Axial anomaly and staggered fermions in the coordinate-space interpretation

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The U(1) axial anomaly is derived in the framework of the coordinate-space interpretation for staggered fermions. The derivations are performed explicitly on a two-dimensional Euclidean lattice.

I. INTRODUCTION

Symmetries are of vital importance for physical theories even when they are broken. In particular, a certain symmetry of a theory can be broken anomalously when the theory is promoted to the quantum level. Anomalies so produced have helped us to understand the physical world and have guided us in the building of new theories. In this work we are concerned with the anomaly of the global U(1) axial symmetry in the staggered-fermion regularization scheme.

The global U(1) axial invariance of a Dirac action is anomalously broken when the theory is quantized and this is known to be the result of the incompatibility of chiral and gauge symmetries.[1] An example in the class of lattice fermionic regularizations is the Wilson fermion action,[2] where gauge invariance is maintained but chiral symmetry is broken explicitly by a term different from the usual mass term. It has been shown that[3,4] one cannot recover the axial symmetry in the continuum limit; it is precisely the chiral-breaking term in the action that gives rise to the anomaly.

The study of chiral anomalies on a lattice is important and has been successful. Not only does it provide a check on the validity of certain lattice regularization, the appearance of anomalies is also the foundation of the well-known Nielsen-Ninomiya no-go theorem,[5] which excludes the realization of chiral fermions on the lattice under reasonable assumptions.

As an illustration, we consider the so-called naive fermion action on an even-d-dimensional Euclidean lattice

$$S=(a^d)^2\sum_{x,y}\bar{\psi}(x)[\not{D}(x,y)+m\,\delta_{x,y}]\psi(y)\ , \tag{1}$$

where a is the lattice spacing and $\not{D}$ is the lattice Dirac operator

$$\not{D}(x,y)\equiv\sum_{\mu}\frac{1}{2a}\gamma_\mu[U_\mu(x)\delta_{x+a\hat{\mu},y}-U_\mu^\dagger(y)\delta_{x-a\hat{\mu},y}]\ .$$

The gauge link is parametrized as

$$U_\mu(x)=\exp[igaA_\mu(x)]\ .$$

It is clear that both chiral and gauge invariances are preserved for this action. As a result, a straightforward definition of chiral current

$$J_\mu^5(x)_{\text{naive}}\equiv\tfrac{1}{2}[\bar{\psi}(x)\gamma_\mu\gamma_5U_\mu(x)\psi(x+a\hat{\mu})+\text{H.c.}]\ , \tag{2}$$

cannot reproduce the anomaly. In fact, we have

$$\begin{aligned}\langle\Delta_\mu J_\mu^5(x)\rangle &\equiv Z_\psi^{-1}\int_\psi e^{-S}\sum_\mu\frac{1}{a}[J_\mu^5(x)-J_\mu^5(x-a\hat{\mu})]\\ &=\left\langle\bar{\psi}(x)\gamma_\mu\gamma_5\frac{1}{2a}[U_\mu(x)\psi(x+a\hat{\mu})-U_\mu^\dagger(x-a\hat{\mu})\psi(x-a\hat{\mu})]+\text{H.c.}\right\rangle\\ &=\text{tr}\{[\gamma_5\not{D},(\not{D}+m)^{-1}](x,x)\}=\text{tr}[\gamma_5\{\not{D},(\not{D}+m)^{-1}\}(x,x)]\\ &=2m\langle\bar{\psi}(x)\gamma_5\psi(x)\rangle\ ,\end{aligned} \tag{3}$$

where

$$Z_\psi\equiv\int_\psi e^{-S}\ .$$

Here and in what follows we use Hermitian γ matrices,

$$\begin{aligned}&\{\gamma_\mu,\gamma_\nu\}=2\delta_{\mu,\nu}\ ,\\ &\gamma_5\equiv\begin{cases}i\gamma_1\cdots\gamma_d, & d=4k+2\ ,\\ \gamma_1\cdots\gamma_d, & d=4k\ ,\end{cases}\end{aligned} \tag{4}$$

together with dimensionful coordinates and fields. Furthermore, repeated indices are summed over unless specified otherwise.

The absence of the anomaly term in (3) is reached at the expense of extra species. As the naive fermion action describes 2^d fermionic species instead of one, Karsten and Smit[3] have shown that these doublers are generated in such a way as to cancel the would-be anomaly.

To recover the anomaly the cancellation has to be removed. This can be done by decoupling all but one of the species as in the case of the Wilson fermions mentioned above. Alternatively, one can introduced a lattice chiral transformation such that all species transform in the *same* way. That is, the transformation is no longer a symmetry of the action although it reduces to the usual form of its counterpart in the continuum limit. The modification is possible due to the freedom we have on the lattice to add irrelevant terms. Sharatchandra, Thun, and Weisz[6] adopted the latter strategy for naive fermions. They have introduced, by means of a doubler interpretation, another current different from (2) so that the contributions from various species add up. In either way of removing the cancellation, the important role of the species interpretation cannot be more emphasized.

The staggered-fermion formulation[6–8] is yet another class of multispecies lattice gauge theories. Because of a certain remnant of chiral symmetry, the formulation is useful for the study of dynamical behavior relating to the symmetry. As will be seen in the next section staggered fermions enjoy a close link with the naive ones. This, in turn, enables the authors of Ref. 6 to reduce their naive-fermion chiral current to that of the staggered fermions. The correct anomaly was then subsequently derived. However, staggered fermions themselves also admit an elegant flavor interpretation in coordinate space,[8] which has been employed extensively in numerical simulations. It is, therefore, important and interesting to investigate the U(1) chiral anomaly in the framework of this interpretation.

We devote the next section to a cursory review of staggered fermions and their coordinate-space flavor interpretation. There we introduce a new formalism suitable for subsequent calculations. In Sec. III we give a form for the lattice chiral current in the flavor interpretation. This current is then proved to possess the right anomaly explicitly in two dimensions with a *background* Abelian gauge field. In most other anomaly derivations on the lattice, the Ward-Takahashi identity is employed. The identity is the expression of the invariance of the generating function under some change of the field variables. One then still has to identify the chiral current to express the identity as the lattice divergence of the current on one side and the anomaly-producing term on the other. In this work, however, after identifying the current we calculate the continuum limit of the lattice divergence *directly*. Some general criteria for choosing the appropriate axial-vector current will also be mentioned. The last section contains some concluding remarks.

II. STAGGERED FERMIONS

Without loss of generality we will work with massless theories. There are two approaches to derive the staggered-fermion action from that of the naive fermions (1). Sharatchandra, Thun, and Weisz[6] use a projection operator to restrict the Grassmann measure; namely, naive-fermion fields are projected onto a one-dimensional subspace. We, however, follow Kawamoto and Smit[9] and use their unitary transformation

$$\psi(r)=\Gamma_{r/a}\phi(r),\quad \bar\psi(r)=\bar\phi(r)\Gamma^\dagger_{r/a}\ ,\qquad \Gamma_{r/a}=\gamma_1^{r_1/a}\cdots\gamma_d^{r_d/a}\ , \tag{5}$$

to diagonalize the γ matrices of the naive action. The resulting form then contains a piece bilinear in $\phi_\alpha(r)$, which is replicated for $2^{d/2}$ values of the spinor index α. The desired action for the one-component Grassmann fields $\phi(r)$ and $\bar\phi(r)$ at each site is thus obtained by discarding all but one of the copies:

$$\begin{aligned} S&=a^d\sum_{\mu,r}\alpha_\mu(r)\bar\phi(r)\frac{1}{2a}[U_\mu(r)\phi(r+a\hat\mu)\\ &\qquad -U^\dagger_\mu(r-a\hat\mu)\phi(r-a\hat\mu)]\ ,\\ \alpha_\mu(r)&=\frac{1}{2^{d/2}}\mathrm{tr}(\Gamma^\dagger_{r/a}\gamma_\mu\Gamma_{r/a\pm\hat\mu})\\ &=(-1)^{(r_1+\cdots+r_{\mu-1})/a}\ . \end{aligned} \tag{6}$$

This action describes $2^{d/2}$ species and these can be identified as different flavors either in momentum space[7] or in coordinate space.[8] We will work in the (quasi)local coordinate-space interpretation. A proof of the equivalence of these two interpretations in the continuum limit has been given elsewhere.[10]

First, we partition the lattice into hypercubes

$$r_\mu=x_\mu+a\eta_\mu,\quad \frac{x_\mu}{2a}\in Z,\quad \eta_\mu=0\ \text{or}\ 1 \tag{7}$$

and introduce the notation

$$\chi_\eta(x)\equiv\frac{1}{2^{d/2}}\phi(x+a\eta)\ ,\qquad \bar\chi_n(x)\equiv\frac{1}{2^{d/2}}\bar\phi(x+a\eta)\ . \tag{8}$$

Then by means of a unitary transformation, gauge-covariant quark fields with spinor (α) and flavor (a) indices are defined[8] as

$$q^{\alpha a}(x)=\frac{1}{2^{d/4}}\sum_\eta\Gamma^{\alpha a}_\eta\mathcal{U}_\eta(x)\chi_\eta(x)\ ,\qquad \bar q^{\,\alpha a}(x)=\frac{1}{2^{d/4}}\sum_\eta\bar\chi_\eta(x)\mathcal{U}^\dagger_\eta(x)\Gamma^{*\alpha a}_\eta\ , \tag{9}$$

where

$$\mathcal{U}_\eta(x)\equiv[U_1(x)]^{\eta_1}\cdots[U_d(x+a(\eta_1+\cdots+\eta_{d-1}))]^{\eta_d}$$

is the product of link variables $U_\mu(r)$ along a definite path going from x to $x+a\eta$. Lattice operators can now be constructed in terms of these local (hypercube-confined) quark fields. In fact, in the next section lattice chiral currents are formed this way. But when (8) and (9) are substituted into the action (6) the resulting form is rather complicated due to the appearance of $\mathcal{U}_\eta(x)$. So, instead, we work directly with the one-link action (6).

We define the vectors $\chi(x)$ [$\bar\chi(x)$] whose η-component is $\chi_\eta(x)$ [$\bar\chi_\eta(x)$], and the matrix $\overline{\Gamma_A\otimes T_B}$ with the (η,η')-component

$$(\overline{\Gamma_A \otimes T_B})_{\eta\eta'} \equiv \frac{1}{2^{d/2}} \mathrm{tr}(\Gamma_\eta^\dagger \Gamma_A \Gamma_{\eta'} \Gamma_B) , \tag{10}$$

where

$$T_B = \Gamma_B^* = \Gamma_B^t .$$

The multiplication rule is

$$(\overline{\Gamma_A \otimes T_B}\;\overline{\Gamma_C \otimes T_D})_{\eta\eta'} \equiv \sum_\theta (\overline{\Gamma_A \otimes T_B})_{\eta\theta} (\overline{\Gamma_C \otimes T_D})_{\theta\eta'} . \tag{11}$$

The definition follows the usual matrix definition of lattice propagator with positional vectors (x,y) being the indices; the only difference here is that (η,η') are dimensionless; hence, the multiplication rule has to be modified accordingly. It can then be shown that

$$\overline{\Gamma_A \otimes T_B}\;\overline{\Gamma_C \otimes T_D} = \overline{\Gamma_A \Gamma_C \otimes T_B T_D} , \tag{12}$$

and, when taking the generalized trace Sp,

$$\mathrm{Sp}(\overline{\Gamma_A \otimes T_B}) \equiv \sum_\eta (\overline{\Gamma_A \otimes T_B})_{\eta\eta} = (\mathrm{tr}\Gamma_A)(\mathrm{tr}\Gamma_B^t) = (\mathrm{tr}\Gamma_A)(\mathrm{tr}T_B) , \tag{13}$$

with tr as the ordinary trace. These matrices thus behave in exactly the same way as direct products of ordinary matrices. To arrive at the results, we have used the orthogonality identity

$$\sum_A \Gamma_A^{a\alpha} \Gamma_A^{\dagger b\beta} = 2^{d/2} \delta_{\alpha\beta} \delta_{ab} . \tag{14}$$

We next rewrite the action (6) as

$$S = (2a)^{2d} \sum_{x,y} \bar{\chi}(x) G^{-1}(x,y) \chi(y) , \tag{15}$$

where we have substituted into (6) the identities

$$\chi_\eta(x \pm 2a\hat{\mu}\eta_\mu) = \eta_\mu \chi_\eta(x \pm 2a\hat{\mu}) + (1-\eta_\mu)\chi_\eta(x)$$

to get the inverse propagator

$$[G^{-1}(x,y)]_{\eta\eta'} = \frac{1}{2^{d/2}} \mathrm{tr}(\Gamma_\eta^\dagger \gamma_\mu \Gamma_{\eta'}) \frac{1}{2a} \{ U_\mu(x + a\eta)\eta_\mu \delta_{x+2a\hat{\mu},y} - U_\mu^\dagger(x + a\eta - a\hat{\mu})\eta'_\mu \delta_{x-2a\hat{\mu},y} + [U_\mu(x + a\eta)\eta'_\mu - U_\mu^\dagger(x + a\eta - a\hat{\mu})\eta_\mu]\delta_{x,y} \} . \tag{16}$$

Assuming that A_μ is slowly varying and $agA_\mu \ll 1$ as usual in the anomaly derivation on the lattice. (These assumptions can be derived from the single assumption of $a^2 gF_{\mu\nu} \ll 1$ with an appropriate gauge choice. I am grateful to Sen[11] for pointing this out to me.) We can, after the expansion

$$U_\mu(x + a\eta - a\hat{\mu}) = 1 + igaA_\mu(x + a\eta - a\hat{\mu}) + \cdots ,$$

expand $A_\mu(x + a\eta - a\hat{\mu})$ around x up to the necessary orders in a to obtain

$$\begin{aligned} G^{-1}(x,y) &= \frac{1}{2a} \int_p e^{ip(x-y)} \{ [\overline{\gamma_\mu \otimes 1} i \sin 2ap_\mu + \overline{\gamma_5 \otimes t_\mu t_5}(\cos 2ap_\mu - 1)] \\ &\quad + igaA_\mu(x)[\overline{\gamma_\mu \otimes 1}(\cos 2ap_\mu + 1) + \overline{\gamma_5 \otimes t_\mu t_5} i \sin 2ap_\mu] \} + \text{higher-order terms in } a \\ &\equiv \frac{1}{2a} \int_p e^{ip(x-y)} [S_0^{-1}(pa) + igaA_\mu(x) Q_\mu(pa)] + \text{higher-order terms in } a , \end{aligned} \tag{17}$$

where the shorthand notation

$$\int_p \equiv \int_{-\pi/2a}^{\pi/2a} \frac{d^d p}{(2\pi)^d}$$

has been used. Note that as x and y are the coordinates on the coarse lattice with spacing $2a$, the resulting Brillouin zone is $[-\pi/2a, \pi/2a]^d$.

We have concealed in "higher-order terms" in (17) terms of order $O(g^2)$ and derivative terms, with appropriate powers of a, coming from the Taylor expansion of $A_\mu(x + a\eta - a\hat{\mu})$ around x. This is equivalent to the Taylor expansion in momentum space of the interacting vertex, up to $O(g)$, in al where l is the gauge-field momentum. Slowly varying field A_μ means that it only has low-momentum components; its Fourier transform is only nonvanishing for $|al| \ll 1$. The vertex $Q_\mu(pa)$ is then arrived at as the lowest order of the expansion and all the other terms containing explicit powers of a (from powers of al) are represented by the "higher-order terms." The expansion is permissible for the anomaly derivation since there is no gauge-field propagator contribution. That is, up to one-loop order as the anomaly is one-loop effect, we only have internal fermion loops.

The first term on the right-hand side of Eq. (17) is the free inverse propagator denoted by $G_0^{-1}(x,y)$, and the rest of the terms are denoted by $V(x,y)$ with the definitions $t_\mu=\gamma_\mu^*=\gamma_\mu^t$, $t_5=\gamma_5^*=\gamma_5^t$. In the derivation of this equation, the anticommutators (4) together with the representation of η_μ,

$$\eta_\mu=\tfrac{1}{2}[1-(-1)^{\eta_\mu}] ,$$

enable us to cast the various structures into the newly defined matrix notation. For example,

$$\frac{1}{2^{d/2}}\mathrm{tr}(\Gamma_\eta^\dagger\gamma_\mu\gamma_5\Gamma_{\eta'})(-1)^{\eta'_\nu}$$
$$=\frac{1}{2^{d/2}}\mathrm{tr}[\Gamma_\eta^t\gamma_\mu\gamma_5(\gamma_5\gamma_\nu)\Gamma_{\eta'}(\gamma_\nu\gamma_5)]$$
$$=(\overline{\gamma_\mu\gamma_\nu\otimes t_5 t_\nu})_{\eta\eta'} .$$

In the series expansion of the full propagator

$$G(x,y)=\left[G_0\sum_{n=0}^{\infty}(-VG_0)^n\right](x,y) , \tag{18}$$

only the first few terms contribute to the chiral anomaly as will be seen in the next section. In fact, we only require two terms; so from (17) and (18) we get

$$G(x,y)=2a\int_p e^{ip(x-y)}\frac{S_0^{-1}(pa)}{\mathrm{den}(pa)}-2iga^2\int_p\int_k e^{ipx+k(x-y)}\tilde{A}_\mu(p)\frac{S_0^{-1}(pa+ka)Q_\mu(ka)S_0^{-1}(ka)}{\mathrm{den}(pa+ka)\mathrm{den}(ka)}$$
$$+\text{higher-order terms in } a , \tag{19}$$

with

$$\mathrm{den}(pa)\equiv-4\sum_\sigma \sin^2 ap_\sigma$$

and

$$\tilde{A}_\mu(l)\equiv(2a)^d\sum_x e^{-ilx}A_\mu(x),\quad l\in\left[-\frac{\pi}{2a},\frac{\pi}{2a}\right]^d .$$

III. U(1) AXIAL ANOMALY IN $d=2$ DIMENSIONS

We restrict ourselves to the case of $d=2$ as it already captures the features of the calculation without having to tackle the complexity of the mathematics in higher dimensions. The calculation can be extended readily to $d=4$, say, and also to non-Abelian gauge fields.

Some time ago Sharatchandra, Thun, and Weisz[6] derived the axial anomaly for naive fermions with the chiral current

$$J_\mu^5(r)_{\mathrm{STW}}\equiv\frac{1}{2^d}\bar{\psi}(r)\gamma_\mu\gamma_5\sum_{\{\xi:\xi_\nu=\pm1,\ \forall\nu\}}\psi(r+a\hat{\mu}+a\xi) , \tag{20}$$

where we have put all the gauge links equal to one. The corresponding current for staggered fermions can be obtained in the same way as in the derivation of the staggered fermion action (6) from the naive one (1), i.e., via the transformation (5). In the context of the coordinate-space interpretation (9), we can write this current as

$$J_\mu^5(x)_{\mathrm{STW}}=\tfrac{1}{8}\bar{q}(x)\{\gamma_\mu\gamma_5\otimes\mathbb{1}[2q(x+2a\hat{\mu})+2q(x)+q(x+2a\hat{\mu}+2a\hat{\nu})+q(x+2a\hat{\mu}-2a\hat{\nu})+q(x+2a\hat{\nu})+q(x-2a\hat{\nu})]$$
$$+\sigma_{\mu\nu}\otimes t_\nu t_5[q(x+2a\hat{\mu}+2a\hat{\nu})-q(x+2a\hat{\mu}-2a\hat{\nu})+q(x+2a\hat{\nu})-q(x-2a\hat{\nu})]\},\quad \mu\neq\nu . \tag{21}$$

Being interpreted in this way, the current clearly is not of minimal form as it contains extra Dirac-flavor structure compared to that of the chiral current in the continuum. This correction, at the tree level, is $O(a)$ which vanishes in the continuum limit. When interactions are switched on, however, the current does not have a simple expression.

On the other hand, the first term on the right-hand side of (21) is of familiar form for lattice chiral current, corresponding to a particular split-point definition, which has just the required Dirac-flavor structure. To illustrate the use of the coordinate-space interpretation in the derivation of the anomaly, we will work with, say, the gauge-invariant current, up to renormalization,

$$J_\mu^5(x)\equiv\tfrac{1}{2}\bar{q}(x)\gamma_\mu\gamma_5\otimes\mathbb{1}U_\mu(x)U_\mu(x+a\hat{\mu})q(x+2a\hat{\mu})+\mathrm{H.c.} \tag{22}$$

This is the gauge-invariant generalization of the current obtained by Kluberg-Stern, Morel, Napoly, and Petersson[8] in the free case from the lattice Ward-Takahashi identity associating with a lattice chiral transformation. In the block-variable formulation of Susskind fermions, namely, the Dirac-Kähler theory, and its generalization, a current of this form but with gauge links defined on the block lattice has also been employed.[12]

We can define a current in several ways on the lattice which all reduce to the same continuum form naively. However, they are not all equivalent. To reproduce the anomaly additional criteria have to be met. The current (22) is chosen on the bases of locality, gauge invariance, and, at least in the free theory, it corresponds to the correct assignment of the chiral charge to the species. Such an assignment should hold for interacting theories, as will be confirmed in the calculation below, for interactions do not affect the global axial transformation.

We now evaluate directly the continuum limit of the vacuum expectation value of the lattice divergence

$$\Delta_\mu J^5_\mu(x)=\frac{1}{2a}\sum_\mu [J^5_\mu(x)-J^5_\mu(x-2a\hat{\mu})] \tag{23}$$

in a background gauge field. From (22) together with the formalism developed in the preceding section we can rewrite the vacuum expectation value of (23) as

$$\begin{aligned}\langle \Delta_\mu J^5_\mu(x)\rangle &=\frac{1}{4a}\langle \bar{q}(x)\gamma_\mu\gamma_5\otimes \mathbb{1}[U_\mu(x)U_\mu(x+a\hat{\mu})q(x+2a\hat{\mu})-U^\dagger_\mu(x-a\hat{\mu})U^\dagger_\mu(x-2a\hat{\mu})q(x-2a\hat{\mu})]+\text{H.c.}\rangle \\ &=-\text{Sp}\{[K,G](x,x)\}\ ,\end{aligned} \tag{24}$$

where we have defined

$$[K(x,y)]_{\eta\eta'}\equiv\frac{1}{4a}\frac{1}{2^{d/2}}\text{tr}(\Gamma^\dagger_\eta\gamma_\mu\gamma_5\Gamma_{\eta'})[W_{\eta\eta'}(x)\delta_{x+2a\hat{\mu},y}-W^\dagger_{\eta'\eta}(y)\delta_{x-2a\hat{\mu},y}]\ , \tag{25}$$

and, from (9),

$$W_{\eta\eta'}(x)\equiv\mathcal{U}^\dagger_\eta(x)U_\mu(x)U_\mu(x+a\hat{\mu})\mathcal{U}_{\eta'}(x+2a\hat{\mu})\ . \tag{26}$$

Taylor expanding the gauge links around x up to the necessary orders in a as in the derivation of the inverse propagator before, we get

$$\begin{aligned}K(x,y)=\tfrac{1}{2}\int_k e^{ik(x-y)}&\left[\overline{\gamma_\mu\gamma_5\otimes\mathbb{1}}\frac{i}{a}\sin 2ak_\mu+igA_\mu(x)(\overline{\gamma_\mu\gamma_5\otimes \mathbb{1}}2\cos 2ak_\mu+\overline{\sigma_{\nu\mu}\otimes t_\mu t_5}i\sin 2ak_\nu)\right] \\ &+\text{higher-order terms in } a\ ,\end{aligned} \tag{27}$$

where once again the "higher-order terms" also contain various partial derivatives of $A_\mu(x)$ with appropriate factors of a.

In the commutator of (24) the first term (the second term) on the right-hand side of (27) combines with the second term (the first term) of the propagator (19) to yield the only nonvanishing contributions as $a\to 0$ as will be shown later. We thus have, from the cyclic property of taking the trace,

$$\begin{aligned}\langle\Delta_\mu J^5_\mu(x)\rangle=\int_p\int_k e^{ipx}iga^2\tilde{A}_\mu(p)\text{Sp}&\left[\overline{\gamma_\lambda\gamma_5\otimes\mathbb{1}}\frac{i}{a}[\sin 2a(k_\lambda+p_\lambda)-\sin 2ak_\lambda]\frac{S_0^{-1}(ka+pa)Q_\mu(ka)S_0^{-1}(ka)}{\text{den}(pa+ka)\text{den}(ka)}\right.\\ &\left.+\frac{1}{a}\left[\frac{S_0^{-1}(pa+ka)}{\text{den}(pa+ka)}-\frac{S_0^{-1}(ka)}{\text{den}(ka)}\right](\overline{\gamma_\mu\gamma_5\otimes \mathbb{1}}2\cos 2ak_\mu+\overline{\sigma_{\nu\mu}\otimes t_\mu t_5}i\sin 2ak_\nu)\right]\\ &+\text{irrelevant terms}\ .\end{aligned} \tag{28}$$

We now expand the above expression in ap, the external gauge-field momentum, and rescale the internal momentum $k\to k/2a$. As will be seen in Eq. (31) later, there is in fact no infrared singularity in the expression above to prevent such a Taylor expansion. Then in the continuum limit the integration over p gives the partial derivative of $A_\mu(x)$:

$$\lim_{a\to 0}\langle\Delta_\mu J^5_\mu(x)\rangle=\langle\partial_\mu J^5_\mu(x)\rangle=gI_{\lambda\mu}\partial_\lambda A_\mu(x)\ , \tag{29}$$

where $I_{\lambda\mu}$ is independent of x. After calculating the various traces of γ matrices, we get

$$\begin{aligned}I_{\lambda\mu}=\frac{\epsilon_{\lambda\mu}}{\pi^2}\int_{-\pi}^{\pi}d^2k&\left\{\cos k_\lambda\left[\sin k_\mu\frac{\partial}{\partial k_\mu}\left[\frac{1}{\text{den}(k/2)}\right]+\frac{1}{2}\frac{\cos k_\mu+1}{\text{den}(k/2)}\right]\right.\\ &\left.+\cos k_\mu\frac{\partial}{\partial k_\lambda}\left[\frac{\sin k_\lambda}{\text{den}(k/2)}\right]-\frac{\sin k_\lambda}{2}\frac{\partial}{\partial k_\lambda}\left[\frac{\cos k_\mu-1}{\text{den}(k/2)}\right]\right\}\ .\end{aligned} \tag{30}$$

To arrive at the last expression, trivially vanishing integrals, i.e., those whose integrands are odd functions, have been omitted.

Direct integrations by parts in (30) would naively yield a vanishing result. However, such an operation is illegitimate since the integrands on the right-hand side (RHS) are singular at the origin except the last term which has a removable singularity there. We thus partition the integration domain as shown in Fig. 1. In the region excluding the origin we can now integrate by parts and only some of the surface terms survive; Taylor expansions of the integrands in the inner region, on the other hand, cancel each other:

$$I_{\lambda\mu}=\frac{\epsilon_{\lambda\mu}}{\pi^2}\left[\int_{-\epsilon}^{\epsilon}dk_\lambda\frac{\cos k_\lambda \sin k_\mu}{\mathrm{den}(k/2)}\Bigg|_{k_\mu=\epsilon}^{k_\mu=-\epsilon}+\int(\mu\leftrightarrow\lambda)+\int_{-\epsilon}^{\epsilon}\int_{-\epsilon}^{\epsilon}dx\,dy\left[x\frac{2x}{(x^2+y^2)^2}-\frac{1}{x^2+y^2}-\frac{x^2-y^2}{(x^2+y^2)^2}\right]\right]$$

$$=\frac{\epsilon_{\lambda\mu}}{\pi^2}4\int_{-\epsilon}^{\epsilon}dk_\mu\frac{\epsilon}{\epsilon^2+k_\mu^2}=2\frac{\epsilon_{\lambda\mu}}{\pi}\ . \tag{31}$$

We thus obtain the correct anomaly for a theory containing two flavors.

To complete the proof we now show that this is the only nonvanishing contribution by a a power-counting argument. This is expected as the anomaly is a quantum effect induced by the singularities only.

Generically, a term of $K(x,y)$ can be written as

$$\int_{p,q,l}e^{i(px-qy)}\kappa(p,q,l)\delta\left[p-q-\sum l\right],$$

and a typical term of $G(x,y)$ as

$$\int_{m,n,l'}e^{i(mx-ny)}g(m,n,l')\delta\left[m-n-\sum l'\right],$$

where la and $l'a$ are the dimensionless gauge-field momenta in which we can Taylor expand. The first term of the commutator in the generalized trace

$$\mathrm{Sp}\{[K,G](x,x)\}=\mathrm{Sp}\left[a^2\sum_y[K(x,y)G(y,x)-G(x,y)K(y,x)]\right]$$

is then

$$\begin{aligned}&\sim\int_{p,l,l',n}e^{ix(p-n)}\kappa\left[p,p-\sum l,l\right]g\left[p-\sum l,n,l'\right]\delta\left[p-n-\sum l-\sum l'\right]\\&\sim\int_{p,l,l'}\exp\left[ix\left[\sum l+\sum l'\right]\right]\kappa\left[p,p-\sum l,l\right]g\left[p-\sum l,p-\sum l-\sum l',l'\right].\end{aligned}\tag{32}$$

Similarly, the other term of the commutator is

$$\sim\int_{p,l,l'}\exp\left[ix\left[\sum l+\sum l'\right]\right]g\left[p+\sum l',p,l'\right]\times\kappa\left[p,p-\sum l,l\right].\tag{33}$$

Thus the commutator of the first term on the RHS of Eq. (27) and the free propagator $G_0(x,y)$ do not contribute to the divergence equation (24), as can be seen by setting l and l' to zero in the last two expressions and from the cyclic property of taking the trace.

We next show that the trace of the commutator of the second terms of (19) and (27) vanishes in the limit and so do the ones involving the "higher-order terms." From the Taylor expansions of the functions g,κ of Eqs. (32) and (33) in $\sum la$ and $\sum l'a$, we see that extra factors of a are gained for higher-derivative terms in the trace of the commutator. Thus all the other contributions are, at worst, $O(a)$.

Note that in this power-counting argument we have to introduce an infrared regulator. However, since the infrared behavior is the same on an infinite lattice as in the continuum, we can appeal to the continuum result of the vanishing, in two dimensions, of diagrams with more gauge fields attached (see Bodwin and Kovacs,[12] for example) to justify the removal of this regulator.

Lastly, the anomaly result (29) can now be shown to be independent of the choice of paths of gauge links going from $x+a\eta$ to $y+a\eta'$. The difference between any two

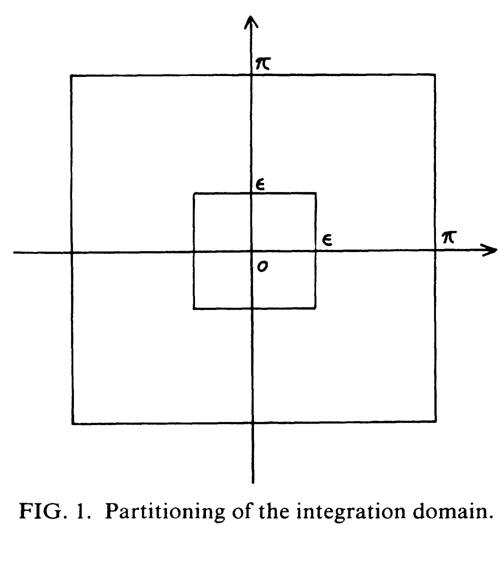

FIG. 1. Partitioning of the integration domain.

such paths amounts to the difference of a closed path of product of gauge links from unity, that is of order $O(a^2)$ (i.e., $\sim a^2 g F_{\mu\nu}$). Such a term can be absorbed into "higher-order terms" of $K(x,y)$ which, in turn, has just been demonstrated to have no effect on the result.

IV. CONCLUDING REMARKS

We have cast the staggered fermion formulation into a suitable formalism to derive the correct U(1) axial anomaly in the framework of the coordinate-space interpretation. The derivation makes use of a lattice chiral current which contains just the required spin-flavor structure and assumes some point-split definition which spreads over two hypercubes. However, we have also checked that the current corresponding to the second term on the RHS of Eq. (21),

$$J_\mu^5(x)=\bar{q}(x)\gamma_\mu\gamma_5\otimes \mathbb{1} q(x) , \qquad (34)$$

which is confined to one hypercube, can also be used. To incorporate gauge invariance, we can insert the gauge links in a specific way as

$$J_\mu^5(x)=\frac{1}{2}\sum_{\eta,\eta'}\bar{\chi}_\eta(x)(\overline{\gamma_\mu\gamma_5\otimes\mathbb{1}})_{\eta\eta'}(V_{\eta\eta'}+V_{\eta'\eta}^\dagger)\chi_{\eta'}(x) , \qquad (35)$$

where

$$V_{\eta\eta'}\equiv[U_1(x+a\eta)]^{(\eta_1'-\eta_1)}\cdots[U_d(x+a(\eta_1'-\eta_1+\cdots+\eta_{d-1}'-\eta_{d-1}+\eta_d))]e^{(\eta_d'-\eta_d)} .$$

The results obtained in this work provide a "validity check" on, from the point of view of anomaly reproduction, and illustrate the role of the flavor interpretation of staggered fermions in coordinate space. They also agree with the conclusion of a recent work.[13]

All calculations have been carried out in two dimensions, but the representation of staggered fermions developed in Sec. II is readily extendable to higher dimensions and non-Abelian gauge groups. We expect, in particular, that corresponding results would hold for QCD_4. The formalism in Sec. II can also be useful for the renormalization calculation of staggered-fermion two- and four-point operators, especially the ones for weak matrix element evaluation on the lattice.[14]

Not long ago, Oshima[15] proved that a reduced version of staggered fermions, which has only $2^{(d-2)/2}$ flavors, could reproduce the anomaly. Similar to the work presented here, one can also address the problem using the coordinate-space flavor interpretation for that reduced version.[8]

Finally, the investigation of the Abelian anomaly on the lattice is closely related to that of the non-Abelian anomaly.[16]

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