Charged-vortex solution to spontaneously broken gauge theories with Chern-Simons term

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We study the charged-vortex solution to the classical equations of motion for a non-Abelian Higgs model with a Chern-Simons term. We determine the energy of the vortex and the values of the magnetic and electric fields by means of a power-series expansion and also using a variational approach.

I. INTRODUCTION AND RESULTS

It is by now well established that gauge theories in $2+1$ dimensions present a variety of very interesting phenomena,[1] particularly when a Chern-Simons term is included, as first proposed by Deser, Jackiw, and Templeton[2] and Schonfeld.[3]

Since at high temperatures a relativistic quantum field theory becomes effectively three dimensional[4-6] these theories have physical applications at high temperatures. In this context it has been shown[7] that fermionic, *CP*-violating effects induce at high temperature the Chern-Simons action.

An interesting effect of the addition of a Chern-Simons term to spontaneously broken gauge theories is the possibility of having topologically stable solutions both with magnetic and electric charge[8-11] (without Chern-Simons term, it can be shown that Nielsen-Olesen vortices[12] do not admit electrically charged generalizations[13,14]). In particular, it has been shown in Refs. 9 and 10 that only charged vortices are admissible when a Chern-Simons term is present. Moreover, not only the magnetic flux but also the electric charge (and the angular momentum) of the vortex solution are quantized already at the classical level.

Although various *Ansätze* have been proposed for these charged vortices[8-11] no detailed analysis of the corresponding solution has yet been presented.

It is the aim of this work to give such a detailed study of the charged-vortex solution for the case of an SU(2) spontaneously broken gauge theory with the addition of a Chern-Simons term.

The plan of the paper is as follows. After a review of the charged-vortex *Ansatz* proposed in Refs. 9 and 10 (Sec. II) we study the resulting radial equations of motion in Sec. III. Although two different vector-meson masses are in principle possible when a Chern-Simons term is present,[15] we show (details of the proof are given in the Appendix) that only one charged-vortex solution survives: namely, the one which corresponds to a larger mass (which we call m_+).

In order to analyze the properties of the solution we first compare it with the neutral one, since for the latter an exact solution is known for a particular relation between coupling constants[16] (which corresponds to the Ginsburg-Landau parameter $\lambda=1$) and very accurate numerical calculations exist for various values of λ (Ref. 17). In particular, we obtain a perturbative expansion for the energy of the charged vortex [Eqs. (3.9)–(3.11)] with the Chern-Simons coefficient as expansion parameter.

We also study the behavior of the solution expressed as an expansion in powers of the radial variable, showing that it corresponds to a well-defined vortexline (when both the scalar- and vector-meson masses are of the same order). It carries both electric and magnetic fields decreasing monotonically with characteristic length $1/m_+$ and the scalar field increasing with the same length from zero at the origin to its vacuum value at infinity.

We also present a variational solution using an approach developed in Ref. 18 which significantly reduces the number of degrees of freedom and makes the analysis tractable for arbitrary values of the various physical parameters of the model. As an application of the solution we discuss at the end of this section the possible relevance of the charged-vortex solution when considered as a cosmic string formed at the grand unification phase transition.[19]

Finally in the Appendix we discuss the boundary conditions obeyed by the vortex solution and also sketch the proof on the nonexistence of the lower-vector-meson-mass solution.

II. REVIEW OF THE CHARGED-VORTEX *ANSATZ*

We describe for simplicity the SU(2) vortex *Ansatz*[19] but the SU(N) generalization can be obtained following the same steps.[10]

We start from the $(2+1)$-dimensional Lagrangian

$$\mathcal{L}=-\tfrac{1}{2}\mathrm{tr}F_{\mu\nu}F^{\mu\nu}+\mathrm{tr}D_\mu\phi D^\mu\phi+\mathrm{tr}D_\mu\psi D^\mu\psi - V(\phi,\psi)+\frac{\mu}{2}\epsilon_{\alpha\beta\gamma}\mathrm{tr}(F^{\alpha\beta}A^\gamma-\tfrac{2}{3}eA^\alpha A^\beta A^\gamma)\ . \tag{2.1}$$

Here A_μ is the gauge field, taking values in the Lie algebra of SU(2), $A_\mu=A_\mu^a t^a$ with t^a the SU(2) generators normalized according to

$$\mathrm{tr}(t^a t^b)=\frac{\delta^{ab}}{2}\ ,\quad [t^a,t^b]=i\epsilon^{abc}t^c\ , \tag{2.2}$$

ψ and ϕ are two Higgs fields in the adjoint representation needed in order to have a complete symmetry breaking [N Higgs fields in the adjoint representation are needed in SU(N) (Ref. 10)]. The covariant derivative D_μ is defined as

$$D_\mu = \partial_\mu + e[A_\mu, \ ] \tag{2.3}$$

and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + e[A_\mu, A_\nu] . \tag{2.4}$$

SU(2) vortex solutions are associated with spontaneously broken gauge symmetries via Higgs fields. In order to have topologically stable vortices, the relevant homotopy group $\pi_1(G/H)$ must be nontrivial (G stands for the gauge group and H for the invariance group of the vacuum.) For $G = \mathrm{SU}(N)$ and the Higgs fields in the adjoint representation it is convenient to have maximum symmetry breaking of G so that the vacuum is only invariant under the unit matrix in the adjoint representation. Then $H = Z_N$, $\pi_1(\mathrm{SU}(N)/Z_N) = Z_N$, and one has $N-1$ topologically nontrivial homotopy classes besides the ordinary vacuum. This leads to Z_2 vortices for $G = \mathrm{SU}(2)$. Maximum symmetry breaking is achieved in the SU(2) case with the two Higgs fields [N in the SU(N) case] in the adjoint representation and a potential of the form

$$V(\phi,\psi) = \tfrac{1}{8}g^2(\phi^a\phi^a - \eta^2)^2 + \tfrac{1}{8}g'^2(\psi^a\psi^a - \eta'^2)^2 + \tfrac{1}{2}g''^2(\phi^a\psi^a)^2 . \tag{2.5}$$

The last term in (2.1) is the Chern-Simons term.[2] Because of its presence, the corresponding action is not invariant under gauge transformations

$$S = \int d^3x\, \mathcal{L} \to \int d^3x\, \mathcal{L} + \mu \frac{8\pi^2}{e^2} w(g) , \tag{2.6}$$

where $w(g)$ is the winding number of g:

$$w(g) = \frac{1}{24\pi^2} \int d^3x\, \epsilon^{\gamma\beta\alpha} \mathrm{tr}(g^{-1}\partial_\gamma g g^{-1}\partial_\beta g g^{-1}\partial_\alpha g) . \tag{2.7}$$

Equation (2.7) can be converted to a surface integral which is not zero but takes an integer value m, $w(g) = m$, which characterizes the homotopy equivalence class to which g belongs.

Only for homotopically trivial g does $w(g)$ vanish. Then the requirement that the phase exponent of the action be gauge-invariant enforces a quantization condition of the parameters

$$\frac{4\pi\mu}{e^2} = n . \tag{2.8}$$

A simple *Ansatz* that separates the equations of motion into radial and angular parts is[14,9]

$$\phi = f(r) \begin{bmatrix} \cos\phi \\ \sin\phi \\ 0 \end{bmatrix} , \quad \psi = \eta' \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} ,$$
$$\mathbf{A}_\phi = A(r) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} , \quad \mathbf{A}_0 = A_0(r) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} . \tag{2.9}$$

Finite energy requires the following boundary conditions:

$$A(\infty) = -1/e , \quad A_0(\infty) = 0 , \quad f(\infty) = \eta , \tag{2.10}$$

where η and η' correspond to the minimum of V.

One also has

$$A(0) = 0 , \quad f(0) = 0 , \quad A_0(0) = c \neq 0 . \tag{2.11}$$

It is important to stress that the charged-vortex solutions exist only if $A_0(0) \neq 0$ (cf. Refs. 9–11).

Otherwise it can be proved (see the Appendix) that the coupled nonlinear radial equations of motion have no nontrivial solution.

With *Ansatz* (9) the equations of motion

$$D_\mu D^\mu \phi = -\frac{\delta V}{\delta\phi} , \quad D_\mu D^\mu \psi = -\frac{\delta V}{\delta\psi} , \tag{2.12a}$$

$$D_\mu F^{\mu\nu} = j^\nu + \tfrac{1}{2}\mu \epsilon^{\nu\alpha\beta} F_{\alpha\beta} , \tag{2.12b}$$

$$j^\nu \equiv ie([D^\nu\phi, \phi] + [D^\nu\psi, \psi]) \tag{2.12c}$$

become

$$\frac{d^2x}{d\tau^2} - \frac{1}{\tau}\frac{dx}{d\tau} - xz^2 = \delta \frac{dy}{d\tau} \tau , \tag{2.13a}$$

$$\frac{d^2y}{d\tau^2} + \frac{1}{\tau}\frac{dy}{d\tau} - yz^2 = \frac{\delta}{\tau}\frac{dx}{d\tau} , \tag{2.13b}$$

$$\frac{d^2z}{d\tau^2} + \frac{1}{\tau}\frac{dz}{d\tau} - \frac{x^2z}{\tau^2} + y^2z + \frac{\lambda^2}{2}z(1-z^2) = 0 , \tag{2.13c}$$

where we had defined dimensionless quantities

$$x(\tau) = 1 + eA(\tau) , \quad y(\tau) = A_0(\tau)/\eta , \quad z(\tau) = f(\tau)/\eta , \tag{2.14}$$

with

$$\tau = \rho e \eta , \tag{2.15}$$

$$\delta = \mu / e\eta . \tag{2.16}$$

We now define an "electromagnetic field" $\mathcal{F}^{\mu\nu}$ (Refs. 20, 21, and 9):

$$\mathcal{F}^{\mu\nu} = \frac{\psi^a F^{\mu\nu a}}{\psi^a\psi^a} = F^{\mu\nu 3} . \tag{2.17}$$

The electric and magnetic fields then take the form

$$E_i = \mathcal{F}_{0i} , \quad H = \tfrac{1}{2}\epsilon_{ij}\mathcal{F}^{ij} . \tag{2.18}$$

The magnetic flux of the vortex is given by

$$\Phi = \int d^2x\, H \tag{2.19}$$

from the Stokes theorem and the boundary condition one gets

$$\Phi = -\frac{2\pi}{e} \; . \tag{2.20}$$

The magnetic flux is related to the topological number k associated to the homotopy classes of G/H:

$$k = -\frac{e\Phi}{2\pi} \; . \tag{2.21}$$

The present SU(2) *Ansatz* corresponds to $k=1[\pi_1(G/H)=\pi_1(\mathrm{SU}(2)/z_2)=Z_z]$. Concerning the electric charge of the vortex configuration note that Eq. (2.12b) for $\nu=0$ [or Eq. (2.13b)] can be written in the form

$$\partial^i E_i - \mu H = \sigma \; , \tag{2.22}$$

where the charge density σ reads

$$\sigma \equiv \hat{\mathbf{e}}_3 \cdot \mathbf{J}_0 = e^2 f^2 A_0 \; . \tag{2.23}$$

Since $\lim_{\rho\to\infty} E_i = 0$ one gets from Eq. (2.13b) that there is a relation between the electric charge Q,

$$Q = \int d^2x \; \sigma \; , \tag{2.24}$$

and the magnetic flux:

$$Q = -\mu\Phi \; . \tag{2.25}$$

This important relation first obtained in Ref. 2 implies a quantization condition for Q. Indeed from Eqs. (2.8), (2.20), and (2.25) we get

$$Q = \frac{n}{2} e \; . \tag{2.26}$$

The charge quantization can be connected with the angular momentum of the vortex

$$J = \int d^2x \; \epsilon^{ij} x_i T_{0j} \; . \tag{2.27}$$

For the vortex *Ansatz* one has[9]

$$J = \frac{Q}{2e} = \frac{n}{4} \; . \tag{2.28}$$

Concerning the energy per unit length of the vortex solution one can easily see that it takes the form

$$E = \int T_{00} d^2x = \pi\eta^2 \int_0^\infty \tau \, d\tau \left[\frac{1}{\tau^2} \left[\frac{dx}{d\tau} \right]^2 + \left[\frac{dy}{d\tau} \right]^2 + \left[\frac{dz}{d\tau} \right]^2 + y^2 z^2 + \frac{x^2 z^2}{\tau^2} + \frac{\lambda^2}{4}(1-z^2)^2 \right] . \tag{2.29}$$

Compare this expression with that corresponding to the action, which in terms of x,y,z reads

$$S = -E + \pi\eta^2 \int_0^\infty \tau \, d\tau \left[2 \left[\frac{dy}{d\tau} \right]^2 + y^2 z^2 + 2\frac{\delta}{\tau} y \dot{x} \right] . \tag{2.30}$$

III. THE CHARGED-VORTEX SOLUTION

In order to find a solution to the vortex equations (2.13) let us first analyze their asymptotic behavior. There are in principle two possible behaviors for large τ but as we show in the Appendix only one of them corresponds to a solution. Indeed, let us propose for large τ the following forms:

$$\begin{aligned} x(\tau) &= c_1 \tau K_1(\beta\tau)[1+O(e^{-\beta\tau})] \; , \\ y(\tau) &= c_2 K_0(\gamma\tau)[1+O(e^{-\gamma\tau})] \; , \\ z(\tau) &= 1 + c_3 K_0(\alpha\tau) + O(e^{-2\alpha\tau}) \; . \end{aligned} \tag{3.1}$$

It is easy to derive from Eq. (2.13) the following relations:

$$\alpha = \lambda \; , \quad \beta = \gamma \; , \tag{3.2}$$

$$\begin{aligned} c_1(\beta^2 - 1) &= -\delta c_2 \beta \; , \\ c_2(\beta^2 - 1) &= -\delta c_1 \beta \; . \end{aligned} \tag{3.3}$$

Relations (3.3) are consistent if one of the following conditions hold:

$$c_1 = c_2 \tag{3.4a}$$

or

$$c_1 = -c_2 \tag{3.4b}$$

which then lead to

$$\begin{aligned} \beta &= \frac{\delta + \sqrt{\delta^2+4}}{2} \quad \text{if } c_1 = c_2 \; , \\ \beta &= \frac{-\delta + \sqrt{\delta^2+4}}{2} \quad \text{if } c_1 = -c_2 \; . \end{aligned} \tag{3.5}$$

In terms of the original parameters of the model an equivalent equation to (3.5) can be found:

$$m_{\pm} = (\tfrac{1}{4}\mu^2 + e^2\eta^2)^{1/2} \pm \tfrac{1}{2} |\mu| \; . \tag{3.6}$$

The possibility of having two different vector-meson masses in spontaneously broken gauge theories when a Chern-Simons term is present is due to the fact that this term is parity odd as first observed in Ref. 15. One can prove, however, that only one vortex solution, namely, that corresponding to m_+ exists, as is shown in the Appendix. Note that this mass is bigger than that of the neutral vortex one which corresponds to Eq. (3.6) with $\mu=0$.

Before proceeding to the numerical study of the charged vortex, let us compare this solution with the neutral one. For the latter, an exact solution is known for $\lambda=1$ (Ref. 16) and very accurate numerical calculations exist for various values of λ (Ref. 17).

Since for $\delta=0$, Eqs. (2.13) become those of the neutral vortex we propose the following expansion for the charged solution:

$$\begin{aligned} x(\tau,\delta) &= \sum_{n=0}^{\infty} x_n(\tau)\delta^n \; , \\ y(\tau,\delta) &= \sum_{n=0}^{\infty} y_n(\tau)\delta^n \; , \\ z(\tau,\delta) &= \sum_{n=0}^{\infty} z_n(\tau)\delta^n \; , \end{aligned} \tag{3.7}$$

with $x_0(\tau)$ and $z_0(\tau)$ obtained from a neutral-vortex solution[16] [$y(\tau,0)=0$]. Invariance of the equations of motion under $\delta\rightarrow-\delta$ and $y\rightarrow -y$ implies that

$$x_{2k+1}(\tau)=z_{2k+1}(\tau)=0\ , \qquad y_{2k}(\tau)=0\ . \tag{3.8}$$

Inserting (3.7) in Eq. (2.29) one gets, for the charged-vortex energy,

$$\epsilon=\frac{E}{\pi\eta^2}=\epsilon_0(\lambda)+\epsilon_2\delta^2+\cdots\ , \tag{3.9}$$

where $\epsilon_0(\lambda)$ is the neutral-vortex energy. In particular, for the exact ($\lambda=1$) solution

$$\epsilon_0(1)=1\ . \tag{3.10}$$

Concerning ϵ_2 one can show that it does not depend on λ and it takes the value

$$\epsilon_2=\tfrac{1}{4}\ . \tag{3.11}$$

Indeed one can easily see that

$$y_1(\tau)=\tfrac{1}{2}x_0(\tau)\ , \qquad x_2(\tau)=\frac{\tau^2}{8}x_0(\tau)\ , \qquad z_2(\tau)=0\ . \tag{3.12}$$

Then for the explicit expression for ϵ_2 one gets

$$\epsilon_2=\tfrac{1}{2}\int\tau\,d\tau\left[\left[\frac{dx_0}{d\tau}\right]^2+\frac{1}{\tau}\frac{dx_0}{d\tau}x_0+z_0^2x_0^2\right] \tag{3.13}$$

which can be integrated to give (3.11).

We then have an expression for the energy of the charged vortex [given by Eq. (3.9) with $\epsilon_0(\lambda)$ known from Ref. 16 and ϵ_2 given by (3.11)] reliable for small values of $\delta=\mu/e\eta$.

In order to study the behavior of the charged vortex for arbitrary δ we can follow Ref. 16 seeking an expansion of the form

$$x(\tau)=\sum_{n=0}^{\infty}a_{2n}\tau^{2n}\ , \qquad y(\tau)=\sum_{n=0}^{\infty}b_{2n}\tau^{2n}\ , \qquad z(\tau)=\sum_{n=0}^{\infty}c_{2n+1}\tau^{2n+1}\ , \tag{3.14}$$

where we have used the equations of motion to determine parity of each expansion.

One also has, from (2.11),

$$a_0=1\ . \tag{3.15}$$

A recursion relation giving all a_{2n},b_{2n},c_{2n+1} in terms of $a_2,b_0=c,c_1$, can now be established. One then has to determine the last three coefficients using the boundary conditions at infinity [Eqs. (2.10)].

We have solved this problem numerically for $\lambda=1$ in order to compare our solution with the exact neutral-vortex one.[16]

We quote the values obtained for a_2, b_0, and c_1 when $\delta=0.05$, a case which can also be compared with the expansion in the power of δ:

$$a_2=-0.2495\ , \qquad b_0=0.0250\ , \qquad c_1=0.6030\ . \tag{3.16}$$

We have computed for these values the energy of the vortex solution:

$$\epsilon=1.0006 \tag{3.17}$$

which can be written in the form (3.9) with $\epsilon_2=0.24$.

In Fig. 1 we plot the modulus of the scalar field and the magnetic and electric fields as a function of τ. One can see that if the scalar mass $m=\eta g$ is of the order of the vector mass $m_+(\lambda\sim\beta)$ then we have a well-defined vortex line.

The procedure described above becomes tedious when one tries to determine the solution for large δ, since in this case ones does not have the neutral-vortex parameters as a guide for the initial choice of a_2 and c_1.

We shall then describe an alternative (variational) approach[18] which has shown to be in excellent agreement in the cases when one has direct comparison, with analytic[16] or very accurate[18] numerical results.

Following Hill *et al.*[18] we shall then use a combination of powers of exponentials so as to engineer functions with the short- and long-distance behavior given by (2.10) and (2.11):

$$x(\tau)=[1-(1-e^{-a\tau})^2]\ , \qquad y(\tau)=c[1-(1-e^{-b\tau})^2]\ , \qquad z(\tau)=1-e^{-d\tau}\ , \tag{3.18}$$

where a, b, c, and d are four variational parameters. These parameters have to be determined by minimizing

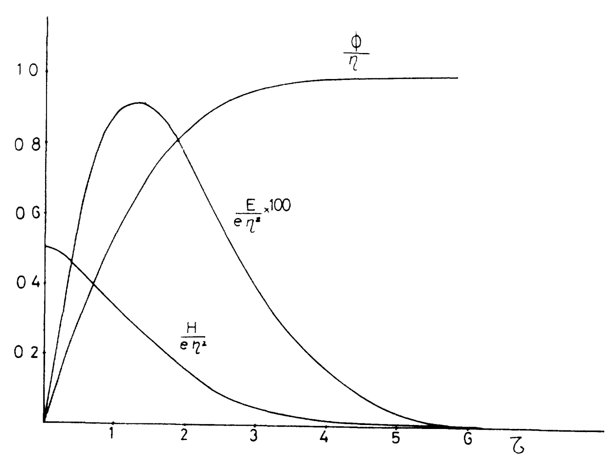

FIG. 1. We plot in the figure the radial dependence of the magnetic field H, the radial electric field Er (multiplied by a factor of 100), and the norm of the Higgs field. The variable is defined in Eq. (2.15). The parameters are $\delta=0.05$ and $\lambda=1$.

the action [note that this does not correspond to minimizing the energy, compare Eqs. (2.29) and (2.30)]. Inserting (3.18) in Lagrangian (2.1) one obtains, for the action the following variational equations,

$$\begin{aligned} &\frac{\partial S}{\partial b}=0 \Longrightarrow -4a^2\ln\tfrac{9}{8}-\delta\sigma c\frac{\partial N_3}{\partial \sigma}+\frac{s}{2}\frac{\partial N_2}{\partial s}=0\ , \\ &\frac{\partial S}{\partial b}=0 \Longrightarrow 2\delta a^2 s^3\frac{\partial N_3}{\partial \sigma}+c\frac{\partial N_4}{\partial \theta}=0\ , \\ &\frac{\partial S}{\partial d}=0 \Longrightarrow c^2N_4+\frac{c^2\theta}{2}\frac{\partial N_4}{\partial \theta}-\lambda^2\tfrac{89}{576}+\frac{a^2s^3}{2}\frac{\partial N_2}{\partial s}=0\ , \\ &\frac{\partial S}{\partial c}=0 \Longrightarrow \tfrac{13}{36}ca^2s^2+\delta a^2s^2N_3+cN_4=0\ , \end{aligned} \tag{3.19}$$

where N_2, N_3, and N_4 are the contributions of different terms appearing in the action (2.30):

$$N_2(s)=\int_0^\infty \frac{x^2z^2}{\tau}d\tau\ , \tag{3.20a}$$

$$N_3(\sigma)=\frac{1}{c}\int_0^\infty y\frac{dx}{d\tau}d\tau\ , \tag{3.20b}$$

$$N_4(\theta)=\frac{d^2}{c^2}\int_0^\infty y^2z^2\tau\, d\tau\ , \tag{3.20c}$$

and

$$\theta=\frac{b}{d}\ ,\quad \sigma=\frac{b}{a}\ ,\quad s=\frac{\sigma}{\theta}\ . \tag{3.21}$$

Equations (3.19) can be solved by selecting values of two of the variational parameters, θ and σ, and then determining the other two variational parameters as well as the two free parameters of the model, λ and δ.

The energy of the vortex is easily computed in this manner. We give in Table I this energy for different values of λ and δ and we plot ϵ as a function of δ in Fig. 2. For small values of δ we find agreement of these results with those arising for expansion (3.9).

As an application of the solution that we have found we briefly discuss the relevance of charged vortices as cosmic strings.[19]

Topological objects such as domain walls, vortices, and monopoles have been playing a role of increasing importance in cosmology.

In the context of the hot big-bang cosmology the grand unified theory used to describe elementary-particle interactions undergoes a series of spontaneous symmetry

TABLE I. Energies of the charged vortex [normalized according to Eq. (3.9)] as a function of δ and λ obtained from the variational approach.

δ \ λ	0.5	1.0	1.5
0.7	0.87	1.12	1.31
1.3	1.02	1.30	1.50
3.0	1.39	1.74	2.01
8.0	1.96	2.43	2.76
12.6	2.26	2.81	3.16
17.7	2.51	3.11	3.47

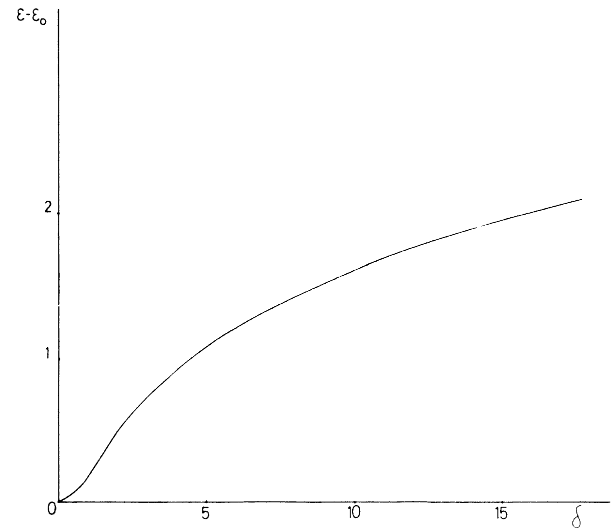

FIG. 2. We plot the dependence of $\epsilon(\lambda,\delta)-\epsilon_0(\lambda)$ as a function of δ for $\lambda=1$.

breakings which can give rise to topologically stable defects.

Concerning vortices (known in this context as cosmic strings), they can lead to very interesting cosmological consequences. In particular, they can generate density fluctuations sufficient to explain the galaxy formation.

Both local and global strings have been investigated in this context.[9] We shall show in this section that also charged-vortex solutions can be of relevance in connection with these problems.

As is well known, the symmetry-breaking pattern is determined from the effective potential of the scalar fields at finite temperature, $V_{\rm eff}$.[4–6] Since $V_{\rm eff}$ acquires additional temperature-dependent terms, one can adjust the parameters so as to have a critical temperature $T_c(T_c \sim \eta)$ which corresponds to a second-order phase transition between a symmetric $(T > T_c)$ and a broken-symmetry $(T < T_c)$ phase.

In the cosmological context, as the Universe cools through the critical temperature T_c, the Higgs field will tend to develop an expectation value corresponding to some point in the manifold G/H of equivalent vacua. The choice of a point of this manifold is random and may be different in different regions of space. Much of this structure will tend towards spatial uniformity unless prevented from doing so by trapped singularities: if the angle of $\langle\phi\rangle$ changes by $2\pi n$ as one goes around a loop, a thin tube with $\langle\phi\rangle \sim 0$ (a vortex line) becomes trapped within the ordered phase.

The mass per unit length $M=E$ of such string is given by an expression which, for charged vortices, corresponds to Eq. (2.29). During the subsequent Hubble expansion the strings will occasionally self-interact and loops will be cut off and subsequently decay.

It has been found that the ratio of the contribution of strings to the total mass density is[19]

$$\frac{\rho_s}{\rho}=GM\ , \tag{3.22}$$

where ρ_s is the energy density due to strings and ρ the energy density of matter (G is the Newton constant).

For closed loops one has, instead of (3.22),

$$\frac{\rho_{\text{loops}}}{\rho}=\sqrt{GM} \ . \tag{3.23}$$

If one considers the grand unification phase transition, $T_c \sim 10^{15}-10^{16}$ GeV, $\eta^2 \sim m_+^2/\alpha$, $\alpha=e^2/4\pi \sim 0.02$. If we take $\lambda=1$ and $m_+ \sim 10^{15}$ GeV we still have to fix δ in order to determine M. For a value of $\delta \sim 1$ (which corresponds to $\mu \sim 10^{14}$) we have, using our numerical results, $M=1.2\pi\eta^2$, which then gives

$$\frac{\rho_s}{\rho} \sim 10^{-6} \ , \quad \frac{\rho_{\text{loops}}}{\rho} \sim 10^{-3} \ . \tag{3.24}$$

One then gets a density contrast which, as in the case of neutral vortices, is of the order of magnitude needed to trigger galaxy formation.[19]

One can then envisage the study of evolution of these charged vortices following the lines described in Ref. 19 for the case of neutral ones since the existence of these charged-vortex solutions may find interesting applications in the early Universe.

Note added. After this work was completed, we received a paper (Ref. 22) where the results that we present in the Appendix and expansion (3.9) are also discussed.

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APPENDIX

We shall first prove in this appendix that if $A_0(0)=0$ then no charged-vortex solution exists, thus showing the necessity of choosing boundary conditions at the origin, in the form (2.11). We then sketch the proof on the nonexistence of the m solution, as advanced in Sec. III.

In terms of fields $x(\tau)$, $y(\tau)$, $z(\tau)$ boundary conditions (2.10) and (2.11) read

$$\begin{aligned} &x(0)=1 \ , \quad x(\infty)=0 \ , \\ &y(0)=c \ , \quad y(\infty)=0 \ , \\ &z(0)=0 \ , \quad z(\infty)=1 \ . \end{aligned} \tag{A1}$$

Let us suppose that $c=0$. Then assuming analyticity we have the following expansions around the origin:

$$\begin{aligned} &x(\tau)=1+a_2\tau^2+O(\tau^4) \ , \\ &y(\tau)=\delta\frac{a_2}{2}\tau^2+O(\tau^4) \ , \\ &z(\delta)=c_1\tau+O(\tau^3) \ . \end{aligned} \tag{A2}$$

We consider a positive δ ($\delta<0$ can be treated analogously). There are two possible behaviors of $x(\tau)$ according to the sign of a_2.

If $a_2>0$ both $x(\tau)$ and $y(\tau)$ are increasing functions in a neighborhood of the origin, $y(\tau)$ then has at least one maximum at finite τ.

Let us call r the position of the first maximum; one also has $y(r)>0$. Equation (2.13b) then implies

$$\left.\frac{dx}{d\tau}\right|_{\tau=r}<0 \ . \tag{A3}$$

We see from (A3) that $x(\tau)$ has to have at least a maximum at some point $r_1, r_1 \epsilon (o,r)$. Let us suppose that r_1 is the first of these maxima. But from (2.13a) we have

$$\left.\frac{d^2x}{d\tau^2}\right|_{\tau=r_1}=xz^2+\delta\left.\frac{dy}{d\tau}\tau\right|_{\tau=r_1}>0 \tag{A4}$$

and hence r_1 cannot correspond to a maximum. We then see there is no solution for $\delta>0$ when $a_2>0$. An analogous proof holds for $a_2<0$.

We shall now prove the nonexistence of the monotonically decreasing solution with the behavior (3.1a) with $\beta=\beta_-$ as given by (3.5b).

The behavior corresponds to $c_1=-c_2$ [see Eq. (3.4b)]. This implies that either

$$\lim_{\tau\to\infty} x(\tau)=0^+ \ , \quad \lim_{\tau\to\infty} y(\tau)=0^- \ , \tag{A5}$$

or

$$\lim_{\tau\to\infty} x(\tau)=0^- \ , \quad \lim_{\tau\to\infty} y(\tau)=0^+ \ . \tag{A6}$$

Let us consider possibility (A5). Since $y(0)>0$ (as can be easily seen following the same steps as in the precedent proof) $y(\tau)$ has at least one negative minimum.

Let us call τ_1 the last value of τ corresponding to such a minimum (in case there is more than one). From Eq. (2.13a) $x(\tau)$ is an increasing function at τ_1. Now, because of Eq. (A5) $x(\tau)$ has then to have a positive maximum in (r_1,∞). However, this is not possible as can be easily seen from (2.13a). Then the solution is excluded if (A5) holds. The same can be shown for the case (A6).

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[1]For a review on (2 + 1)-dimensional gauge theories, see R. Jackiw, in *Relativity, Groups and Topology II*, proceedings of the Les Houches Summer School, Les Houches, France, 1983, edited by R. Stora and B. S. DeWitt (Les Houches Summer School Proceedings, Vol. 40) (North-Holland, Amsterdam, 1984).

[2]S. Deser, R. Jackiw, and S. Templeton, Phys. Rev. D **23**, 2291 (1981); Phys. Rev. Lett. **48**, 975 (1982); Ann. Phys. (N.Y.) **140**, 372 (1982).

[3]J. Schonfeld, Nucl. Phys. **B185**, 157 (1981).

[4]D. A. Kirzhnitz and A. D. Linde, Phys. Lett. **42B**, 471 (1972).

[5]L. Dolan and R. Jackiw, Phys. Rev. D **9**, 3320 (1974).

[6]S. Weinberg, Phys. Rev. D **9**, 3357 (1974).

[7]A. N. Redlich and L. C. R. Wijewardhana, Phys. Rev. Lett. **54**, 970 (1985).

[8]S. K. Paul and A. Khare, Phys. Lett. B **174**, 420 (1986); **182**,

414(E) (1986).

[9]H. de Vega and F. A. Schaposnik, Phys. Rev. Lett. **56**, 2564 (1986).

[10]H. de Vega and F. A. Schaposnik, Phys. Rev. D **34**, 3206 (1986).

[11]S. K. Paul and A. Khare, Phys. Lett. B **174**, 420 (1986).

[12]H. B. Nielsen and P. Olesen, Nucl. Phys. **B61**, 45 (1973).

[13]B. Julia and A. Zee, Phys. Rev. D **11**, 2227 (1975).

[14]M. V. Manias, C. M. Naon, F. A. Schaposnik, and M. Trobo, Phys. Lett. B **171**, 199 (1986).

[15]R. D. Pisarski and S. Rao, Phys. Rev. D **32**, 2081 (1985).

[16]H. de Vega and F. A. Schaposnik, Phys. Rev. D **14**, 1100 (1976).

[17]L. Jacobs and C. Rebbi, Phys. Rev. B **19**, 4486 (1978).

[18]C. T. Hill, H. M. Hodges, and M. S. Turner, Phys. Rev. D **37**, 263 (1988).

[19]For reviews, see T. W. B. Kibble, Phys. Rep. **67**, 183 (1980); A. Vilenkin, *ibid.* **121**, 263 (1985); R. Brandenberger, Proceedings of the U. K. Institute for Theoretical Physics, 1987 (unpublished), and references therein.

[20]G. 't Hooft, Nucl. Phys. **B79**, 276 (1974).

[21]A. M. Polyakov, Pis'ma Zh. Eksp. Teor. Fiz. **20**, 430 (1974) [JETP Lett. **20**, 194 (1974)].

[22]V. I. Inozemtsev, Europhys. Lett. **5**(2), 113 (1988).