Holonomy transformation and Aharonov-Bohm effect in an Einstein-Maxwell space-time

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We obtain the exact expression for the holonomy transformation, for a circle, that corresponds to the most general cylindrically symmetric metric. In a particular case that corresponds to solutions of the combined Einstein-Maxwell field equations for this metric, a discussion about the gravitational Aharonov-Bohm effect is given.

The loop variable in the theory of gravity are matrices representing parallel transport along contours in a space-time with a given affine connection. This is connected with the holonomy transformation which is a mathematical object that contains information about how vectors change when parallel transported around a closed curve. Associated with the holonomy transformation we have a set of numbers (deficit angles), which give us the angular deviations experienced by the vectors when parallel transported, and these are associated with the asymptotically conical space-times under consideration.

In this paper we use the loop variable in the theory of gravity to obtain the exact expression for the holonomy transformation, for a circle, that corresponds to the most general cylindrically symmetric metric[1] and a particular case of it[2,3] showing that it is possible to write it in terms of the deficit angle and the generator of spatial rotation about the local z axis. We also determine the holonomy transformation for the following types of solutions[4] for the combined Einstein-Maxwell field with cylindrical symmetry:[5] solutions of type I, in which case we have an axial-vector current producing a magnetic field whose lines are circles in the plane perpendicular to the axis and centered about the axis; and solutions of type II, in which case we have an angular current producing a magnetic field parallel to the axis. For a particular model given by Safko and Witten[5] of a tubular matter source with axial interior magnetic field and vanishing exterior magnetic field, we examine the effect of the parallel transport of vectors around the closed path (circles, in this case), showing that these combined gravitational and electromagnetic fields provide an Aharonov-Bohm effect[6,7] at the purely classical level.

Let us define the loop variable by the path-ordered product[8]

$$U(C)=P\exp\left[\int_C \Gamma_\mu dx^\mu\right], \tag{1}$$

where P means ordered product along a curve C and Γ_μ is the tetradic connection. The mathematical object defined by Eq. (1) gives the geometrical phase acquired by a vector when it is parallel transported around a given curve C.

The most general cylindrically symmetric metric can be written in the form[1]

$$ds^2=e^{2\gamma-2\psi}(-dt^2+d\rho^2)+\rho^2e^{-2\psi}d\phi^2+e^{2\psi+2\mu}dz^2, \tag{2}$$

where γ, ψ, and μ are functions of ρ alone and $-\infty<t<\infty$, $0\le\rho<\infty$, $0\le\phi<2\pi$, and $-\infty<z<\infty$.

To consider the general line element for the most general cylindrically symmetric metric [Eq. (2)] and for type-I and -II solutions, all together, we shall write Eq. (2) as

$$ds^2=f^2(\rho)(-dt^2+d\rho^2)+g^2(\rho)d\phi^2+h^2(\rho)dz^2, \tag{3}$$

where f, g, and h are given by the obvious identification and for type-I and -II solutions they are given by

$$\begin{aligned}
f_{\rm I}(\rho)&=(\rho+\rho_0)^{c^2-c}[k_{i,e}+(\rho+\rho_0)^{2c}]e^{\gamma_0-\psi_0},\\
g_{\rm I}(\rho)&=(\rho+\rho_0)^{1-c}[k_{i,e}+(\rho+\rho_0)^{2c}]e^{-\psi_0},\\
h_{\rm I}(\rho)&=(\rho+\rho_0)^{c}[k_{i,e}+(\rho+\rho_0)^{2c}]^{-1}e^{\psi_0+\mu_0},\\
f_{\rm II}(\rho)&=(\rho+\rho_0)^{\delta+\delta^2}[1+k_{i,e}(\rho+\rho_0)^{2+2\delta}]e^{\gamma_0-\psi_0},\\
g_{\rm II}(\rho)&=(\rho+\rho_0)^{\delta}[1+k_{i,e}(\rho+\rho_0)^{2+2\delta}]^{-1}e^{-\psi_0},
\end{aligned} \tag{4}$$

and

$$h_{\rm II}(\rho)=(\rho+\rho_0)^{-\delta}[1+k_{i,e}(\rho+\rho_0)^{2+2\delta}]e^{\psi_0+\mu_0},$$

where ρ_0, ψ_0, γ_0, and μ_0 are constants, c, δ, and $k_{i,e}$ are non-negative; c and δ are related to the mass per unit length of the source and $k_{i,e}$ to the magnetic field strength (subscript i is related to the internal magnetic field and e to the external magnetic field). The subscripts I and II correspond to type-I and -II solutions, respectively.

In order to compute the tetradic connections, we start by defining the one-forms θ^A ($A=1,2,3,4$) as

$$\begin{aligned}
&\theta^1=f(\rho)d\rho, \quad \theta^2=g(\rho)d\phi,\\
&\theta^3=h(\rho)dz, \quad \theta^4=f(\rho)dt.
\end{aligned} \tag{5}$$

The geometry given by Eq. (3) is obtained by the expression $ds^2=\eta_{AB}\theta^A\theta^B$, where η_{AB} is the Minkowski tensor diag$(+,+,+,-)$.

Defining a tetrad frame by $\theta^A=e_\alpha^{(A)}dx^\alpha$, in the coordinate system ($x^1=\rho$, $x^2=\phi$, $x^3=z$, and $x^4=t$) the $e_\alpha^{(A)}$ vectors are given by $e_4^{(4)}=f(\rho)$, $e_1^{(1)}=f(\rho)$, $e_2^{(2)}=g(\rho)$, and $e_3^{(3)}=h(\rho)$.

From the Cartan structure equations $d\theta^A = -\omega^A_B \Lambda \theta^B$, a straightforward calculation gives us the following non-null $\Gamma^A_{\mu B} dx^\mu = \omega^A_B$ (A,B tetradic indices):

$$\Gamma^4_{\mu 1} dx^\mu = \frac{1}{f}\frac{df}{d\rho} dt = \Gamma^1_{\mu 4} dx^\mu ,$$
$$\Gamma^2_{\mu 1} dx^\mu = \frac{1}{f}\frac{dg}{d\rho} d\phi = -\Gamma^1_{\mu 2} dx^\mu , \tag{6}$$

and

$$\Gamma^3_{\mu 1} dx^\mu = \frac{1}{f}\frac{dh}{d\rho} dz = -\Gamma^1_{\mu 3} dx^\mu .$$

We shall consider circles with the center at the origin and fixed values of t, ρ, and z. So, in this case,

$$\Gamma^A_{\mu B} dx^\mu = \Gamma_\phi d\phi , \tag{7}$$

where

$$\Gamma_\phi = i\frac{1}{f}\frac{dg}{d\rho}\begin{bmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} . \tag{8}$$

As Γ_ϕ is independent of ϕ, then, for a circle,

$$U(C) = P\exp\left[\int_0^{2\pi} \Gamma_\phi d\phi\right] = e^{2\pi\Gamma_\phi} . \tag{9}$$

Putting Eq. (8) into Eq. (9) we obtain the following expression for the holonomy transformation:

$$U(C) = \exp\left[2\pi i\left[\frac{1}{f}\frac{dg}{d\rho}\right] J\right] , \tag{10}$$

where J is one of the generators of the SO(3,1) group that corresponds to rotations about the local z axis and is given by

$$J = \begin{bmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} .$$

If we consider a tetrad of fixed orientation and take into account that part of the rotation given by Eq. (10) is due to the rotation of the local tetrad frame, we can write Eq. (10) as

$$U(C) = \exp\left[2\pi i\left[\frac{1}{f}\frac{dg}{d\rho} - 1\right] J\right] , \tag{11}$$

where the factor that we have introduced $\exp(-2\pi i J)$ is equal to $\mathbb{1}$ (4×4 identity matrix).

Equation (11) is the simple and exact expression for the holonomy transformation that corresponds to the class of solutions under consideration.

If we use the prescription[8] to calculate the deficit angles associated with the holonomy transformation, we obtain for the two non-null and equal angular deviations the expression

$$\alpha(\rho) = 2\pi\left[\frac{1}{f}\frac{dg}{d\rho} - 1\right] \tag{12}$$

for a chosen orientation. Therefore, $U(C)$ can be written as exp($i\times$deficit angle$\times$generator of rotation about the local z axis), or explicitly as

$$U(C) = \exp\begin{bmatrix} 0 & -\alpha(\rho) & 0 & 0 \\ \alpha(\rho) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \cos\alpha(\rho) & -\sin\alpha(\rho) & 0 & 0 \\ \sin\alpha(\rho) & \cos\alpha(\rho) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} ,$$

which describes a rotation through

$$\alpha(\rho) = 2\pi\left[\frac{1}{f}\frac{dg}{d\rho} - 1\right]$$

about the local z axis. So, in this case, the holonomy transformation corresponds to this rotation.

Putting expressions for f and g corresponding to the general metric with cylindrical symmetry and to type-I and -II solutions into Eqs. (11) and (12) we obtain the expressions of the holonomy transformation and the deficit angle for each of these cases.

In particular, for the general metric cylindrically symmetric [Eq. (2)] we have

$$U(C) = \exp\left\{2\pi i\left[e^{-\gamma}\left[1 - \rho\frac{d\psi}{d\rho}\right] - 1\right] J\right\} . \tag{13}$$

Using the expression for the deficit angle[8] α associated with this axially symmetric asymptotically conical spacetime, we can write $U(C)$ as

$$U(C) = e^{i\alpha J} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} , \tag{14}$$

where

$$\alpha = 2\pi\left[e^{-\gamma}\left[1 - \rho\frac{d\psi}{d\rho}\right] - 1\right]$$

is the deficit angle. Equation (14) is the exact expression for the holonomy transformation for a circle that corresponds to the most general cylindrically symmetric metric. As we can see it is written as exp($i\times$deficit angle$\times$generator of rotation about the local z axis), and therefore describes a rotation through an angle α about the local z axis. If we consider the particular case given by[2]

$$ds^2 = dt^2 - \rho^2 d\phi^2 - A^2(d\rho^2 + dz^2) \tag{15}$$

we obtain for the holonomy transformation the same expression given by Eq. (11) where the deficit angle is $\alpha=2\pi(1/A-1)$. This result was also obtained by Dowker.[9]

The string solution given by Vilenkin[3] is the metric of Eq. (15). In this case the deficit angle α satisfies the relation $\alpha=8\pi\mu$, where μ is the linear mass density of the string.

Consider now a model given by Safko and Witten[4] of a tubular matter source, with interior axial magnetic field and a circular exterior magnetic field. The exterior line element is given by[5]

$$ds^2=(\rho+\rho_0)^{2c^2-2c}[k_e+(\rho+\rho_0)^{2c}]^2 e^{2a}(-dt^2+d\rho^2)+(\rho+\rho_0)^{2-2c}[k_e+(\rho+\rho_0)^{2c}]^2 d\phi^2+(\rho+\rho_0)^{2c}[k_e+(\rho+\rho_0)^{2c}]^{-2}dz^2 , \tag{16}$$

where k_e is the parameter concerned with the exterior magnetic field.

Identifying the functions $f(\rho)$ and $g(\rho)$ in Eq. (16), we obtain

$$\alpha(\rho)=2\pi\left[\left[(1-c)(\rho+\rho_0)^{-c^2}+\frac{2c(\rho+\rho_0)^{-c^2+2c}}{k_e+(\rho+\rho_0)^{2c}}\right]e^{-a}-1\right] . \tag{17}$$

Consider the special case $c=k_e=0$. In this case, the metric (16) takes the form

$$ds^2=e^{2a}(-dt^2+d\rho^2)+(\rho+\rho_0)^2 d\phi^2+dz^2 \tag{18}$$

which is everywhere locally flat, as we can see by an appropriate changing of variables. The cross section $z=$const is topologically equivalent to a cone. In this Universe, any observer outside the tube of matter would see a flat space. However, if we transport a vector around a closed curve in the flat region, it acquires a phase given by

$$U(C)=\exp\{[2\pi i(e^{-a}-1)]J\} \tag{19}$$

with the two non-null and equal deficit angles given by

$$\alpha=2\pi(e^{-a}-1) , \tag{20}$$

where a is given by[5]

$$a=\ln\left[\frac{(1+k_i\rho_1^2)^2}{\rho_2}\right]+\frac{4k_i\rho_1}{1+k_i\rho_1^2}\,\frac{\rho_2-\rho_1}{\bar{q}+1} \tag{21}$$

and depends on the interior magnetic field and on the mass. The quantities ρ_1,ρ_2 are the interior and exterior radius of the tube of matter and $\bar{q}$ is an arbitrary constant.

From Eq. (20) we conclude that there will be no Aharonov-Bohm effect if and only if e^{-a} is an odd integer. However, this condition is not always satisfied because e^{-a} is not necessarily an odd integer. Then, we have shown that if we parallel transport a vector around a closed path (circle, in our case) lying in the flat region, the transported one does not necessarily coincide with the original. Therefore, the parallel transported vector, in a region in which the curvature vanishes, exhibits physical effects arising from the region of nonzero curvature associated with the axial interior magnetic field and the tubular matter source. This is an example of an analogue of the Aharonov-Bohm effect that these combined fields (gravitational and electromagnetic) provide. This effect should be regarded as basically classical associated with the nontriviality of the holonomy transformation due to the tubular matter source and the magnetic field confined to this tube. As in the present case the geometry is locally flat, the phase shift acquired by a particle when parallel transported around the source may be regarded as due to the coupling of the energy-momentum of the particle to the global geometrical properties[10] of this Universe.

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