Functional representation of the superconformal group in two dimensions

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A functional representation for the local supersymmetry transformations of the superconformal group in two dimensions is explicitly constructed. The representation kernel acts on functionals of both a commuting field and a Grassmann field.

Superconformal transformations in two dimensions form a doubly infinite transformation group.[1] Its action on the coordinates $(x,t,\theta,\bar\theta)$ of Minkowski superspace, with θ and $\bar\theta$ Grassmann variables, takes the pairs of coordinates $(x+t,\theta)$ and $(x-t,\bar\theta)$ into pairs of arbitrary functions of the corresponding variables: $(x+t,\theta)\to(\mathcal{F}_+(x+t,\theta),\Theta(x+t,\theta))$, $(x-t,\bar\theta)\to(\mathcal{F}_-(x-t,\bar\theta),\bar\Theta(x-t,\bar\theta))$. $\mathcal{F}_\pm$ and $\Theta,\bar\Theta$ are, respectively, commuting and anticommuting functions. In the following, we restrict the discussion to one component of the group, transforming $(x+t,\theta)$ but not $(x-t,\bar\theta)$. $[\mathcal{F}_-(x-t,\bar\theta)=x-t,\ \bar\Theta(x-t,\bar\theta)=\bar\theta.]$

The infinitesimal transformation of the coordinates $(x+t,\theta)$,

$$\begin{aligned}&\delta_{f,\xi}(x+t)=-f(x+t)+i\theta\xi(x+t)\ ,\\&\delta_{f,\xi}\theta=-\tfrac{1}{2}\theta f'(x+t)-\xi(x+t)\ ,\end{aligned}\tag{1}$$

with f and ξ, respectively, commuting and anticommuting functions of $x+t$, realizes a Lie algebra with structure relations given by

$$\begin{aligned}&[\delta_{f,\xi},\delta_{g,\eta}]=-\delta_{h,\zeta}\ ,\\&h=fg'-gf'+2i\eta\xi,\quad \zeta=\tfrac{1}{2}(g'\xi-f'\eta)+f\eta'-g\xi'\ .\end{aligned}\tag{2}$$

(The prime will always represent differentiation with respect to the space variable. For the sake of conciseness, we shall often suppress the space-time arguments and use a matrixlike notation when spatial integrals appear.)

The minimal two-dimensional field theory which is invariant under the superconformal transformations (1) involves a scalar $\chi(t,x)$ and a Majorana-Weyl spinor $\psi(t,x)$ satisfying the self-dual equations of motion

$$\dot\chi=\chi',\quad \dot\psi=\psi'\ .\tag{3}$$

(The overdot means time differentiation.) The theory is governed by the Lagrangian[2]

$$\begin{aligned}L=&\ \tfrac{1}{4}\int dx\,dy\ \chi(t,x)\mathrm{sgn}(x-y)\dot\chi(t,y)-\tfrac{1}{2}\int dx\ \chi^2(t,x)\\&+\frac{i}{2}\int dx\ \psi(t,x)[\dot\psi(t,x)-\psi'(t,x)]\end{aligned}\tag{4}$$

from which Eqs. (3) arise as Euler-Lagrange equations. The Hamiltonian density is simply $\mathcal{H}=\frac{1}{2}\chi^2+(i/2)\psi\psi'$. Canonical quantization of this first-order theory leads to the following equal-time relations:

$$\begin{aligned}&[\chi(t,x),\chi(t,y)]=i\delta'(x-y)\equiv k(x,y)\ ,\\&\{\psi(t,x),\psi(t,y)\}=\delta(x-y)\ ,\\&[\chi(t,x),\psi(t,y)]=0\ .\end{aligned}\tag{5}$$

At fixed time, the generators of conformal and local supersymmetry transformations are formally given by

$$\begin{aligned}Q_f^c=&\ \tfrac{1}{2}\int dx\ \chi(x)f(x)\chi(x)\\&+\frac{i}{4}\int dx[\psi(x)f(x)\psi'(x)-\psi'(x)f(x)\psi(x)]\ ,\end{aligned}\tag{6a}$$

$$Q_\xi^s=i\int dx\ \chi(x)\xi(x)\psi(x)\ .\tag{6b}$$

They yield the following infinitesimal transformations of the field operators:

$$\begin{aligned}&\delta_f\chi=i[Q_f^c,\chi]=(f\chi)'\ ,\\&\delta_f\psi=i[Q_f^c,\psi]=(f\psi)'-\tfrac{1}{2}f'\psi\ ,\end{aligned}\tag{7a}$$

$$\delta_\xi\chi=i[Q_\xi^s,\chi]=i(\xi\psi)',\quad \delta_\xi\psi=i[Q_\xi^s,\psi]=-\xi\chi\ ,\tag{7b}$$

and it is immediate to verify that the Lagrangian (4) is invariant under these transformations.

It is well known, however, that the conformal generators Q_f^c are singular as they involve products of field operators at the same point. Here, a c-number subtraction q_f^c will suffice to well define them. Observe that the supersymmetry generator Q_ξ^s being linear in χ and ψ need no such renormalization. Using the Schrödinger picture the form of the subtraction q_f^c is intrinsically obtained by requiring that the representation kernel for the *finite* conformal transformations be free from ultraviolet divergences.[3,4] One thus obtains

$$q_f^c=\lim\frac{1}{4}\left[\mathrm{tr}(F\,|\,k\,|\,)-\mathrm{tr}\left[\tilde F\frac{k}{|\,k\,|}\right]\right]\ ,\tag{8}$$

where the limit operation amounts to the substitution $F(x,y) \rightarrow f(x)\delta(x-y)$, $\tilde{F}(x,y) \rightarrow (i/2)[f(x)+f(y)] \times \delta'(x-y)$, and the renormalized generator is defined as

$$:Q_f^c: = Q_f^c - q_f^c \ . \tag{9}$$

The two terms in (8) correspond, respectively, to the renormalizing subtractions of the bosonic and fermionic parts of the charge (6a).

Since the subtraction is a c number, the transformation laws (7a) are not affected when Q_f^c is replaced by $:Q_f^c:$. On the other hand, the commutation rules of the generators $Q_{f,\xi} = Q_f^c + Q_\xi^s$, which from (5) appear to represent (2), do get modified under $Q_f^c \rightarrow :Q_f^c:$ by the occurrence of central elements. One has

$$[:Q_f^c:, :Q_g^c:] = i :Q_{(f,g)}^c: + \frac{i}{48\pi}(\tfrac{3}{2}) \int dx\,(f'''g - g'''f) \ , \tag{10a}$$

$$[Q_\xi^s, Q_\eta^s] = i :Q_{(\xi,\eta)}^c: + \frac{1}{8\pi} \int dx\,(\eta\xi'' - \xi\eta'') \ , \tag{10b}$$

$$[Q_\xi^s, :Q_f^c:] = iQ_{(\xi,f)}^s \ , \tag{10c}$$

with

$$(f,g) = fg' - gf', \quad (\xi,\eta) = 2i\eta\xi \ ,$$

$$(\xi,f) = \tfrac{1}{2}f'\xi - f\xi' \ .$$

Notice that the central charge arising in the commutator of two supersymmetry generators is nothing else than the renormalizing subtraction of $Q_{(\xi,\eta)}^c$.

Functional representations for the finite elements of the superconformal group can be obtained using the Schrödinger picture. The representation kernels act in the space of all functionals $\Psi(\phi,u)$ of a commuting field $\phi(x)$ and a Grassmann field $u(x)$. In this picture, the canonical relations (5) are realized by identifying $\chi(x)$ and $\psi(x)$ at fixed time with the following Hermitian differential operators:[4]

$$\chi(x)\Psi(u,\phi) \equiv \frac{1}{\sqrt{2}} \left[\phi'(x) - i\frac{\delta}{\delta\phi(x)} \right] \Psi(u,\phi) \ , \tag{11a}$$

$$\psi(x)\Psi(u,\phi) \equiv \frac{1}{\sqrt{2}} \left[u(x) + \frac{\delta}{\delta u(x)} \right] \Psi(u,\phi) \ . \tag{11b}$$

The kernel that represents simple conformal transformations has been recently obtained (see Refs. 3 and 4 for its explicit expression). Here we shall devote our attention to the kernel $U_\xi^s(u,u_0;\phi,\phi_0;\tau)$ that represents the finite supersymmetry transformation $e^{-i\tau Q_\xi^s}$ in the functional space. This kernel obeys a Schrödinger-type equation,

$$i\partial_\tau U_\xi^s = Q_\xi^s U_\xi^s \tag{12a}$$

with

$$U_\xi^s(u,u_0;\phi,\phi_0;0) = \delta(\phi-\phi_0)\delta(u-u_0) \tag{12b}$$

as a boundary condition.

Information on the solution of this equation can be obtained by considering the action of finite supersymmetry transformations on the field variables:

$$\chi_\tau(x) \equiv e^{i\tau Q_\xi^s}\chi(x)e^{-i\tau Q_\xi^s} = \chi(x) + i\tau[\xi(x)\psi(x)]' \ , \tag{13a}$$

$$\psi_\tau(x) \equiv e^{i\tau Q_\xi^s}\psi(x)e^{-i\tau Q_\xi^s} = \psi(x) - \tau\xi(x)\chi(x) - \frac{i\tau^2}{2}\xi(x)\xi'(x)\psi(x) \ . \tag{13b}$$

These transformations imply the following two constraint equations on U_ξ^s:

$$\left[\phi'(x) - i\frac{\delta}{\delta\phi(x)} \right] U_\xi^s = \left\{ \phi_0'(x) + i\frac{\delta}{\delta\phi_0(x)} + i\tau \left[\xi(x) \left[u_0(x) - \frac{\delta}{\delta u_0(x)} \right] \right]' \right\} U_\xi^s \ , \tag{14a}$$

$$\left[u(x) + \frac{\delta}{\delta u(x)} \right] U_\xi^s = \left[u_0(x) - \frac{\delta}{\delta u_0(x)} - \tau\xi(x) \left[\phi_0'(x) + i\frac{\delta}{\delta\phi_0(x)} \right] - \frac{i\tau^2}{2}\xi(x)\xi'(x) \left[u_0(x) - \frac{\delta}{\delta u_0(x)} \right] \right] U_\xi^s \ . \tag{14b}$$

It is convenient to write the kernel U_ξ^s in the form

$$U_\xi^s(u,u_0;\phi,\phi_0;\tau) = \exp\left[-\int \phi k \phi_0 \right] \exp\left[-\int u u_0 \right] \times V_\xi(v_+,v_-;\phi_+,\phi_-;\tau) \ , \tag{15}$$

where we have introduced the notation $\phi_\pm = \phi \pm \phi_0$, $v_\pm = u \pm u_0$. Using the anticommuting character of the parameter $\xi(x)$, $\{\xi(x),\xi(y)\} = 0$, it is easy to see that (14a) and (14b) amount to the following constraints on V_ξ:

$$\left[\int dy\, k(x,y) \left[\phi_-(y) - \frac{i\tau}{2}\xi(y)\frac{\delta}{\delta v_-(y)} \right] + \frac{\delta}{\delta\phi_+(x)} \right] V_\xi = 0 \ , \tag{16a}$$

$$\left[v_-(x) + \frac{\delta}{\delta v_+(x)} - \frac{i\tau}{2}\xi(x)\frac{\delta}{\delta\phi_-(x)} \right] V_\xi = 0 \ . \tag{16b}$$

With the help of these conditions, the Schrödinger equation (12) is simplified to

$$i\partial_\tau V_\xi=\frac{1}{2}\int dx\frac{\delta}{\delta\phi_-(x)}\xi(x)\frac{\delta}{\delta v_-(x)}V_\xi\ , \tag{17a}$$

$$V_\xi(v_+,v_-;\phi_+,\phi_-;0)=\delta(\phi_-)\delta(v_-)\ , \tag{17b}$$

and we observe that V_ξ can be taken to depend only on the variables ϕ_- and v_-. The following power series in τ,

$$V_\xi(v_-,\phi_-;\tau)=\delta(\phi_-)\delta(v_-)-\frac{i\tau}{2}\int dx\,\xi(x)\frac{\delta}{\delta\phi_-(x)}\delta(\phi_-)\times\frac{\delta}{\delta v_-(x)}\delta(v_-)+\cdots\ , \tag{18}$$

clearly satisfies (17). The series can be summed by using a "functional integral" representation

$$V_\xi(v_-,\phi_-;\tau)=\int \mathcal{D}\rho\,\mathcal{D}\theta\exp\left[-i\int\rho\phi_-\right]\times\exp\left[\int\theta v_-\right]\exp\left[\frac{\tau}{2}\int\rho\xi\theta\right] \tag{19}$$

with $\theta(x)$ an anticommuting function. Performing the $\rho(x)$ and $\theta(x)$ integrations, one finally obtains

$$U^s_\xi(u,u_0;\phi,\phi_0;\tau)=\exp\left[-\int\phi k\phi_0\right]\exp\left[-\int uu_0\right]\times\delta\left[u-u_0-\frac{i\tau}{2}\xi\frac{\delta}{\delta\phi}\right]\delta(\phi-\phi_0)\ . \tag{20}$$

We have not succeeded in writing down an expression for U^s_ξ which does not involve the derivative $\delta/\delta\phi$. The difficulty resides in the fact that the solution of the constraints (16) requires the "inversion" of the anticommuting parameter $\xi(x)$, an operation which we could only define implicitly through Eq. (19).[5]

The above expression for U^s_ξ can be transformed into equivalent formulas that sometimes prove more practical. From the representation

$$\delta\left[u-u_0-\frac{i\tau}{2}\xi\frac{\delta}{\delta\phi}\right]=\int\mathcal{D}\theta\exp\left[\int\theta\left[u-u_0-\frac{i\tau}{2}\xi\frac{\delta}{\delta\phi}\right]\right] \tag{21}$$

and the fact that

$$\exp\left[-\int\phi k\phi_0\right]\exp\left[-\frac{i\tau}{2}\int\theta\xi\frac{\delta}{\delta\phi}\right]=\exp\left[\frac{\tau}{2}\int\theta\xi\left[\phi_0'-i\frac{\delta}{\delta\phi}\right]\right]\exp\left[-\int\phi k\phi_0\right]\ , \tag{22}$$

one obtains

$$U^s_\xi=\exp\left[-\int uu_0\right]\delta\left[u-u_0+\frac{\tau}{\sqrt{2}}\chi\xi\right]\delta(\phi-\phi_0)\ . \tag{23}$$

This can also be recast in the form

$$U^s_\xi=\delta\left[u-u_0+\frac{\tau}{\sqrt{2}}\xi\chi(1-uu_0)\right]\delta(\phi-\phi_0) \tag{24}$$

using

$$\exp\left[-\int uu_0\right]=\prod_x[1-u(x)u_0(x)]\equiv\det(1-uu_0)\ .$$

Note that the Grassmanian δ functions in (23) and (24) are well defined despite the presence of the operator $\chi(x)$ in their arguments. Indeed, for $x\neq y$, $\xi(x)\chi(x)$ anticommutes with $\xi(y)\chi(y)$; when $x=y$, one can safely say that $[\xi(x)\chi(x)]^2=0$ irrespectively of the singular products of two $\chi(x)$ because $\xi(x)^2=0$, strictly. The coherence of this rule within our framework is established as follows. On the one hand, if one accepts that $(\xi\chi)^2=0$, the expression (24) for U^s_ξ is seen to verify the Schrödinger equation (12) by direct substitution. On the other hand, we know that (24) is equivalent to (20) which in turn has been proven to satisfy the Schrödinger equation without recourse to the relation $(\xi\chi)^2=0$. No inconsistencies therefore arise. Finally, notice that the above representation for U^s_ξ is unitary with respect to the natural inner product on functional space.[4]

Let us now say a few words on the convolution of two supersymmetry kernels. In general, the composition of two local supersymmetry transformations gives as a result a superposition of conformal and local supersymmetry transformations. However, when the parameters of the two supersymmetry transformations are proportional to each other, composition yields again a pure supersymmetry transformation. Explicitly one has

$$\int\mathcal{D}\bar\phi\,\mathcal{D}\bar u\ U^s_\xi(u,\bar u;\phi,\bar\phi;\tau_1)U^s_\xi(\bar u,u_0;\bar\phi,\phi_0;\tau_2)=U^s_\xi(u,u_0;\phi,\phi_0;\tau_1+\tau_2) \tag{25}$$

which can be easily checked using, for example, the form (23) for the kernel. When the parameters of the two supersymmetry kernels are different, the projective composition rule

$$e^{-i\tau Q^s_\xi}e^{-i\tau Q^s_\eta}=e^{-i\tau q^c_h}e^{-i\tau(Q^s_\zeta+:Q^c_h:)} \tag{26}$$

with $\zeta=\xi+\eta-(i\tau^2/4)(\xi\eta\eta'+\eta\xi\xi')$ and $h=i\tau\eta\xi$, can be established in a τ power series. [Equation (26) has been explicitly checked up to $O(\tau^2)$.]

Finally, let us observe that the kernel U^s_ξ can be used to compute how states transform under local supersymmetry transformations:

$$\Psi(u,\phi)\to\tilde\Psi(u,\phi)=\int\mathcal{D}\bar\phi\,\mathcal{D}\bar u\ U^s_\xi(u,\bar u;\phi,\bar\phi;\tau)\Psi(\bar u,\bar\phi)\ . \tag{27}$$

In particular, for $\xi(x)=\alpha f(x)$, with α an anticommuting constant and f an arbitrary function, the Gaussian

$$\Psi_{\Omega,\omega}(u,\phi)=\exp\left[-\tfrac{1}{2}\int\phi\Omega\phi\right]\exp\left[\tfrac{1}{2}\int u\omega u\right]$$

is transformed into

$$\tilde{\Psi}_{\Omega,\omega}(u,\phi)=\exp\left[\frac{i\tau}{2}\int u(\omega-1)\xi(\Omega-k)\phi\right]\times\Psi_{\Omega,\omega}(u,\phi)\ . \quad (28)$$

The supersymmetric vacuum of the theory (4) has for wave functional the Gaussian with covariances $\Omega=|k|$ and $\omega=-k/|k|$. From (28) one can check that this state is invariant under the OSp(1,1) subgroup of the superconformal transformations generated by $\xi(x)=(\alpha,\alpha x)$.

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[5]We could also write

$$U^s_\xi=\lim_{\lambda\to 0}N\exp\left[-\int\phi k\phi_0\right]\exp\left[-\int uu_0\right]\times\delta\left[u-u_0+\frac{i\tau}{2\lambda}\xi(\phi-\phi_0)\right]\times\exp\left[-\frac{1}{2\lambda}\int(\phi-\phi_0)^2\right],$$

where N is a λ-dependent normalization factor such that

$$\lim_{\lambda\to 0}N\exp\left[-\frac{1}{2\lambda}\int(\phi-\phi_0)^2\right]=\delta(\phi-\phi_0)\ .$$

One can check that this expression for U^s_ξ satisfies (12a) for finite λ, although the boundary condition is recovered only in the limit of small λ.