking at graphs with two or more dressed loops, which we will not do here.

Since the Hamiltonian H is gauge invariant, direct solution of the FSE automatically yields Ω functions obeying the Ward identities. We write the relevant equation for the three-point function Ω_{ijk}^{abc} as

$$\Omega_{il}(1)\Omega_{ljk}^{abc}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) + \Omega_{jl}(2)\Omega_{lik}^{bac} + \Omega_{kl}(3)\Omega_{lij}^{cab} = \epsilon^{abc}\Gamma_{ijk}$$
$$[\Omega_{il}(1) = \Omega_{il}(\mathbf{k}_1), \text{ etc.}] \ . \tag{2.35}$$

The left-hand side (LHS) of this equation comes from the cubic term in $(\delta S/\delta A)^2$ in $H\psi_v$, and the RHS comes from the cubic term in H itself plus a term from the five-point Ω function, which is itself determined by an infinite sequence of other equations. When only the cubic term in H is saved, Γ_{ijk} reduces to the perturbative vertex Γ_{ijk}^0:

$$\Gamma_{ijk}^0 = i[(k_1 - k_2)_k \delta_{ij} + (k_2 - k_3)_i \delta_{jk} + (k_3 - k_1)_j \delta_{ik}] \ . \tag{2.36}$$

It is necessary that Γ_{ijk} obey a Ward identity of its own if the Ward identity (2.34) is to be satisfied. This identity for Γ_{ijk} is

$$k_{1i}\Gamma_{ijk} = [\Omega_{jk}^2(3) - \Omega_{jk}^2(2)] \ , \tag{2.37}$$

which is readily checked in lowest order [where $\Omega^0(\mathbf{k}) = |\mathbf{k}|$]. Equation (2.37) follows from (2.34) and the transversality of Ω_{ij}. Below we will show how to solve (2.37) (modulo identically conserved term) for Γ_{ijk} in terms of Ω. First, we express Ω_{ijk}^{abc} in terms of Ω_{ij} and Γ_{ijk}. This is easily done by writing out the $k_i k_j \mathbf{k}^{-2}$ terms in Ω_{ij}, using (2.34), and the result is

$$\Omega^{abc}_{ijk}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)=[\Omega(1)+\Omega(2)+\Omega(3)]^{-1}\epsilon^{abc}\left[\Gamma_{ijk}+\left\{\Omega(1)\frac{k_{1i}}{\mathbf{k}_1^2}[\Omega(2)_{jk}-\Omega(3)_{jk}]+\text{c.p.}\right\}\right], \tag{2.38}$$

where c.p. stands for cyclic permutations. This should be compared to (2.17), the corresponding result for scalar Φ^3 theory; a naive guess based on (2.17) would give the first term in large square brackets in (2.38), but not the other terms, which are necessary to maintain the Ward identities. The reader should check that the divergence of (2.38) does yield (2.34), as it must, if (2.37) holds.

It only remains to construct Γ_{ijk} explicitly as a function which is totally antisymmetric, obeys (2.37), and reduces to (2.36) in lowest-order perturbation theory. In fact, this construction has already been given (Cornwall and Hou, Ref. 3), with the result

$$\Gamma_{ijk}=\Gamma^0_{ijk}-\left\{\frac{1}{2}\frac{k_{1i}k_{2j}}{\mathbf{k}_1^2\mathbf{k}_2^2}(k_1-k_2)_p\hat{\Pi}_{kp}(3)+[P_{ip}(1)\hat{\Pi}_{pj}(2)-\hat{\Pi}_{ip}(1)P_{jp}(2)]\frac{k_{3j}}{\mathbf{k}_3^2}+\text{c.p.}\right\}, \tag{2.39}$$

where P_{ip} is the transverse projection operator

$$P_{ip}(1)\equiv P_{ip}(\mathbf{k}_1)=\delta_{ip}-\frac{k_{1i}k_{1p}}{\mathbf{k}_1^2} \tag{2.40}$$

and $\hat{\Pi}_{kp}$ expresses the difference between the free and dressed versions of Ω_{kp}:

$$\Omega^2P_{kp}=\Omega^{02}P_{kp}+\hat{\Pi}_{kp}\ . \tag{2.41}$$

Equations (2.38) and (2.39) are major results of this section; they are the minimum constructs which are consistent with gauge invariance and the structure of the FSE. In lowest-order perturbation theory they give the correct solution to the FSE. Of course, (2.39) is not a unique solution to (2.37), since totally conserved terms can be added, and indeed in higher order of g the FSE requires such transverse terms. These transverse terms are of $O(g^2)$ in Ω^{abc}_{ijk}, and, hence, contribute only at two (or more) dressed loops; vanish in the infrared limit, where nonperturbative effects come into play; and are relatively unimportant in the ultraviolet where g^2 is replaced by a small running coupling constant. For these reasons, we will drop transverse terms in Γ_{ijk} from now on, although they can easily be included.

In principle, constructions similar to (2.38) and (2.39) could be carried out for four- and higher-point Ω functions. In practice, we will not need to record the explicit formulas, which become very lengthy.

III. $0^{\pm+}$ QUANTUM SOLITONS

The development of these solitons is somewhat analogous to the corresponding treatment for scalar field theories,[2] but complicated by the need to maintain gauge invariance. The strong-coupling soliton picture is very difficult because of entropylike effects (see the Introduction), and we will carry out explicit calculations only to the point of finding an effective Hamiltonian which (presumably) describes solitons in the presence of a condensate which we do not describe self-consistently. Our main line of approach will be through a trial wave functional, but we begin by discussing properties of the exact functional.

A. 0^{++} solitons

This soliton we will term S (for scalar); it is to be identified with the similarly named phenomenological glueball of Ref. 18. In a scalar field theory one would proceed from the vacuum wave functional (2.15) to a trial S functional by shifting ϕ by an amount $\phi(\mathbf{x})$. Thus, if S in (2.15) describes a ϕ^4 theory, i.e., the sum is over even N only, we have for the vacuum state $\langle\phi\rangle=0$ and for the S state $\langle\phi\rangle=\bar{\phi}$. This introduces terms with odd N in S, notably a term linear in ϕ. Likewise, in the gauge theory (2.31) is supplemented with a term linear in A^a_i in the exponent, and the kinematical structure of the Ω functions is different from, e.g., (2.33) or (2.38). The Ω functions will now have the kinematic structure of N-point functions for a gauge theory in a background gauge potential.

For the S wave functional ψ_S we write

$$\psi_S\{A^a_i\}=\exp\left[-\sum g^{-1}A^a_iM\bar{A}^a_i-\frac{1}{2!}\sum A^a_i\Omega^{ab}_{ij}A^b_j-\frac{g}{3!}\sum A^a_iA^b_jA^c_k\Omega^{abc}_{ijk}+\cdots\right]. \tag{3.1}$$

Here M is an operator to be chosen so that $\langle A^a_i\rangle=\bar{A}^a_i$; the Ω functions appearing here are *not* the same as in (2.31), but we do not note this explicitly to avoid cumbersome notation. The background gauge potential $\bar{A}_i$ is unchanged under the gauge transformation (2.28).

At this point, one should be concerned about the orthogonality of ψ_S and ψ_v, the vacuum wave functional. If the S soliton were topological (it is not), ψ_S of (3.1) and ψ_v of (2.31) would be automatically orthogonal in the infinite-volume limit. It is easy in principle to orthogonalize ψ_S to ψ_v by mixing of these two states, but in practice one encounters delicate problems similar to those encountered in discussing the entropy of condensation. We will not discuss orthogonalization further here, but will simply use (3.1) as it stands.

We can, just as we did for ψ_v, find a series of Ward identities for the coefficient functions in (3.1) which ensure that ψ_S is gauge invariant. Under the infinitesimal gauge transformation (2.32), we require

$$\partial_i\bar{A}^a_i=0\ , \tag{3.2a}$$

$$\partial_i \Omega_{ij}^{ab} = M \epsilon_{abc} \overline{A}_j^c \tag{3.2b}$$

as the first two in an infinite sequence of equations. The rest of these equations are formally the same as before, but with different solutions.

Assuming that (3.2a) holds, we solve (3.2b) by

$$\Omega_{ij}^{ab} = M \epsilon_{abc} \frac{1}{\nabla^2}(\partial_i \overline{A}_j^c - \partial_j \overline{A}_i^c) + \Omega \delta_{ab} \left[\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2} \right] . \tag{3.3}$$

The second term on the RHS of (3.3) is of the kinematic form already discovered in (2.33). The FSE shows that M and Ω both depend on $\overline{A}_i^a$, so (3.3) and its successors are not linear relations in the external potential. If one saves only the $N=1$ and 2 terms in the expansion of S, then the choice $M=\Omega$ ensures that $\langle A_i^a \rangle = \overline{A}_i^a$.

At first it may seem unnatural to introduce a potential $\overline{A}_i^a$ which is gauge invariant and identically conserved. In fact, such potentials do occur naturally in gauge theories, in the form of a decomposition into a gauge potential in an arbitrary gauge and an auxiliary group matrix.[6] We write (using matrix instead of component notation)

$$\overline{A}_i = U^{-1} \mathcal{A}_i U + g^{-1} U^{-1} \partial_i U , \tag{3.4}$$

where U is a unitary matrix, and assume the gauge transformation laws

$$\mathcal{A}_i \rightarrow V \mathcal{A}_i V^{-1} + g^{-1} V \partial_i V^{-1} , \tag{3.5a}$$

$$U \rightarrow VU , \tag{3.5b}$$

the first of which has already been given in (2.28). It is an elementary calculation to show that $\overline{A}_i$ is formally gauge invariant. To discuss the conserved nature of $\overline{A}_i$ it is convenient to introduce the auxiliary potential C_i,

$$C_i = \mathcal{A}_i + g^{-1}(\partial_i U) U^{-1} = U \overline{A}_i U^{-1} , \tag{3.6}$$

which transforms homogeneously:

$$C_i \rightarrow V C_i V^{-1} . \tag{3.7}$$

We impose[6] the condition that C_i be covariantly conserved:

$$[\mathcal{D}_i, C_i] = 0 , \tag{3.8}$$

where $\mathcal{D}_i$ is the covariant divergence based on $\mathcal{A}_i$. Another elementary calculation shows that this implies the conservation law (3.2a) for $\overline{A}_i$. The reason for imposing (3.8) is that C_i will appear as a source [see (3.23) below] for the classical Yang-Mills equations and, therefore, must be covariantly conserved. This reason aside, imposition of (3.8), which may be understood as a set of equations determining the matrix U, allows for the construction of U as a formal series in $\mathcal{A}_i$ such that if $\mathcal{A}_i$ obeys its transformation law (3.5a), then U automatically satisfies (3.56) (Ref. 6). With the notation

$$U = \exp \left[\frac{i}{2} \lambda_a \theta^a \right] \tag{3.9}$$

the first few terms of the series are

$$\theta^a = g \frac{1}{\nabla^2} K - g^2 \frac{1}{\nabla^2} \left[\frac{1}{2} K \times \frac{1}{\nabla^2} K + \mathcal{A}_j \times \partial_j \frac{1}{\nabla^2} K \right]_a + \cdots , \tag{3.10}$$

where

$$K = \partial_i \mathcal{A}_i , \tag{3.11}$$

$$(A \times B)_a \equiv \epsilon_{abc} A_b B_c . \tag{3.12}$$

One may check by direct substitution from (3.9)–(3.12) that U satisfies (3.5b) if $\mathcal{A}_i$ satisfies (3.5a). From these developments it follows that a gauge-invariant conserved potential such as $\overline{A}_i$ can always be constructed formally in terms of a potential $\mathcal{A}_i$ subject to no restrictions, transforming in the usual way.

At this point we could proceed by direct substitution in the FSE, just as in Sec. II. We leave that exercise to the reader, and instead give our explicit form for a trial wave functional to be used variationally. This trial wave functional, analogous to the soliton wave functional of scalar field theory which is found[2] by shifting the argument ϕ of the trial vacuum wave functional, is guaranteed to be gauge invariant.

Begin by simply shifting a wave functional of the form (2.31), yielding the functional $\psi\{A_i^a - \overline{A}_i^a\}$. Even though the Ω functions in (2.31) satisfy their Ward identities and $\overline{A}_i^a$ is gauge invariant, this wave functional is not; instead, it transforms as

$$\begin{aligned} \psi\{V A_i V^{-1} + g^{-1} V \partial_i V^{-1} - \overline{A}_i\} &= \psi\{V(A_i - V^{-1}\overline{A}_i V)V^{-1} + g^{-1} V \partial_i V^{-1}\} \\ &= \psi\{A_i - V^{-1} \overline{A}_i V\} . \end{aligned} \tag{3.13}$$

The unwanted homogeneous transformation of $\overline{A}_i$ can be undone by replacing $\overline{A}_i$ with $U(A)\overline{A}_i U^{-1}(A)$, where $U(A)$ is the construction (3.9)–(3.12) but in terms of A_i, not $\mathcal{A}_i$. So a suitable gauge-invariant trial wave functional for the S soliton is just

$$\psi_S\{A_i\} = \psi\{A_i - U(A)\overline{A}_i U^{-1}(A)\} , \tag{3.14}$$

where ψ is of the form given in (2.31). As we have already said, the Ω functions in (3.14) are not the same as those which describe the vacuum with no additional soliton; their values will depend on the $\overline{A}_i$. However, they obey the same Ward identities, which ensure gauge invariance. If only the quadratic term in the exponent of (3.14) is saved, the expectation value of A_i is $\overline{A}_i$, but cubic and other odd-N terms change this result; we will not bother to distinguish $\langle A_i \rangle$ from $\overline{A}_i$ in what follows because it adds nothing new in principle.

Let us recapitulate: a trial wave functional for a 0^{++} soliton which is locally gauge invariant is given by

$$\psi_S\{A_i\}=e^{-S},$$

$$S=\frac{1}{2!}\sum \Omega_{ij}^{ab}Q_i^aQ_j^b+\frac{g}{3!}\sum Q_i^aQ_j^bQ_k^c\Omega_{ijk}^{abc}+\cdots, \tag{3.15}$$

$$Q_i=A_i-U(A)\overline{A}_iU^{-1}(A), \tag{3.16}$$

where the Ω functions obey Ward identities such as (2.33) and (2.34), and U is given in (3.9)–(3.12) as a function of A_i. The soliton potential $\overline{A}_i$ is gauge invariant and conserved, and can be written in the form (3.5) in terms of an unrestricted extended potential $\mathcal{A}_i$ and associated gauge matrix $U(\mathcal{A})$.

It now remains to parametrize the Ω functions in a way which satisfies the Ward identities, and to calculate the expectation value of the Hamiltonian H. This expectation value is then minimized with respect to the independent functions which appear in it. We will consider only the case where the Ω functions are constructed according to the gauge technique as in (2.33) and (2.38), so there is only one independent scalar function Ω in addition to the external gauge potential $\overline{A}_i$. (Similarly in a scalar field theory with a Gaussian trial functional, there is one propagator; the soliton field $\overline{\phi}$ is the other functional variable.) The calculation of $\langle H\rangle$ is quite analogous to that given in Ref. 2 for a ϕ^4 field theory, differing only in the tedious details of the extra terms in (3.15) and (3.16) which ensure gauge invariance, and in the need to account for a cubic interaction in H. It is, in fact, easy to envisage the results of this calculation without actually carrying it out, by considering the results of Ref. 2, as given there in Eq. (5.23) [in that equation, $G=(2\Omega)^{-1}$ in our present notation]. The variational equation for Ω is the same as for the vacuum state with additional terms of quadratic and higher order in $\overline{A}_i$. Neglect these additional terms for the moment; then the Ω equation shows that $\Omega^2-\Omega_0^2$ contains some terms that generate a gluon mass, as we have already discussed in covariant form in Ref. 4. In the ϕ^4 theory of Ref. 2, the mass-generation mechanism is the seagull graph. In gauge theory, the seagull term is supplemented by extra contributions from the other one-loop graph of Fig. 2, which are necessary for gauge invariance and for asymptotic freedom. This mass contribution needs regularization, and some relevant approaches are given in Ref. 4, which we do not repeat here. There are also (divergent) terms in $\Omega^2-\Omega_0^2$ which are removed by wave-function renormalization and which also contribute to coupling-constant renormalization. The only point which is of real concern to us is the nonperturbative generation of a gluon mass term in Ω, so we will use the simple form

$$\Omega^2=-\nabla^2+m^2+O(\overline{A}_i^2) \tag{3.17}$$

for further discussion.

Next, consider the terms in $\langle H\rangle$ which are quadratic in $\overline{A}_i$. It is easy to show that these are of the form

$$\langle H\rangle=\tfrac{1}{2}\sum \overline{A}_i^a(\Omega^2)_{ij}^{ab}\overline{A}_j^b$$

$$=\tfrac{1}{2}\sum \overline{A}_i^a(-\nabla^2+m^2)\left[\delta_{ij}-\frac{\partial_i\partial_j}{\nabla^2}\right]\overline{A}_j^a. \tag{3.18}$$

At $m=0$ these give the usual kinetic energy tensor

$$\tfrac{1}{4}\sum(\partial_i\overline{A}_j^a-\partial_j\overline{A}_i^a)^2. \tag{3.19}$$

In view of the conserved nature of $\overline{A}_i$ and with the help of the decomposition (3.4), the mass term is

$$\tfrac{1}{2}m^2(-2\,\mathrm{Tr})[\mathcal{A}_i+g^{-1}(\partial_iU)U^{-1}]^2. \tag{3.20}$$

It is not hard to appreciate that there are extra cubic and quartic terms in the $\overline{A}_i$ which complete (3.19) to the usual Yang-Mills form. According to (3.4), $\overline{A}_i$ differs from the unrestricted external potential $\mathcal{A}_i$ by a gauge transformation, so the Yang-Mills form can be written in terms of $\mathcal{G}_{ij}$, the field strength constructed from $\mathcal{A}_i$ as in (2.27). As a result we can identify a pseudoclassical contribution to $\langle H\rangle$ containing all the terms of the form

$$\langle H\rangle=g^{-2}F(g\mathcal{A}_i,U) \tag{3.21}$$

[we consider m^2 to be formally independent of g in writing (3.21)]. This pseudoclassical contribution is

$$\langle H\rangle_{\rm pc}=(-2\,\mathrm{Tr})\int d^3x\left[\tfrac{1}{4}\mathcal{G}_{ij}^2+\frac{m^2}{2}[\mathcal{A}_i+g^{-1}(\partial_iU)U^{-1}]^2\right] \tag{3.22}$$

plus terms independent of $\mathcal{A}_i$, which we will not consider here. In (3.21) and (3.22), the coupling constant g should be thought of as evaluated at a renormalization point of order m. A more exact treatment of the FSE than we have given here would replace this simple prescription for g by a complicated and nonlocal dependence on $\mathcal{A}_i$ in the spirit of the well-known one-loop correction[22] to the effective action.

Equation (3.22) is the simplest gauge-invariant approximation to $\langle H\rangle$ which allows us to see quantum solitons. In fact, $\langle H\rangle$ contains terms of all orders in $\mathcal{A}_i$ including also terms which behave like $\ln\mathcal{A}_i$. The omitted terms contain, essentially by definition, all those effects which we referred to as entropy effects in the Introduction, and they are by no means minor corrections to the soliton phenomena contained in the simple approximation (3.22). We are unable to deal convincingly with these effects now, and perforce must ignore them. However, the presumed chain of effects is that entropy dominates, leading to a condensate of solitons, and the world-line entropy will appear as an effective soliton (not gluon) wrong-sign mass term.[17] The soliton condensate then furnishes the vacuum fluctuations that drive the gluon mass term, closing the chain.

Even though (3.22) cannot be a quantitative description of the S soliton, it certainly goes beyond purely classical QCD, which has no finite-energy static solitons at all.

We have described elsewhere[12,13] this nontopological soliton in considerable detail and thus, will be very brief here. The equations found by varying $\mathcal{A}_i$ in (3.22) are

$$0=-[\mathcal{D}_i,\mathcal{G}_i]+m^2[\mathcal{A}_i+g^{-1}(\partial_i U)U^{-1}] , \quad (3.23)$$

where $\mathcal{D}_i$ is the covariant derivative based on $\mathcal{A}_i$. The equations of motion for U can be found[6] in varying U in (3.22), or more simply by taking the covariant divergence of (3.23). The result is just (3.8), which we have already noted is equivalent to $\partial_i \overline{A}_i=0$. The SU(2) hedgehog ansatz

$$\mathcal{A}_i=\frac{1}{2i}\epsilon_{iak}\tau_a\hat{\mathbf{x}}_k\left[\frac{\phi_1(r)-1}{r}\right]+\frac{1}{2i}(\tau_i-\hat{\mathbf{x}}_i\hat{\mathbf{x}}\cdot\tau)\frac{\phi_2(r)}{r}$$
$$+\frac{1}{2i}\hat{\mathbf{x}}_i\hat{\mathbf{x}}\cdot\tau H_1(r) , \quad (3.24a)$$

$$U=\exp\left[\frac{i}{2}\tau\cdot\hat{\mathbf{x}}\beta(r)\right] \quad (3.24b)$$

allows[12,13] for a finite-energy solution to (3.23). The form of the ansatz (3.24) is important in deducing the J^{PC} properties of the S soliton; it will not be necessary to repeat the detailed structure of the soliton, as found numerically in Refs. 12 and 13.

Although four functions appear in (3.24), only three combinations have physical (gauge-invariant) relevance. These are

$$I_1\equiv\phi_1\sin\beta-\phi_2\cos\beta , \quad (3.25a)$$

$$I_2\equiv\phi_1\cos\beta+\phi_2\sin\beta , \quad (3.25b)$$

$$I_3\equiv\beta'-\frac{H_1}{r} . \quad (3.25c)$$

It is an instructive exercise to show that the field $\overline{A}_i$ of (3.4) depends only on the I_j, as does the energy density in (3.23). The S soliton is characterized by $I_1=I_3=0$, a specific realization of which turns out to be[13] $\phi_2=H_1=0$, $\beta=\pi$. In this realization, or gauge choice, it is evident from (3.24) that

$$\mathcal{A}_i(-\mathbf{x})=-\mathcal{A}_i(\mathbf{x}) , \quad (3.26a)$$

$$[\partial_i U(-x)]U^{-1}(-\mathbf{x})=-[\partial_i U(\mathbf{x})]U^{-1}(\mathbf{x}) . \quad (3.26b)$$

Now consider the behavior of $\psi\{A_i(\mathbf{x})\}$ under the parity operation, which replaces $A_i(\mathbf{x})$ by $-A_i(-\mathbf{x})$, modulo a gauge transformation. One finds, using (3.26), that the soliton wave functional ψ_S of (3.24) is even under parity, as expected. As is well known, ψ_S is a spatial scalar, since a spatial rotation of $\mathcal{A}_i$ and U can be compensated for by a gauge transformation. Similarly, ψ_S is even under C, so we are describing a 0^{++} soliton.

B. 0^{-+} soliton

We will construct a trial wave functional ψ_p for a pseudoscalar soliton called (as in Ref. 4) P, by a simple modification of ψ_S of (3.14). The P soliton will be heavier than the S soliton by an amount which is related to $M_S^{-3}\langle g^2G_{\mu\nu}^2\rangle$, but which is impossible for us to calculate accurately.

The properties of P have simple analogs in the harmonic oscillator. In the potential $V=k^2x^2$, the lowest state is even parity, with wave function $\psi_S\sim\exp[-(2\hbar)^{-1}kx^2]$. The first excited state has odd parity, and its wave function is $\psi_P\sim x\psi_S$. Note that in the classical limit $\hbar\rightarrow 0$, ψ_S is proportional to $\delta(x)$ while $\psi_P\sim x\delta(x)=0$. Thus, parity is a concept lacking a classical interpretation.

In QCD, we proceed by constructing the trial wave functional

$$\psi_P\{A_i\}=\sin\pi W\psi_S\{A_i\} , \quad (3.27)$$

where ψ_S is the scalar wave functional in (3.15), and W is the Chern-Simons secondary class:

$$W=-\frac{g^2}{4\pi^2}\epsilon^{ijk}\int d^3x\, \mathrm{Tr}\left[\tfrac{1}{2}A_i\partial_j A_k-\frac{g}{3}A_iA_jA_k\right] . \quad (3.28)$$

Under the parity operation, W is odd, so ψ_P is indeed a pseudoscalar wave functional. Moreover, the (pseudo-) classical limit of ψ_p vanishes, because the S-soliton wave function $\mathcal{A}_i$, satisfying (3.23) and having $\phi_2=H_1=0$, $\beta=\pi$, yields zero for the integral in (3.28). The reason for using $\sin\pi W$ instead of any other odd function of W is that under a gauge transformation with nonzero winding number N, $W\rightarrow W+N$ and thus, under large gauge transformations $\sin\pi W$ suffers at most a change of sign. (We have not investigated the behavior of ψ_S under gauge transformation with $N\neq 0$, for which considerations of infinitesimal transformations are irrelevant.[15]) One might consider constructing ψ_P by multiplying ψ_S by $\int d^3x\, G_{\mu\nu}^* G^{\mu\nu}$ or equivalently by $\int d^3x\, \epsilon^{ijk}E_iG_{jk}$. In the FSE, E_i is equivalent to $-i\delta/\delta A_i$, and acting on ψ_S this generates a complicated wave functional which we have not analyzed in detail.

It is clear that P is heavier than S, on the basis of the trial wave functional (3.27), but by exactly how much is problematical, even if we ignore the effects of entropy of world lines. We can make a very rough estimate by looking at one term in the mass difference $\langle H\rangle_P-\langle H\rangle_S$, the term coming from the functional derivatives in H of (2.26) acting on $\sin\pi W$. This term is

$$\langle H\rangle_P-\langle H\rangle_S=\left\langle \tfrac{1}{2}d^3x\sum\left[\pi\cos\pi W\frac{\delta W}{\delta A_i^a}\right]^2\right\rangle_S + \cdots , \quad (3.29)$$

where $\langle\ \rangle_S$ indicates an expectation value with the wave functional ψ_S. Since

$$\frac{\delta W}{\delta A_i^a}=\left[\frac{g^2}{8\pi^2}\right]\tfrac{1}{2}\epsilon_{ijk}G_{jk}^a \quad (3.30)$$

this term in the mass difference will involve the mean-square magnetic field fluctuations integrated over a soliton. The covariant form of these remarks will be

$$M_P - M_S \simeq \frac{\left\langle g^2 \sum G^a_{\mu\nu} G^{\mu\nu}_a \right\rangle}{M_S^3} \tag{3.31}$$

with an unknown coefficient; we have assumed that field fluctuations in the vacuum are not too different from field fluctuations in solitons, since the vacuum is supposed to be made of solitons. If we take (3.31) literally (that is, the numerical coefficient is unity), assume $M_S = 1$ GeV, and take the value of Novikov, Shifman, Vainshtein, and Zakharov[22] of 0.47 GeV^4 for the expectation value, we would guess $M_P \simeq 1.5$ GeV, a not unreasonable value (cf. the ι at 1.44 GeV).

Finally, let us discuss one of the most interesting features of the S and P solitons: they mix with each other when the vacuum angle θ is nonzero. Such mixing has been invoked in Ref. 18, as a phenomenological solution to the U(1) problem. The reason for mixing is that in the presence of the θ angle, the electric field and the momentum are related[15] by

$$E_i^a = \Pi_i^a + \theta \left[\frac{g^2}{8\pi^2} \right] \tfrac{1}{2} \epsilon_{ijk} G^a_{jk} \ , \tag{3.32}$$

which implies[15] that the wave function depends on θ as

$$\psi_\theta\{ A_i \} = e^{-i\theta W} \psi_{\theta=0}\{ A_i \} \ . \tag{3.33}$$

We have not progressed far enough in this paper to understand whether the simple phenomenological mixing of S and P under U(1) rotations (or equivalently a change of θ) advocated in Ref. 18 is a reasonable description of what really happens. This will require an intimate understanding of the phenomenon discovered in Ref. 13, wherein soliton world lines can terminate on fractionally charged instantons (i.e., instantons whose winding number is fractional). A *classical* fractionally charged instanton has infinite action. The pseudoclassical massive theory described by the four-dimensional version of (3.23) has finite-action configurations consisting of two fractionally charged instantons whose total winding number is an integer separated by a soliton world line. Of course, as the separation between the two fractionally charged instantons grows, the action of the world line also grows in proportion to its length. The combined system of S and P solitons and fractionally charged instantons, in a vacuum with $\theta \neq 0$, is an extraordinarily rich structure which we only dimly understand now.

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