# Gravitational Faraday rotation induced by a Kerr black hole

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We study the gravitational lens effect caused by a Kerr black hole via the propagation of a polarization vector along a light ray. In particular, we find a rotation of the plane of polarization due to the presence of the black hole's spin. The rotation angle is proportional to the mass and the line-of-sight component of the angular momentum of the black hole. This is a general-relativistic effect analogous to the well-known Faraday rotation.

## I. INTRODUCTION

General relativity predicts that the gravitational field of a massive object can bend a ray of electromagnetic radiation. Some twin quasars are thought to be two (or more than two) images of a single quasar formed by a gravitational lens intervening between the quasar and us.[1] Some effects due to the gravitational lens were investigated by many authors, and most of the works done so far were mainly concerned with the problems such as deflection of light rays, amplification of intensity, and differential time delay. Recently, Bray[2] studied how the angular momentum of a Kerr black hole, as a gravitational lens, contributes to these effects.

In this paper we direct our attention to another effect and study how the polarization vector of a linearly polarized electromagnetic wave propagates in a curved spacetime, i.e., in the presence of a gravitational lens.

Stark and co-workers[3,4] investigated general-relativistic effects on a property of the polarization vector of an x ray emitted from an accretion disk surrounding a Kerr black hole; they first pointed out the fact that the polarization vector rotates in the gravitational field of the black hole. This change in the polarization direction of each light ray can reduce the total degree of polarization.

We here consider a different situation such that both the source of radiation and the observer are remote from the gravitational lens. For this simple situation it is easy to extract purely general-relativistic effects on the propagation of the polarization vector.

Some decades ago Plebansky[5] shed light on this problem by solving Maxwell's equations in a curved spacetime. The propagation of the polarization vector obeys the parallel transport along a null geodesic. The polarization vector changes its direction as a result of the deflection of the light ray. In addition to this change, a rotation of the polarization vector around the propagation vector may occur. We call it the rotation of the plane of polarization. Such two changes are mixed in the works by Connors, and co-workers.[4] By solving the equation for the parallel transport of the polarization vector derived from the Maxwell's equations in a curved spacetime, Plebansky concluded that the rotation of the plane of polarization occurs only when the ray penetrates into rotating matter. The rotation of the plane of polarization induced by the angular momentum of matter is analogous to the well-known Faraday effect which appears in the magnetized plasma and so it may be called the "gravitational Faraday rotation."[6]

The close analogy between electromagnetic and gravitational effects is an interesting problem which has been discussed from various points of view. For example, a gravitational analogue of the Aharonov-Bohm effect produced by a cosmic-string solution was extensively studied.[7] The gravitational Faraday rotation, different from the deflection of light ray, is another important phenomenon to understand how a non-Newtonian or "magnetic" effect of the Weyl conformal curvature can play the role of a magnetic field.

Does a rotating (Kerr) black hole as the gravitational lens generate this magnetic effect? Any light ray received by a distant observer must pass through a vacuum region outside the event horizon. Then Plebansky's calculation should fail to extract the gravitational Faraday rotation on the propagation of the polarization vector. However, this is due to his rough approximation in the weak-field limit of the metric tensor. In this paper we also assume a small deflection of the path of the ray, but we include the higher-order terms in the weak-field approximation to study this point in more detail.

Since the spacetime we consider here (the Kerr metric) has a symmetry of Petrov type D, there exists a (complex) conserved quantity along a null geodesics, which was discovered by Walker and Penrose.[8] As was done in Ref. 4, one can deal with the parallel transport of the polarization vector in terms of the Walker-Penrose constant. Then the problem becomes quite simple because the equation for the parallel transport reduces to an algebraic equation which requires us to give only the propagation vector of a light ray.

In Sec. II we apply the procedure based on the Walker-Penrose constant to the parallel transport along a null geodesic in a gravitational lens. We calculate the difference between the initial and final polarization direction of a light ray emitted from the source and received by the observer. Our main conclusion obtained in Sec. III is that the gravitational Faraday rotation is really induced by the presence of black-hole spin. The amplitude of the rotation angle is evaluated. Brief discussions are contained in Sec. IV. Necessary integration of the equation of motion for a light ray in the Kerr metric is given in the Appendix.

## II. PARALLEL TRANSPORT OF POLARIZATION VECTOR

We consider the Kerr geometry described by the Boyer-Lindquist coordinates

$$ds^2=\frac{\rho^2}{\Delta}dr^2+\rho^2 d\theta^2+\frac{\sin^2\theta}{\rho^2}[a\,dt-(r^2+a^2)d\phi]^2 - \frac{\Delta}{\rho^2}(dt-a\sin^2\theta\, d\phi)^2 , \tag{2.1}$$

where

$$\rho^2=r^2+a^2\cos^2\theta$$

and

$$\Delta=r^2-2mr+a^2 .$$

The parameters $m$ and $a$ ($m \geq a$) mean the mass and the angular momentum per unit mass of a Kerr black hole. The polarization four-vector $f^i$ of a linearly polarized light ray, which is orthogonal to the direction of propagation, must be parallel transported along a null geodesic with its tangent vector $k^i$. We obtain the relation between the propagation vector at the source and at the observer by the technique used in the works by Connors and co-workers.[4]

Since the Kerr geometry is of Petrov type D, the complex quantity

$$K_{\rm WP}=(A+iB)(r-ia\cos\theta) \tag{2.2}$$

is conserved along a null geodesic,[8] where

$$\begin{aligned} A&=(k^t f^r-k^r f^t)+a\sin^2\theta(k^r f^\phi-k^\phi f^r) , \\ B&=(r^2+a^2)\sin\theta(k^\phi f^\theta-k^\theta f^\phi)-a\sin\theta(k^t f^\theta-k^\theta f^t) . \end{aligned} \tag{2.3}$$

Since $f^i$ is determined up to a multiple of the null vector $k^i$, we can always set $f^t=0$ without loss of generality.

We denote the positions of the source and of the observer in the three-dimensional space by $(r_s,\theta_s,\phi_s)$ and $(r_o,\theta_o,\phi_o)$, respectively (see Fig. 1). We consider a situation that the source and the observer are remote enough from the black hole, i.e.,

$$r_{\min}/r_o \ll 1, \quad r_{\min}/r_s \ll 1 , \tag{2.4}$$

where $r_{\min}$ is the distance of closest approach of the path of ray. From the equation of motion for a light ray the asymptotic behavior of $k^i$ near the position of the source or of the observer is given by

$$\begin{aligned} &k^t \to 1, \quad k^r \to k^r/|k^r| , \\ &k^\theta \to \beta/r^2, \quad k^\phi \to \lambda/(r^2\sin^2\theta) , \end{aligned} \tag{2.5}$$

where

$$\beta=(\eta-\lambda^2\cot^2\theta+a^2\cos^2\theta)^{1/2}k^\theta/|k^\theta| \tag{2.6}$$

and the parameters $\lambda$ and $\eta$ are constants of motion. If

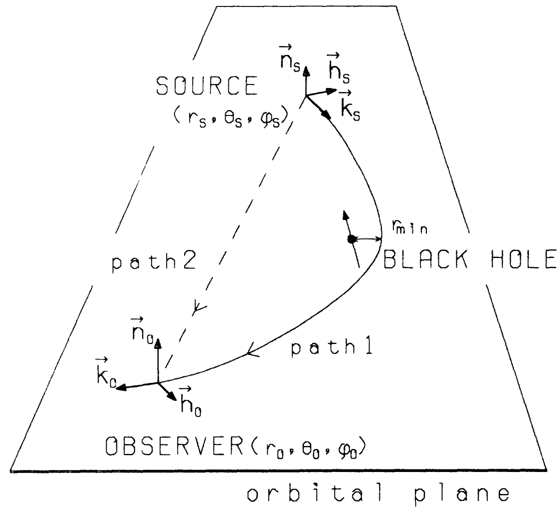


FIG. 1. A schematic diagram of the paths of the light ray from the source $(r_s,\theta_s,\phi_s)$ to the observer $(r_o,\theta_o,\phi_o)$. The solid curve (path 1) denotes a path deflected by the gravitational-lens effect. The dashed line (path 2) is far from the Kerr black hole. The orbital plane is drawn as a plane including the source, the black hole and the observer. The three-dimensional vector **n** is normal to this orbital plane, and $\mathbf{h}=\mathbf{n}\times\mathbf{k}/|\mathbf{k}|$ where **k** is the propagation vector of path 1.

$f^r$ is eliminated by the use of the orthogonal condition $k^i f_i=0$, the real and imaginary parts of Eq. (2.2) become

$$K_1=\gamma\hat{f}^\theta-\beta\hat{f}^\phi, \quad K_2=-(\beta\hat{f}^\theta+\gamma\hat{f}^\phi)k^r/|k^r| , \tag{2.7}$$

where

$$\gamma=\lambda\csc\theta-a\sin\theta \tag{2.8}$$

and

$$\hat{f}^r\equiv f^r, \quad \hat{f}^\theta\equiv rf^\theta, \quad \hat{f}^\phi\equiv r\sin\theta f^\phi .$$

Equations (2.7) hold both at the source and at the observer. Paying attention to the fact that

$$\beta_s^2+\gamma_s^2=\beta_o^2+\gamma_o^2=\eta+(\lambda-a)^2 \tag{2.9}$$

is a constant of motion, we obtain the matrix $R$ transforming the $\theta$ and $\phi$ components of the polarization vector at the source into the ones at the observer as follows:

$$\begin{bmatrix}\hat{f}^\theta\\ \hat{f}^\phi\end{bmatrix}_o=R\begin{bmatrix}\hat{f}^\theta\\ \hat{f}^\phi\end{bmatrix}_s , \tag{2.10}$$

where

$$R=(1+x^2)^{-1/2}\begin{bmatrix}1 & -x\\ -x & -1\end{bmatrix} . \tag{2.11}$$

The parameter $x$ is defined by

$$x\equiv\frac{\beta_s\gamma_o+\gamma_s\beta_o}{\gamma_s\gamma_o-\beta_s\beta_o} . \tag{2.12}$$

Hereafter, we always understand that the value with the subscript $s$ or $o$ is evaluated at the position of the source or of the observer.

On the assumption of Eq. (2.4), we can ignore the radial component of the linear polarization vector at both the source and at the observer. Then the matrix $R$ describes the full change in the polarization vector. The parameter $x$ contains the constants of motion $\lambda,\eta$, the position angles $(\theta_s,\phi_s),(\theta_o,\phi_o)$, and the spin parameter $a$. If we obtain a null geodesic joining the source and the observer (see the Appendix), we can evaluate the amplitude of $x$ which describes the change in the polarization vector.

## III. ROTATION OF THE PLANE OF POLARIZATION

Let us call a plane which includes the source, the Kerr black hole, and the observer orbital plane (see Fig. 1). From the assumption Eq. (2.4), the orbit should be placed on the plane in the regions near the source and near the observer. We introduce three-dimensional unit vectors $\mathbf{n}$ which is normal to the orbital plane and $\mathbf{h}$ defined by $\mathbf{h}=\mathbf{n}\times\mathbf{k}/|\mathbf{k}|$. Here $\mathbf{k}$ is a three-dimensional propagation vector whose components are $(k^r,rk^\theta,r\sin\theta k^\phi)$. In the asymptotic region the components $k^r$, $k^\theta$, $k^\phi$ behave as Eq. (2.5). The symbol $\times$ means the usual vector product in three dimensions. Since the positions of the source and of the observer are far from the black hole, the Boyer-Lindquist coordinate system given by Eq. (2.1) reduces there to the usual spherical coordinate system in the flat spacetime. Hereafter, the carets are dropped. Thus, at the distant places from the black hole any linear polarization vector $\mathbf{f}$ can be expressed as

$$\mathbf{f}=f_\perp\mathbf{n}+f_\parallel\mathbf{h}\ . \tag{3.1}$$

The component $f_\perp$ perpendicular to and $f_\parallel$ projected onto the orbital plane are given by

$$\begin{bmatrix} f_\parallel \\ f_\perp \end{bmatrix}=N\begin{bmatrix} f^\theta \\ f^\phi \end{bmatrix}, \tag{3.2}$$

where $N$ is a matrix of the form

$$N=\begin{bmatrix} h^\theta & h^\phi \\ n^\theta & n^\phi \end{bmatrix}. \tag{3.3}$$

The deflection of the light ray must change the direction of $\mathbf{h}$ in the orbital plane [i.e., $\mathbf{h}_s-\mathbf{h}_o=\mathbf{n}\times(\mathbf{k}_s-\mathbf{k}_o)/|\mathbf{k}|$]. From Eq. (3.1) this leads to a trivial change (a typical effect of the "electric" portion of the Weyl curvature) in the polarization vector. Our main concern is the change in the components $f_\perp$ and $f_\parallel$ since it corresponds to the rotation of the plane of polarization, i.e., the gravitational Faraday rotation which is a magnetic effect due to the presence of the angular momentum of the black hole. In this section we will estimate the rotation angle.

$$\begin{bmatrix} f_\parallel \\ f_\perp \end{bmatrix}_o=N_o\begin{bmatrix} f^\theta \\ f^\phi \end{bmatrix}_o=N_oR\begin{bmatrix} f^\theta \\ f^\phi \end{bmatrix}_s =N_oRN_s^{-1}\begin{bmatrix} f_\parallel \\ f_\perp \end{bmatrix}_s. \tag{3.4}$$

This implies that the rotation of the plane of polarization is determined by the matrix $N_oRN_s^{-1}$. At the positions of the source and of the observer, the vector $\mathbf{h}$ becomes

$$\begin{bmatrix} h^\theta \\ h^\phi \end{bmatrix}_o=\begin{bmatrix} n^\phi \\ -n^\theta \end{bmatrix}_o, \quad \begin{bmatrix} h^\theta \\ h^\phi \end{bmatrix}_s=\begin{bmatrix} -n^\phi \\ n^\theta \end{bmatrix}_s.$$

Hence, from Eqs. (2.11) and (3.3) we obtain the rotation matrix of the form

$$N_oRN_s^{-1}=\frac{n_s^\theta n_o^\theta}{(1+x^2)^{1/2}}\begin{bmatrix} u_o & -1 \\ 1 & u_o \end{bmatrix}\begin{bmatrix} 1 & -x \\ -x & -1 \end{bmatrix}\times\begin{bmatrix} -u_s & 1 \\ 1 & u_s \end{bmatrix}, \tag{3.5}$$

where $u_s\equiv n_s^\phi/n_s^\theta$ and $u_o\equiv n_o^\phi/n_o^\theta$. Since the unit vector $\mathbf{n}$ has no radial component, $n^\theta$ becomes

$$n_o^\theta=\pm(1+u_o^2)^{-1/2},\quad n_s^\theta=\pm(1+u_s^2)^{-1/2} \tag{3.6}$$

($n_o^\theta n_s^\theta>0$ when the spin vector of Kerr black hole is not just in the orbital plane and we consider this case), and then the rotation matrix reduces to

$$N_oRN_s^{-1}=\begin{bmatrix} \cos\chi & -\sin\chi \\ \sin\chi & \cos\chi \end{bmatrix}, \tag{3.7}$$

where $\chi$ is the rotation angle given by the equation

$$\sin\chi=(1+X^2)^{-1/2}(1+x^2)^{-1/2}(X-x)\ , \tag{3.8}$$

and the parameter $X$ is defined by

$$X=-\frac{u_s+u_o}{1-u_su_o}\ . \tag{3.9}$$

If the difference $\delta x=x-X$ is small, we can see the rotation angle $\chi$ turns out to be

$$\chi=-\frac{\delta x}{1+X^2} \tag{3.10}$$

(see Fig. 2), and when $x=X$ no rotation of the plane of

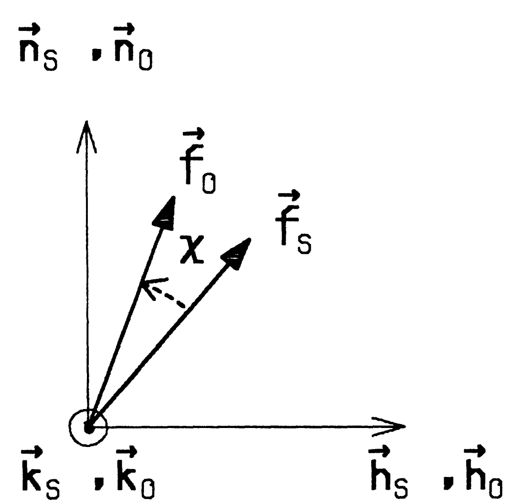


FIG. 2. The rotation of the polarization plane which is a gravitational analogue of the Faraday rotation. The change in the polarization vector $\mathbf{f}$ is shown by matching the direction of the propagation vectors $\mathbf{k}_s$ and $\mathbf{k}_o$ at the source and the observer. The rotation angle $\chi$ is equal to $\frac{5}{4}\pi m^2a\cos\theta_o/r_{\min}^3$.

polarization occurs.

Now we want to calculate the value of $\chi$ and verify the existence of the gravitational Faraday rotation induced by a Kerr black hole. For simplicity we assume that a light ray propagates in the weak gravitational field, i.e.,

$$m/r_{\min},\quad a/r_{\min} \ll 1 \ . \tag{3.11}$$

This means that the bending angles defined by

$$\Delta\theta \equiv \theta_o + \theta_s - \pi,\quad \Delta\phi \equiv \phi_o - \phi_s - \pi \ , \tag{3.12}$$

are small, where we chose $\phi_o > \phi_s$. The further assumption we adopt here is as follows:

$$r_{\min}/r_o, r_{\min}/r_s \ll m/r_{\min}, a/r_{\min} \ . \tag{3.13}$$

We take the small quantities $m/r_{\min}$ and $a/r_{\min}$ into account but ignore $r_{\min}/r_o$ and $r_{\min}/r_s$. In this case there exists another null geodesic joining the source and the observer with a negligibly small deflection (path 2 in Fig. 1). This path of the ray will feel no effective gravitational force, i.e., no gravitational Faraday rotation. We consider only the path of the ray deflected fully by the black hole.

Using these approximations, we express the rotation angle $\chi$ in terms of the position angles and the Kerr parameters $m$ and $a$. Since the vector **n** is orthogonal to the orbital plane, $u_s$ and $u_o$ have the forms

$$\begin{aligned} u_s &= \sin\theta_s[-\cot\theta_s\cos(\phi_o-\phi_s)+\cot\theta_o]/\sin(\phi_o-\phi_s) \ , \\ u_o &= \sin\theta_o[\cot\theta_o\cos(\phi_o-\phi_s)-\cot\theta_s]/\sin(\phi_o-\phi_s) \ . \end{aligned} \tag{3.14}$$

Using Eq. (3.12) and substituting Eq. (3.14) into Eq. (3.9), we get the expansion of $X$ up to the third-order terms in $\Delta\theta$ and $\Delta\phi$:

$$X = \Delta\phi\cos\theta_o + \tfrac{1}{2}\Delta\theta\Delta\phi\sin\theta_o + \tfrac{1}{12}(1+3\cos^2\theta_o)\cos\theta_o\Delta\phi^3 \ . \tag{3.15}$$

We note that $X$ depends on only the bending angles $\Delta\theta, \Delta\phi$ and the position angle $\theta_o$.

Our tedious task is to obtain $x$ in Eq. (2.12) for a light ray in the gravitational field of the Kerr black hole. We assume that along the light ray there exists only one minimum (or maximum) value of $\theta$ [see Eq. (A3)]. Substituting Eqs. (2.6) and (2.8) into Eq. (2.12) and using Eq. (3.12), we obtain the following expansion up to the third-order terms in $m/r_{\min}$ and $a/r_{\min}$:

$$\begin{aligned} x = \pm\{ & \mu^{-1}(\lambda/r^{(0)}_{\min})\cot\theta_o\Delta\theta + \tfrac{1}{2}\mu^{-3}(\lambda/r^{(0)}_{\min})[1+\cos^2\theta_o-(\lambda/r^{(0)}_{\min})^2\csc^2\theta_0]\Delta\theta^2 \\ & +\mu^{-1}\sin\theta_o\cos\theta_o(a/r^{(0)}_{\min})\Delta\theta + \tfrac{1}{6}(\lambda/r^{(0)}_{\min})(2\mu^{-1}+3\mu^{-3}+3\mu^{-5}\cos^2\theta_o\sin^2\theta_o)\cot\theta_o\Delta\theta^3 \\ & +\tfrac{1}{2}[\mu^{-1}\sin^2\theta_o+\mu^{-3}(\lambda/r^{(0)}_{\min})^2\cos^2\theta_o](a/r^{(0)}_{\min})\Delta\theta^2 - \tfrac{1}{2}\mu^{-3}(\lambda/r^{(0)}_{\min})\sin\theta_o\cos^3\theta_o(a/r^{(0)}_{\min})^2\Delta\theta\} \ , \end{aligned} \tag{3.16}$$

where $\mu$ is defined by Eq. (A14) and $r^{(0)}_{\min}$ is related to $r_{\min}$ by Eq. (A7). The upper sign corresponds to the ray which passes through $\theta_{\min}$ and the lower corresponds to $\theta_{\max}$. We can regard, from Eqs. (A15) and (A17), the constants of motion $\lambda$ and $\eta$ as the functions of $\Delta\theta$ and $\Delta\phi$. Thus, eliminating $\lambda$ and $\eta$ from Eq. (3.16) we rewrite $x$ as a function of $\theta_o, \phi_o$, and $\Delta\theta, \Delta\phi$ as follows:

$$\begin{aligned} x &= \Delta\phi\cos\theta_o + \tfrac{1}{2}\Delta\theta\Delta\phi\sin\theta_o + \tfrac{1}{12}(1+3\cos^2\theta_o)\cos\theta_o\Delta\phi^3 \\ &\quad -\pi\tfrac{5}{4}m^2 a\cos\theta_o/r^3_{\min} \ . \end{aligned} \tag{3.17}$$

The difference between $x$ and $X$ turns out to emerge only in the third-order term. Then from Eq. (3.10) we arrive at the final result:

$$\chi = \pi\tfrac{5}{4}m^2 a\cos\theta_o/r^3_{\min} = \pi\tfrac{5}{4}mJ\cos\theta_o/r^3_{\min} \ . \tag{3.18}$$

We can see that the rotation angle of the plane of polarization around the propagation vector is proportional to the line-of-sight component of the angular momentum of the black hole.

## IV. DISCUSSIONS

As we have seen in Sec. III, the angular momentum of a Kerr black hole affects not only the deflection of the null geodesic[2] but also the rotation of the plane of polarization. It gives rise to only the higher correction for deflection angle, but it appears in the leading term of the rotation angle of the plane of polarization. The rotation angle is proportional to its line-of-sight component of the angular momentum. Our calculation for the null geodesic is relevant only when along the path $\theta$ has only one minimum (or maximum) value. Thus, the case $\theta_o \sim 0$ or $\pi$ is excluded. However, since Eq. (3.18) has no singular behavior for all values of $\theta_o$, we can expect the extension of the result obtained here to $\theta_o = 0$ or $\pi$ in which the maximum rotation angle of the plane of polarization is attained.

By observing the rotation of the plane of polarization we can determine the line-of-sight component of the angular momentum of the Kerr black hole. This fact is very similar to the usual Faraday effect, if the angular momentum is replaced by the magnetic field. This similarity makes us justify the fact that the rotation of the plane of polarization due to the angular momentum of the Kerr black hole is just the "gravitational Faraday rotation."

Plebansky has claimed that the rotation of the plane of polarization does not occur when the light ray passes through the vacuum region outside the rotating matter. However, we could point out that it emerges in the weak-field approximation including up to the third order $m^2 a/r^3_{\min}$ which was ignored in Ref. 5.

The symmetry of the Kerr metric (type-D space) allows us the simple treatment, i.e., the method based on the use of the Walker-Penrose constant to study the propagation of the polarization vector. In more general curved spacetime we must solve directly the equation of parallel transport. In our result, however, any property of the black hole such as the existence of an event horizon plays no essential role at all. The rotation of the plane of polarization will occur, in general, when the light ray passes through the gravitational field of any rotating massive object, but the amplitude of the rotation angle will be quantitatively modified from the value of Eq. (3.18).

Our calculation done in Sec. II is based on the assumption that $r_{\min}/r_o$ and $r_{\min}/r_s$ are negligibly small. If it is not the case, the gravitational Faraday rotation would appear in the second-order terms in $a/r_{\min}$ and $m/r_{\min}$ multiplied by the factors $r_{\min}/r_o$ and/or $r_{\min}/r_s$. In a probable black-hole model of the gravitational lens the assumption (3.13) will not be always valid. The contribution of these terms should be investigated in more detail.

There is a serious difference between the magnetic and gravitational Faraday rotations. In the latter effect, the rotation angle has no frequency dependence. Thus, we can observe the effect only by comparison between the planes of polarization of two images produced by gravitational lens. For the situation that we considered in the previous section, we observe a path of the ray with no gravitational influence (path 2) and during the propagation along this path the plane of polarization of a light ray emitted from the source will be preserved. Thus, if we assume the polarization angles (the ratios $f_\perp/f_\parallel$) are the same for two light rays at the source, the difference between the planes of polarization of two images becomes identical with the rotation angle predicted by Eq. (3.18).

However, two light rays which reach the observer are emitted in the slightly different directions at the source, and it is plausible that the polarization properties at the source will depend on the direction of emission. Furthermore, the intrinsic polarization will change during the time delay between two light rays. The rotation angle of the plane of polarization by the gravitational Faraday rotation is at most of the order (deflection angle)$^3 \sim (m/r_{\min})^3$, then the gravitational Faraday rotation is contaminated by such effects and the detection of it is very difficult from the observational viewpoint.

From a viewpoint of general relativity, however, it is interesting to study how large the rotation angle of the plane of polarization is allowed when the light ray experiences a strong gravitational field near the event horizon. An approach without the weak-field approximation is necessary for solving the problem. This point is now under consideration.

## ACKNOWLEDGMENTS

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## APPENDIX: DEFLECTION OF THE LIGHT RAY BY KERR BLACK HOLE

The spatial orbit of a null geodesic joining two positions $(r_s,\theta_s,\phi_s)$ and $(r_o,\theta_o,\phi_o)$ is determined by the equations

$$\int^r \frac{|dr|}{[R(r)]^{1/2}} = \int^\theta \frac{|d\theta|}{[\Theta(\theta)]^{1/2}} , \tag{A1}$$

$$\phi_o-\phi_s = \int^r \frac{r^2+2mr(a-\lambda)}{\Delta[R(r)]^{1/2}}\,|dr| + \int^\theta \frac{\lambda\cot^2\theta}{[\Theta(\theta)]^{1/2}}\,|d\theta| , \tag{A2}$$

where integrations are performed along a path

$$r_s \to r_{\min} \to r_o, \quad \theta_s \to \theta_{\min/\max} \to \theta_o , \tag{A3}$$

and

$$R(r) = r[r(r^2+a^2)+2a^2m] - 4amr\lambda - (r^2-2mr)\lambda^2 - \Delta\eta , \tag{A4}$$

$$\Theta(\theta) = \eta + a^2\cos^2\theta - \lambda^2\cot^2\theta . \tag{A5}$$

We evaluate these integrals up to and including the third-order terms in $m/r_{\min}$ and $a/r_{\min}$. As shown in Sec. III, the third-order terms are essential to the presence of the gravitational Faraday rotation of the polarization vector. In Ref. 2 the geodesic integral was evaluated up to the second-order inclusive, and now we improve it.

First let us integrate the left-hand side of Eq. (A1). If we ignore the terms of order $(r_{\min}/r_s)^2$ and $(r_{\min}/r_o)^2$, it reduces to

$$\int^r \frac{|dr|}{[R(r)]^{1/2}} \sim 2\int_{r_{\min}}^\infty \frac{dr}{[R(r)]^{1/2}} - \frac{r_s+r_o}{r_s r_o} , \tag{A6}$$

where $r_{\min}$ is the largest root of $R(r)=0$. For small deflection, the leading term in $r_{\min}$ is $r_{\min}^{(0)} \equiv (\lambda^2+\eta)^{1/2}$. In order to integrate Eq. (A6) with sufficient accuracy, we must obtain $r_{\min}$ up to and including the second-order terms in $m/r_{\min}^{(0)}$ and $a/r_{\min}^{(0)}$:

$$r_{\min} \sim r_{\min}^{(0)}(1-\tilde{m}-\tfrac{3}{2}\tilde{m}^2-\tfrac{1}{2}\tilde{a}^2+\tfrac{1}{2}\tilde{a}^2\tilde{\eta}+2\tilde{a}\tilde{m}\tilde{\lambda}) . \tag{A7}$$

For economy of notation in Eq. (A7) and hereafter, we understand the quantity with a tilde to be the dimensionless one normalized by $r_{\min}^{(0)}$. With the aid of Eq. (A7), Eq. (A6) becomes

$$\int^r \frac{|dr|}{[R(r)]^{1/2}} \sim r_{\min}^{(0)-1/2}(1+\tfrac{1}{2}\tilde{a}^2)[\pi(1-\tfrac{3}{4}\tilde{a}^2\tilde{\eta}+\tfrac{15}{4}\tilde{m}^2-15\tilde{a}\tilde{m}^2\tilde{\lambda})+4\tilde{m}-8\tilde{m}\tilde{a}\tilde{\lambda}$$
$$+\tfrac{128}{3}\tilde{m}^3+10\tilde{m}\tilde{a}^2-16\tilde{m}\tilde{a}^2\tilde{\eta}]-\frac{r_s+r_o}{r_s r_o} \; . \tag{A8}$$

Next, we integrate the right-hand side of Eq. (A1) to obtain the result

$$\int^\theta \frac{|d\theta|}{[\Theta(\theta)]^{1/2}} = \mp\left[\int_{\theta_s}^{\theta_{\min/\max}} \frac{d\theta}{[\Theta(\theta)]^{1/2}} - \int_{\theta_{\min/\max}}^{\theta_o} \frac{d\theta}{[\Theta(\theta)]^{1/2}}\right]$$
$$\sim r_{\min}^{(0)-1/2}(1+\tfrac{1}{2}\tilde{a}^2)(1-\tfrac{3}{4}\tilde{a}^2\tilde{\eta})\{\pi\mp\arctan[(1+\tfrac{1}{4}\tilde{a}^2\tilde{\eta})\cot\sigma_s]\mp\arctan[(1+\tfrac{1}{4}\tilde{a}^2\tilde{\eta})\cot\sigma_o]\} \; , \tag{A9}$$

where

$$\cos\theta=\tilde{\eta}^{1/2}(1+\tfrac{1}{2}\tilde{a}^2\tilde{\lambda}^2)\cos\sigma \tag{A10}$$

and $\theta_{\min/\max}$ is determined by $\Theta(\theta)=0$:

$$\cos\theta_{\min/\max}\sim\pm\tilde{\eta}^{1/2}(1+\tfrac{1}{2}\tilde{a}^2\tilde{\lambda}^2) \; . \tag{A11}$$

In Eqs. (A9) and (A11) and hereafter, the upper sign corresponds to the path which $\theta$ undergoes through $\theta_{\min}$ and the lower sign to the path through $\theta_{\max}$. This expression is valid only when $\theta$ has only one minimum or maximum value along the path.

Substituting Eqs. (A8) and (A9) into Eq. (A1) we obtain

$$\cos\theta_s \sim -\cos\theta_o\cos\delta\mp\mu\sin\delta$$
$$\mp\tfrac{1}{2}\sin\delta\frac{1}{\mu}\tilde{a}^2\tilde{\eta}\tilde{\lambda}^2\mp\frac{\mu}{4}(\tilde{\eta}-2\mu^2)\sin\delta\tilde{a} \; , \tag{A12}$$

where

$$\delta\equiv4\tilde{m}-8\tilde{m}\tilde{a}\tilde{\lambda}+\pi(\tfrac{15}{4}\tilde{m}^2-15\tilde{a}\tilde{m}^2\tilde{\lambda})$$
$$+\tfrac{128}{3}\tilde{m}^3+10\tilde{m}\tilde{a}^2-13\tilde{m}\tilde{a}^2\tilde{\eta}$$
$$-(1-\tfrac{1}{2}\tilde{a}^2)r_{\min}^{(0)}\frac{r_s+r_o}{r_s r_o} \tag{A13}$$

and

$$\mu\equiv(\tilde{\eta}-\cos^2\theta_o)^{1/2} \; . \tag{A14}$$

By the use of Eq. (3.12), Eq. (A12) leads to

$$\Delta\theta=\pm\mu\csc\theta_o\delta-\tfrac{1}{2}\tilde{\lambda}^2\cos\theta_o\csc^3\theta_o\delta^2$$
$$\mp\tfrac{1}{6}\tilde{\lambda}^2\mu\csc^3\theta_o(1+3\cot^2\theta_o)\delta^3$$
$$\pm\frac{1}{4\mu}\csc\theta_o[2(\tilde{\eta}-\cos^4\theta_o)-3\tilde{\eta}\mu^2]\tilde{a}^2\delta \; . \tag{A15}$$

Finally let us evaluate the integral in Eq. (A2). Following Ref. 2, Eq. (A2) can be rewritten with the help of Eq. (A1) as

$$\phi_o-\phi_s=\int^r \frac{a^2(2mr-a\lambda)}{\Delta[R(r)]^{1/2}}\,|dr|$$
$$+\int^\theta \frac{\lambda\cot^2\theta}{[\Theta(\theta)]^{1/2}}\,|d\theta| \; . \tag{A16}$$

After a straightforward evaluation in a manner similar to Eqs. (A8) and (A9) we obtain

$$\Delta\phi=4\tilde{m}\tilde{a}-6\tilde{m}\tilde{a}^2\tilde{\lambda}+\pi5\tilde{m}^2\tilde{a}$$
$$+\tilde{\lambda}\delta(1\pm\delta\cos\theta_o\csc^2\theta_o\mu)\csc^2\theta_o$$
$$-\tilde{m}\tilde{a}^2\tilde{\lambda}\csc^2\theta_o(\tilde{\eta}-2\mu^2)$$
$$+\tfrac{64}{3}\tilde{m}^3\tilde{\lambda}\csc^6\theta_o[(2\mu^2-\tilde{\eta})\sin^2\theta_o+4\cos^2\theta_o\mu^2] \; . \tag{A17}$$

When the observer and the source are joined by the null geodesic characterized by the constants of motion $\lambda$ and $\eta$, the relation between the position angles $\theta_o,\phi_o$ and $\theta_s,\phi_s$ are given by Eqs. (A15) and (A17).

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