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Spectrum of the Casimir effect

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The problem of assigning a frequency spectrum to the Casimir effect is studied. The specific case of a massless scalar field with periodicity in one spatial direction is investigated in both two- and four-dimensional spacetime. The spectrum is defined by introducing spectral weight functions which distort the original spectrum of quantum fluctuations and hence reveal the contribution of each frequency interval to the finite Casimir energy. The result is a function $\sigma(\omega)$ whose integral over all frequencies is the total vacuum energy. In order to have $\sigma(\omega)\to 0$ as $\omega\to\infty$, it is necessary to specify a nonzero tolerance $\Delta\omega$ which is the allowed uncertainty in the measurement of the spectrum. In the limit that $\Delta\omega\to 0$, $\sigma(\omega)$ approaches a discontinuous function which does not vanish as $\omega\to\infty$.

I. INTRODUCTION

In recent years the Casimir effect[1] has been extensively studied.[2] Finite vacuum energies have been computed for quantized fields which satisfy various boundary conditions in flat spacetime as well as for those on curved background spacetimes.[3] A typical calculation for the vacuum energy in a cavity in flat spacetime might proceed as follows.

(1) Solve the classical wave equation subject to the appropriate boundary conditions and determine the eigenfrequencies ω_n of the normal modes.

(2) The formal vacuum energy $E=\frac{1}{2}\sum_n \omega_n$ is divergent, so introduce an artificial prescription for rendering the sum finite. For example, introduce a cutoff function which suppresses the contribution of high-frequency modes.

(3) Subtract from this finite sum the corresponding result in the absence of boundaries. The latter is an integral containing the same cutoff function as in (2).

(4) Take the limit as the cutoff is removed. This should yield a finite, unambiguous result for the vacuum energy. We interpret this vacuum energy physically as being dominated by the contributions of modes whose wavelengths are of the order of the characteristic dimension of the cavity (more precisely, its smallest dimension). The contribution of shorter-wavelength modes is effectively canceled by the subtraction at step (3).

However, this interpretation is not made quantitative by the above procedure. It is necessary to sum over all modes at step (2) before performing the subtraction. A more precise interpretation would require a spectral function whose integral over all frequencies is the finite vacuum energy. This finite energy sometimes appears at step (4) naturally expressed as such an integral. For example, the energy density of a massless scalar field in $S^1\times R$ (two-dimensional flat spacetime with spatial periodicity of length L) can be expressed as (see Sec. II)

$$\rho=-\pi^{-1}\int_0^\infty \frac{\omega\, d\omega}{e^{L\omega}-1}=-\frac{\pi}{6L^2}\ . \tag{1.1}$$

This form has a suggestive interpretation as an integral over a thermal spectrum with a temperature equal to L^{-1}. However, one must be cautious about this because there are obviously an infinite number of integrals on ω which yield the same result. A more careful treatment is required and will be offered in the subsequent sections of this paper.

II. SCALAR FIELD IN $S^1\times R$

Here we consider a massless scalar field in two dimensions which satisfies periodic boundary conditions:

$$\phi(x,t)=\phi(x+L,t)\ . \tag{2.1}$$

The eigenfrequencies of the normal modes are

$$\omega_n=|k_n|\ ,\quad k_n=\frac{2\pi n}{L},\quad n=0,\pm1,\pm2,\ldots\ . \tag{2.2}$$

The formal, divergent, vacuum energy density is

$$\rho_0=(2L)^{-1}\sum_{n=-\infty}^{\infty}\omega_n\ . \tag{2.3}$$

Let us introduce a spectral weight function $W(\omega)$ which has the effect of weighting some regions of the spectrum differently from others, and define

$$\rho_W=(2L)^{-1}\sum_{n=-\infty}^{\infty}\omega_n W(\omega_n)\ . \tag{2.4}$$

If $W(\omega)$ vanishes sufficiently rapidly as $\omega\to\infty$ then ρ_W is finite.

We can rewrite this sum using the Plana summation formula,[4] which states that

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$$\sum_{n=1}^{\infty} f(n)+\tfrac{1}{2}f(0)=\int_0^{\infty} f(x)dx + i\int_0^{\infty} \frac{f(it)-f(-it)}{e^{2\pi t}-1}dt \ . \tag{2.5}$$

Here $f(z)$ is assumed to be analytic for $\text{Re}z \geq 0$. The result is

$$\rho_W=\frac{1}{2\pi}\int_0^{\infty}\omega W(\omega)d\omega - \frac{2\pi}{L^2}\int_0^{\infty}\frac{t[W(2\pi L^{-1}it)+W(-2\pi L^{-1}it)]}{e^{2\pi t}-1}dt \ . \tag{2.6}$$

If $W \equiv 1$, then the first term above is the usual divergent vacuum energy density of R^2, flat spacetime without periodicity. The second term becomes the finite vacuum energy density ρ of $S^1 \times R$, Eq. (1.1). For a general choice of W, let us call the second term σ:

$$\sigma=-\frac{2\pi}{L^2}\int_0^{\infty}\frac{t[F(it)+F(-it)]}{e^{2\pi t}-1}dt \ , \tag{2.7}$$

where $F(x)=W(2\pi L^{-1}x)$.

So far, one could regard $W(\omega)$ as an arbitrary cutoff function whose sole purpose is an ultraviolet regulator and which should now be eliminated. However, we wish to regard it as having a physical significance as a weighting function which alters the usual contribution of a given frequency interval to the total vacuum energy. In the limit in which W is sharply peaked about one frequency, we can expect to discover that frequency's contribution to the vacuum energy. Note that $W(\omega)$ has an effect equivalent to that of a dielectric function. However, we require that W be analytic for $\text{Re}\omega \geq 0$, whereas dielectric functions are analytic for $\text{Im}\omega \geq 0$. The effects of dielectrics on the Casimir effect for the electromagnetic field were investigated by Lifshitz.[5,6]

We now make an explicit choice for the function $F(x)$:

$$F(x)=F_n(x,x_0) =(2n/x_0)^{2n+1}\frac{x^{2n}}{(2n)!}e^{-2nx/x_0} \ . \tag{2.8}$$

The functions F_n have the properties

$$\int_0^{\infty}F_n(x,x_0)dx=\int_0^{\infty}F_n(x,x_0)dx_0=1 \ . \tag{2.9}$$

Furthermore, F_n is sharply peaked about $x=x_0$ if $n \gg 1$. This is illustrated in Fig. 1. Thus $\lim_{n\to\infty}F_n(x,x_0)=\delta(x-x_0)$. The width Δx may be defined for finite n by

$$F_n(x_0+\tfrac{1}{2}\Delta x,x_0)=\tfrac{1}{2}F_n(x_0,x_0) \ . \tag{2.10}$$

Note that for $n \gg 1$, we have approximately

$$F_n(x,x_0)\approx x_0^{-1}\left[\frac{n}{\pi}\right]^{1/2}e^{-2n[x/x_0-\ln(x/x_0)-1]} \ . \tag{2.11}$$

This form may be used to show that for large n

$$\Delta x \approx 2x_0\left[\frac{\ln 2}{n}\right]^{1/2} \ . \tag{2.12}$$

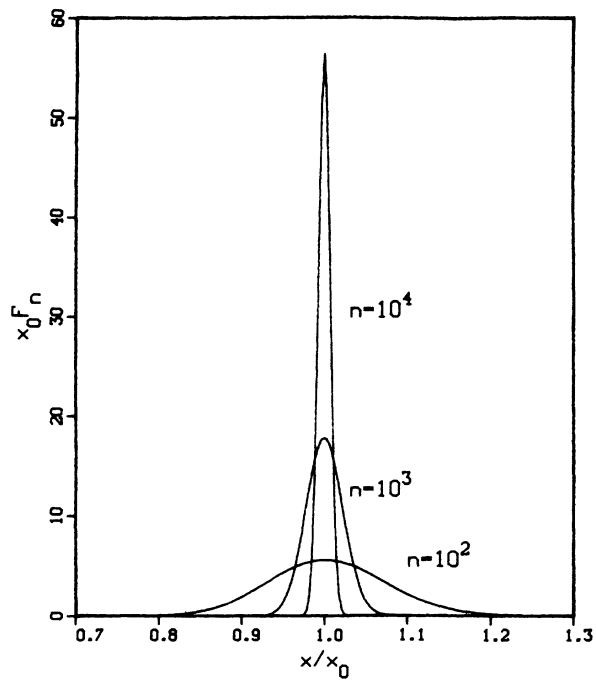

FIG. 1. The weight functions $F_n(x,x_0)$ for various values of n.

If we substitute Eq. (2.8) into Eq. (2.7) for σ the result is

$$\sigma=-\frac{4\pi}{L^2}G(n,x_0) \ , \tag{2.13}$$

where

$$G(n,x_0)=\frac{(-1)^n}{(2n)!}\left[\frac{2n}{x_0}\right]^{2n+1}\times\int_0^{\infty}\frac{t^{2n+1}\cos(2nt/x_0)}{e^{2\pi t}-1}dt \ . \tag{2.14}$$

The integral may be evaluated to write

$$G(n,x_0)=\frac{1}{4(2n)!}\left[\frac{n}{x_0}\right]^{2n+1}\coth^{(2n+1)}\left[\frac{n}{x_0}\right]+\left[\frac{2n+1}{4n}\right]x_0 \ . \tag{2.15}$$

We may use the expansion $\coth(y)=1+2\sum_{l=1}^{\infty}e^{-2ly}$ to write (for $n \gg 1$)

$$G(n,x_0)=-(2x_0)^{-1}\left[\frac{n}{\pi}\right]^{1/2}\sum_{l=1}^{\infty} l e^{-2n[l/x_0+\ln(l/x_0)-1]}+\left[\frac{2n+1}{4n}\right]x_0 \ . \tag{2.16}$$

The latter form is especially useful for numerical evaluation when n is large. The results of such evaluations for various values of n are shown in Fig. 2. We see that for

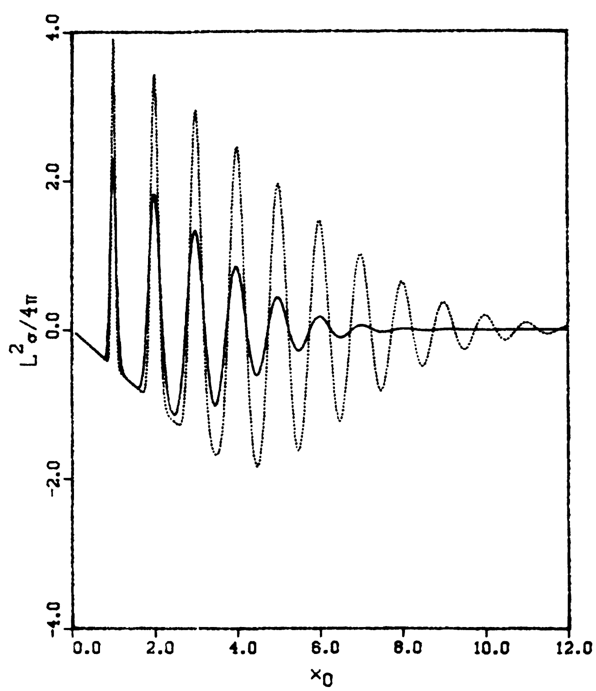

FIG. 2. The spectral function $\sigma(\omega)$ for a scalar field in $S^1\times R$ is plotted as a function of $x_0=L\omega/(2\pi)$ for $n=100$ (solid curve) and $n=250$ (dotted curve).

large but finite n, σ as a function of x_0 contains sharp peaks at integer values of x_0. In the limit $n\to\infty$, σ approaches the difference between a discrete spectrum of δ-function peaks at integer values and a continuous linear spectrum. This is the type of spectrum which one might have naively predicted based on the consideration that the finite vacuum energy is effectively obtained by subtracting a continuous spectrum from a discrete spectrum. However, for finite n, σ is a well-behaved continuous spectrum whose integral over all frequencies is the total vacuum energy [note that $x_0=L\omega/(2\pi)$]

$$\int_0^\infty \sigma\, dx_0=\rho=-\frac{\pi}{6L^2} \; . \tag{2.17}$$

We can interpret σ as the result of measuring the spectrum of the Casimir effect with a finite precision, $\Delta\omega=2\pi L^{-1}\Delta x$. As long as $\Delta\omega>0$, $\sigma\to 0$ as $x_0\to\infty$. What is not well defined is the spectrum that would arise if infinite precision is demanded, so $\Delta\omega=0$.

III. SCALAR FIELD IN $S^1\times R^3$

A. Spectrum from the Green's function

In this section we consider a massless scalar field in four-dimensional flat space-time with periodicity in one spatial direction ($S^1\times R^3$). The total vacuum energy can be calculated using a mode sum with cutoff functions, or by constructing the Green's function in this space. Let us review the latter approach. Let $G_0(x,x_0)$ be the two-point function in R^4:

$$G_0(x,x_0)=\{4\pi^2[(\mathbf{x}-\mathbf{x}')^2-(t-t')^2]\}^{-1} \; . \tag{3.1}$$

The Green's function in $S^1\times R^3$ can be expressed as an image sum:

$$G(x,x')=\sum_{n=-\infty}^{\infty} G_0(\mathbf{x}-\mathbf{x}'+n\hat{\mathbf{z}}L,t-t') \; , \tag{3.2}$$

where $\hat{\mathbf{z}}$ is the direction of periodicity and L is the periodicity length. We see that G is a solution of the homogeneous wave equation, $\Box_x G=0$, and that it satisfies the required periodicity $G(\mathbf{x}-\mathbf{x}'+L\hat{\mathbf{z}},t-t')=G(\mathbf{x}-\mathbf{x}',t-t')$. The renormalized Green's function can be defined as

$$G_R=G-G_0 \; ; \tag{3.3}$$

i.e., G_R is given by Eq. (3.2) with the $n=0$ term omitted. The energy density[7] for a massless scalar field is $\frac{1}{2}(\dot{\phi}^2+|\nabla\phi|^2)$, so we can obtain the finite vacuum energy density as a coincidence limit

$$\rho=\tfrac{1}{2}\lim_{\substack{\mathbf{x}'\to\mathbf{x}\\ t'\to t}}[(\partial_t\partial_{t'}+\nabla\cdot\nabla')G_R]=-\frac{\pi^2}{90L^4} \; . \tag{3.4}$$

This approach to the calculation of ρ suggests an alternative way to obtain a spectral density for the Casimir energy.[8] This is to take the coincidence limit in the spatial coordinates only and then Fourier transform the resulting function of $t-t'$:

$$H(t-t')\equiv\tfrac{1}{2}\lim_{\mathbf{x}\to\mathbf{x}'}[(\partial_t\partial_{t'}+\nabla\cdot\nabla')G_R]$$
$$=-\pi^{-2}\sum_{n=1}^{\infty}\frac{n^2L^2+3(t-t')^2}{[n^2L^2-(t-t')^2]^3} \; . \tag{3.5}$$

Note that $H(0)=\rho$. Define the spectral density by

$$\sigma_0(\omega)=\frac{2}{L}\int_0^\infty H(t)e^{-i\omega t}dt \; . \tag{3.6}$$

This function has the property that

$$\rho=\int_0^\infty \sigma_0 dx_0 \; , \tag{3.7}$$

and is given explicitly as

$$\sigma_0(\omega)=-\frac{\omega^2}{\pi L^2}S(\omega L) \; , \tag{3.8}$$

where

$$S(x)=\sum_{n=1}^{\infty}\frac{\sin nx}{n} \; . \tag{3.9}$$

For $0<x<2\pi$, $S(x)=\frac{1}{2}(\pi-x)$; otherwise $S(x)$ is defined by its periodicity, $S(x+2\pi)=S(x)$, and is hence a discontinuous function. The spectral density $\sigma_0(\omega)$ is also a discontinuous function which is shown in Fig. 3. Its maximum magnitude grows with increasing ω, but it oscillates so that its integral over all ω is finite.

B. Spectrum from weight functions

Let us now study the spectrum of the vacuum energy in $S^1\times R^3$ using the spectral weight functions introduced in Sec. II. Here the formal, divergent, vacuum energy density is expressible as

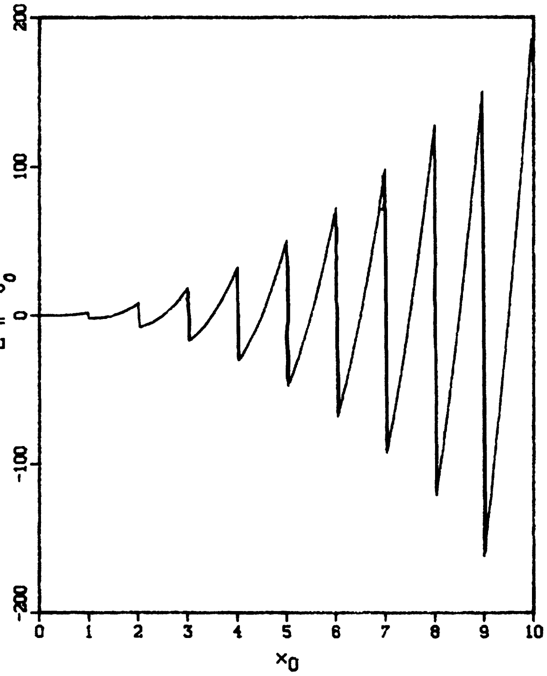

FIG. 3. The discontinuous spectral function σ_0 (corresponding to $n=\infty$) for a scalar field in $S^1\times R^3$.

$$\rho=(4\pi L)^{-1}\int_0^{\infty}dk\,k\sum_{n=-\infty}^{\infty}\omega_n\ , \tag{3.10}$$

where $\omega_n=[k^2+(2\pi n/L)^2]^{1/2}$. We again introduce a spectral weight function $W(\omega)$ and define

$$\rho_W=(4\pi L)^{-1}\int_0^{\infty}dk\,k\sum_{n=-\infty}^{\infty}\omega_n W(\omega_n)\ . \tag{3.11}$$

The weighted energy density is defined as

$$\sigma=\rho_W-(8\pi^2)^{-1}\int_0^{\infty}dk\,k\int_{-\infty}^{\infty}dk_z\omega W(\omega)\ , \tag{3.12}$$

$\omega=(k^2+k_z^2)^{1/2}$. Using the Plana formula, we find

$$\sigma=-4\pi^2L^{-4}\int_0^1 d\beta\beta^2\int_0^{\infty}\frac{dt\,t^3}{e^{2\pi t}-1}[W(2\pi it\beta/L)+W(-2\pi it\beta/L)]\ . \tag{3.13}$$

Here

$$\beta=\frac{L}{2\pi t}\left[\left(\frac{2\pi t}{L}\right)^2-k^2\right]^{1/2} \tag{3.14}$$

and we have used

$$[(\pm 2\pi it/L)^2+k^2]^{1/2}=\begin{cases}[k^2-(2\pi t/L)^2]^{1/2}, & k>2\pi t/L\ ,\\ \pm i[(2\pi t/L)^2-k^2]^{1/2}, & k<2\pi t/L\ .\end{cases} \tag{3.15}$$

Let

$$W(\omega)=F_n\left(\frac{L\omega}{2\pi},x_0\right)\ , \tag{3.16}$$

where the F_n are defined in Eq. (2.8), and perform the t integration in Eq. (3.13) to find

$$\sigma=4\pi^2L^{-4}\int_0^1 d\beta\beta\left[\frac{\beta^{2n}}{8(2n)!}\left(\frac{n}{x_0}\right)^{2n+1}\coth^{(2n+3)}\left(\frac{n}{x_0}\beta\right)+(2n+1)(2n+2)(2n+3)\left(\frac{x_0}{2n}\right)^3\beta^{-4}\right]\ . \tag{3.17}$$

Although each term above diverges as $\beta\to 0$, their sum is finite if we introduce a cutoff at the lower limit of integration and then remove it after performing the β integration. The result is

$$\begin{aligned}\sigma&=\pi^2L^{-4}(2n+1)(n+1)(x_0/n)^2\left[\sum_{j=0}^{2n+2}(-1)^j\frac{1}{j!}\left(\frac{n}{x_0}\right)^j\coth^{(j)}\left(\frac{n}{x_0}\right)-(2n+3)(x_0/n)\right]\\&=\frac{\pi^2(n+1)(2n+1)}{n^2L^4}x_0^2\left[2\sum_{j=1}^{2n+2}\frac{1}{j!}\sum_{l=1}^{\infty}e^{j\ln(2nl/x_0)-2nl/x_0}+\coth\left(\frac{n}{x_0}\right)-\left(\frac{2n+3}{n}\right)x_0\right]\quad(n\gg 1)\ .\end{aligned} \tag{3.18}$$

This spectral function has the property that for any n, its integral on x_0 is the total vacuum energy density

$$\rho=\int_0^{\infty}\sigma\,dx_0\ . \tag{3.19}$$

The result of numerical evaluation of σ for various choices of n is shown in Fig. 4. We see that for large n, and x_0 not too large, σ is closely approximated by the spectral function σ_0, the latter being the $n\to\infty$ limit of σ. However, for finite n, $\sigma\to 0$ as $x_0\to\infty$, as was the case in Sec. II.

IV. DISCUSSION AND CONCLUSION

In the preceding sections we have seen that it is possible to assign a spectrum to the Casimir effect, but that it is quite different from the simple, monotonic spectrum suggested by Eq. (1.1). However, the appearance of a thermal-like spectrum in expressions for the energy density is not entirely coincidental. This may be seen by comparing the Green's function for a nonsimply connected space, such as $S^1\times R^3$ [Eq. (3.2)] with the thermal Green's function in Minkowski space. The latter is periodic in imaginary time with period $\beta=(kT)^{-1}$, and is hence

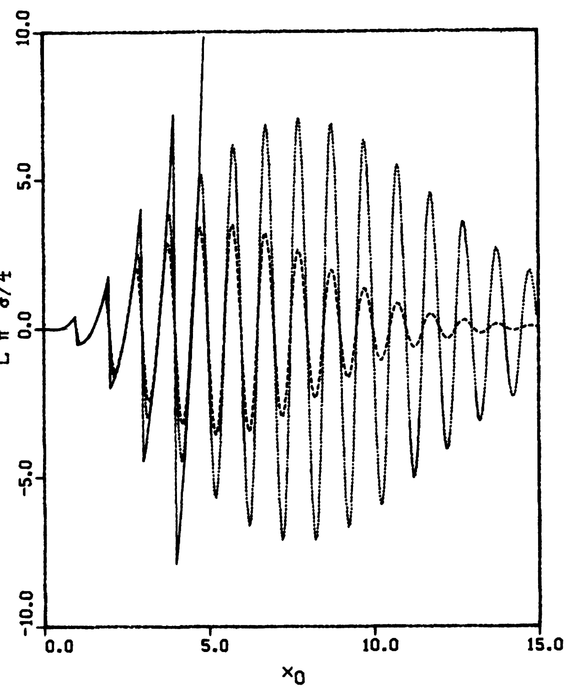

FIG. 4. The spectral function σ for a scalar field in $S^1 \times R^3$ for $n = 300$ (dashed curve) and $n = 600$ (dotted curve). The solid curve shows σ_0, the $n \to \infty$ limit of σ.

$$G_T = \sum_{n=-\infty}^{\infty} G_0(t + i\beta n, \mathbf{x}) , \tag{4.1}$$

where G_0 is the vacuum two-point function, Eq. (3.1). We see that the Euclidean thermal Green's function, $G_{ET}(\tau, \mathbf{x}) = G_T(-i\tau, \mathbf{x})$ is related to the Euclidean Green's function $G_E(\tau, \mathbf{x}) = G(-\tau, \mathbf{x})$ in $S^1 \times R^3$ by

$$G_{ET}(\tau, x, y, z) = G_E(z, x, y, \tau) . \tag{4.2}$$

This relation leads to the fact that the vacuum energy density in $S^1 \times R^3$, Eq. (3.4), is equal in magnitude (but opposite in sign) to the energy density of a thermal bath with $\beta = L$. Another consequence of this correspondence is that discussions of symmetry breaking for field theories in $S^1 \times R^3$ are completely analogous to those of symmetry breaking at finite temperature.[9,10] A simple example is the Goldstone model[11] of global-symmetry breaking based upon the classical potential

$$V_0 = -\tfrac{1}{2} m^2 \phi^2 + \tfrac{1}{12} \lambda \phi^4 , \tag{4.3}$$

which at zero temperature in Minkowski space has a stable minimum at $\phi = (3m^2/\lambda)^{1/2}$. The lowest-order quantum correction to this potential yields

$$V = V_0 + \tfrac{1}{2} \lambda \langle \phi^2 \rangle \phi^2 . \tag{4.4}$$

In a thermal state, $\langle \phi^2 \rangle = T^2/12$, so the symmetry is restored if $T \geq T_c = (12m^2/\lambda)^{1/2}$. Similarly, in $S^1 \times R^3$, $\langle \phi^2 \rangle = 1/(12L^2)$, so the symmetry is restored if $L \leq L_c = (12m^2/\lambda)^{-1/2}$. This correspondence also holds in higher orders of perturbation theory.

The main point of the previous two paragraphs has been to illustrate that there are limited circumstances in which quantum fluctuations in a system of finite spatial extent are similar to thermal fluctuations. Vacuum effects in $S^1 \times R^3$ and thermal effects in R^4 are equivalent only when one is dealing with a quantity (e.g., the magnitude of the energy density or $\langle \phi^2 \rangle$) which is not sensitive to the interchange of arguments displayed in Eq. (4.2). More generally, quantum fluctuations are quite different from thermal fluctuations. As we have seen, the spectral functions σ defined in Secs. II and III are quite different from the Planck spectrum. Not only is σ an oscillatory function, but in order to have σ vanish at high frequencies, one must accept a dependence upon $\Delta\omega$, the tolerance with which frequency measurements are made.

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