Optimization of long-baseline optical interferometers for gravitational-wave detection

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The goal of this paper is to analyze and to evaluate the different configurations currently considered for the interferometric detectors of gravitational waves. We first study the properties of elementary gravito-optic transducers (i.e., delay lines or Fabry-Perot resonators) using an original formalism which allows one to understand and to compare easily the properties of complex interferometers involving these elements, such as recycling or synchronous recycling interferometers. We also describe the new idea of using detuned Fabry-Perot resonators, and we show that, in some cases, it may represent the best compromise between bandwidth and peak sensitivity.

I. INTRODUCTION

Long-baseline interferometers for the detection of gravitational radiation are presently being studied in a few countries (France, Italy, Germany, U.K., and U.S.A.).[1] All these projects are based on the construction of a large, Michelson-type interferometer with an arm length of 1–3 km, containing some kind of gravito-optic transducer in each arm. In order to decrease the shot-noise level, all these interferometers will use high-power lasers, in conjunction with light recycling techniques. The basic idea of recycling was proposed by Drever:[2] it consists in building a resonant optical cavity which contains the interferometer, so that, if the losses are low and if the cavity is kept on resonance with the incoming monochromatic light, there is a power build-up which results in a reduction of the shot noise. This can be realized in different ways, depending on the geometry of the gravito-optic transducer [delay line or Fabry-Perot (FP) resonator]. While the validity of this idea has recently been demonstrated experimentally,[3] its theory remained to be published.

The aim of the present study is to establish several simple models and associated formulas giving the ultimate photon-noise-limited sensitivities of both the current interferometric configurations and their planned extensions. In order to carry out this program some special tools are useful. A common formalism will be developed which allows a straightforward derivation of the properties of arbitrary optical configurations. Comparison of different detector configurations is facilitated by the use of a set of standard parameters.

The cases of nonrecycling delay-line and Fabry-Perot Michelson interferometers will first be treated in order to develop the formalism. Then we will apply these results to various recycling configurations and discuss the relative merits of each configuration according to the frequency range, to the bandwidth of the signal and to the value and the localization of the optical losses which limit the power build-up.

II. OPTICS IN A WEAKLY MODULATING MEDIUM

A. General principles

Consider a plane, transverse, traceless, monochromatic gravitational wave of frequency ν_g, which propagates perpendicularly to the interferometer plane ($z=0$), and is linearly polarized along the directions of the (orthogonal) interferometer arms ($x=0$ and $y=0$, respectively):

$$[h_{ij}(x,y,z,t)]_{z=0}=h_{ij}\cos(\Omega t+\Phi) ,$$

with

$$h_{ij}=\mathrm{diag}(h,-h,0),\quad \Omega=2\pi\nu_g .$$

At every point of the optical path, the light frequency spectrum will resolve in a carrier frequency

$$\nu_{\mathrm{opt}}$$

and two sidebands

$$\nu_{\mathrm{opt}}\pm\nu_g .$$

The enormous ratio between optical and gravitational frequencies allows us to neglect the polarization effects and we shall use a scalar representation of the optical amplitudes. Only first-order effects in h will be considered. The optical amplitudes at an arbitrary point of the interferometer are therefore of the form

$$A(t)=\left(A_0+\tfrac{1}{2}hA_1e^{i(\Omega t+\Phi)}+\tfrac{1}{2}hA_2e^{-i(\Omega t+\Phi)}\right)e^{-i\omega t}\ ,$$

$$\omega=2\pi\nu_{\rm opt}\ .$$

We will represent the action of gravitational transducers upon already modulated light by linear operators **S** acting upon generalized amplitudes

$$\mathbf{A}=(A_0,A_1,A_2)$$

as

$$\mathbf{A}'=\mathbf{S}\cdot\mathbf{A}\ .$$

According to the formalism developed in a previous paper,[4] these operators have the general form

$$\mathbf{S}=\begin{bmatrix} S_{00} & 0 & 0\\ S_{10} & S_{11} & 0\\ S_{20} & 0 & S_{22}\end{bmatrix}.$$

In this formalism, the diagonal elements S_{ii} represent the ordinary reflectance (or transmittance) of the transducer for each frequency (carrier and sidebands) whereas S_{10} and S_{20} characterize the power transfer from the carrier to the sidebands, i.e., the sensitivity to the gravitational wave. Optical elements with dispersion and no gravitational sensitivity will be represented by diagonal matrices, elements without dispersion nor G sensitivity by scalar matrices (mirrors, splitters, etc.). Owing to the $\pi/2$ phase lag between the reflected and the transmitted waves at a mirror, we shall represent the action of a mirror upon the complex amplitude of an optical wave by $i\sqrt{R}$ for a reflection, and by $\sqrt{T}$ for a transmission.

The whole interferometer is itself a gravitational transducer and has therefore an associated global operator **S**. In the following we will encounter three different cases: when S_{10} and S_{20} are of equal moduli, the phase relationships between both and S_{00} will denote either pure phase modulation or pure amplitude modulation, and when S_{10} and S_{20} are not equal, one of them is much larger than the other. Therefore, if the limiting noise reduces to the shot noise, we have, for the signal-to-noise ratio (SNR),

$$\mathrm{SNR}=h\left[\frac{\eta P\tau_i}{\hbar\omega_{\rm opt}}\right]^{1/2}S,\quad S=|S_{10}|+|S_{20}|\ ,$$

where τ_i and η are, respectively, the integration time and quantum efficiency of the photodetector, and P the power of the source. In other words, the minimum, photon-noise-limited, detectable h is

$$h_{\rm PN}=\left[\frac{\hbar\omega_{\rm opt}}{\eta P}\right]^{1/2}\frac{1}{S}\sqrt{\delta\nu}\ ,$$

where $\delta\nu$ is the bandwidth of the detector. In what follows we shall consider the quantity S, that we shall call normalized signal to noise ratio (NSNR), as the quantity to be optimized.

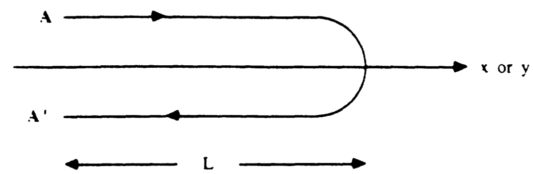

FIG. 1. Round trip in the vacuum (notation).

B. Standard gravito-optic transducers

Gravito-optic transducers are optical devices in which the gravitational wave (GW) is supposed to have a detectable perturbing effect. Current examples are the delay lines and the Fabry-Perot cavities. Both have associated operators D and F which can be related to the elementary propagation operator X corresponding to a round trip in the perturbed vacuum[4] (see Fig. 1). We have $\mathbf{A}'=\mathbf{X}\cdot\mathbf{A}$ with

$$\mathbf{X}=\begin{bmatrix} e^{2i\omega L/c} & 0 & 0\\ i\epsilon\dfrac{\omega L}{c}\dfrac{\sin(\Omega L/c)}{\Omega L/c}e^{i(2\omega-\Omega)L/c} & e^{2i(\omega-\Omega)L/c} & 0\\ i\epsilon\dfrac{\omega L}{c}\dfrac{\sin(\Omega L/c)}{\Omega L/c}e^{i(2\omega+\Omega)L/c} & 0 & e^{2i(\omega+\Omega)L/c}\end{bmatrix}. \tag{1}$$

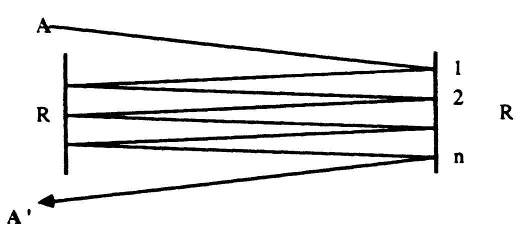

FIG. 2. n-fold delay line (notation).

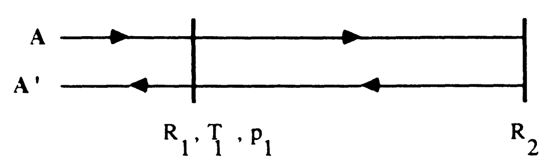

FIG. 3. Reflecting Fabry-Perot cavity (notation).

The value of ϵ is $+1$ for a round trip along the x axis, and -1 along the y axis. Consider now an n-fold delay line with two mirrors of intensity reflection coefficients R (see Fig. 2). It consists in n iterations of the $\mathbf{X}$ operator and $2n-1$ iterations of the operator $i\sqrt{R}$. Its associated operator is thus $i\mathbf{D}$ where

$$\mathbf{D}=(-1)^{n-1}\sqrt{R}^{\,2n-1}\mathbf{X}^n .$$

Consider a Fabry-Perot cavity (see Fig. 3) with a front mirror of intensity reflection and transmission coefficients R_1, T_1, with losses p_1, and with a rear mirror of intensity reflection coefficient R_2. The associated operator $i\mathbf{F}$ looks like the ordinary reflectance of a Fabry-Perot cavity but with the ordinary phase factor replaced by $\mathbf{X}$:

$$\mathbf{F}=[\sqrt{R_1}+(1-p_1)\sqrt{R_2}\mathbf{X}](1+\sqrt{R_1R_2}\mathbf{X})^{-1} .$$

1. Response of delay-line-type detectors

In more detail, the delay-line operator involves the three following elements:

$$D_{00}=(-1)^n\sqrt{R}^{\,2n-1}e^{4in\pi\nu_{\mathrm{opt}}L/c} ,$$

$$D_{10}=(-1)^n\sqrt{R}^{\,2n-1}i\epsilon\frac{\nu_{\mathrm{opt}}}{\nu_g}\sin\left[\frac{2n\pi\nu_g L}{c}\right]e^{2i\pi(2\nu_{\mathrm{opt}}-\nu_g)nL/c} ,$$

$$D_{10}=(-1)^n\sqrt{R}^{\,2n-1}i\epsilon\frac{\nu_{\mathrm{opt}}}{\nu_g}\sin\left[\frac{2n\pi\nu_g L}{c}\right]e^{2i\pi(2\nu_{\mathrm{opt}}+\nu_g)nL/c} .$$

The action of this operator on an unmodulated wave is therefore a pure phase modulation. It is convenient to introduce some parameters which have their counterparts in the case of cavities.

By introducing the storage time

$$\tau_s=\frac{2nL}{c} ,$$

the time constant

$$\tau''=\frac{2L}{cR^*},\quad R^*=1-R ,$$

and the normalized storage time $t=\tau_s/\tau''$ which has the minimum value $t_m=R^*$, we get

$$R^{\,n-1/2}=e^{-(t-t_m/2)} .$$

In what follows we shall consider kilometric interferometers ($L\approx3$ km) so that the minimum value of τ_s, i.e., $2L/c$ is about 2×10^{-5} s, and high reflectivity coatings ($R^*\approx10^{-4}$) so that τ'' is about 0.2 s. This set of parameters will be referred to as the reference antenna. As will be shown later, the best τ_s for a given gravitational frequency $\nu_g^{(0)}$ is of order $1/2\nu_g^{(0)}$. So far as we consider gravitational frequencies smaller than a few kilohertz we can assume $t\gg t_m$. Two more parameters are useful—the normalized gravitational frequency $f=2\pi\nu_g\tau''$ and the maximum quality factor $Q=2\pi\nu_{\mathrm{opt}}\tau''$. (In the reference antenna, when visible light is used, the quality factor Q is about 7.5×10^{14} and $f\approx1.26\nu_g$/Hz.)

The approximate form of the operator D simplifies now to

$$|D_{00}|=e^{-t} ,$$

$$|D_{10}|=|D_{20}|=\frac{Q}{f}e^{-t}\left|\sin\left[\frac{ft}{2}\right]\right| . \tag{2}$$

If the detection system involves two delay lines, it has a NSNR

$$S(f)=\frac{2Q}{f}e^{-t}\left|\sin\left[\frac{ft}{2}\right]\right| . \tag{3}$$

For a given gravitational frequency corresponding to f_0, there exists an optimal normalized storage time:

$$t_0=\frac{2}{f_0}\arctan\left[\frac{f_0}{2}\right] .$$

Note that $f_0\to0$ yields $\tau_S^{(0)}\to\tau''$, and thus τ'' may be interpreted as the maximum value of the optimal storage time.

The optimized NSNR is then

$$S(f)=\frac{2Q}{f}\left|\sin\left[\frac{f}{f_0}\arctan\left[\frac{f_0}{2}\right]\right]\right| \times\exp\left[-\frac{2}{f_0}\arctan\left[\frac{f_0}{2}\right]\right] . \tag{4}$$

We have, at $f=f_0$,

$$S(f)=\frac{2Q}{\sqrt{f_0^2+4}}\exp\left[-\frac{2}{f_0}\arctan\left[\frac{f_0}{2}\right]\right] .$$

Therefore, we can give the limiting value of S when $f_0\to0$:

$$S(0)=Q/e .$$

For $f_0 \gg 1$, a good approximation of the optimal storage time is given by $t_0 = \pi / f_0$: i.e.,

$$\tau_S^{(0)} = \frac{1}{2\nu_g^{(0)}}$$

and the NSNR becomes simply

$$S(f) = \frac{2Q}{f} \left| \sin\left[\frac{\pi}{2}\frac{f}{f_0}\right] \right| ;$$

i.e.,

$$S(\nu_g) = \frac{2\nu_{\text{opt}}}{\nu_g} \left| \sin\left[\frac{\pi \nu_g}{2\nu_g^{(0)}}\right] \right| .$$

2. *Response of Fabry-Perot-type detectors*

For the Fabry-Perot cavity operator, the relevant elements are

$$F_{00} = \frac{(1-p_1)\sqrt{R_2}e^{2i\omega L/c} + \sqrt{R_1}}{1+\sqrt{R_1 R_2}e^{2i\omega L/c}} ,$$

$$F_{10} = i\epsilon T_1 \sqrt{R_2}\frac{\nu_{\text{opt}}}{\nu_g} \sin\left[\frac{\Omega L}{c}\right] \frac{e^{i(2\omega-\Omega)L/c}}{(1+\sqrt{R_1R_2}e^{2i\omega L/c})(1+\sqrt{R_1R_2}e^{2i(\omega-\Omega)L/c})} ,$$

$$F_{20} = i\epsilon T_1 \sqrt{R_2}\frac{\nu_{\text{opt}}}{\nu_g} \sin\left[\frac{\Omega L}{c}\right] \frac{e^{i(2\omega+\Omega)L/c}}{(1+\sqrt{R_1R_2}e^{2i\omega L/c})(1+\sqrt{R_1R_2}e^{2i(\omega+\Omega)L/c})} .$$

The eigenfrequencies of the cavity are determined by the condition $\exp(-2i\omega_0 L/c) = -1$ and consequently, when the optical source is resonant, the preceding operator denotes pure phase modulation. We need now some dimensionless parameters analogous to the delay line's. We may define the time constant of the cavity by

$$\tau_S' = \frac{2L}{c(1-\sqrt{R_1R_2})} .$$

Owing to the constraint $0 < R_1 < 1-p_1$, we have

$$\frac{2L}{c} < \tau_S' < \tau'' \equiv \frac{2L}{c[1-\sqrt{(1-p_1)R_2}]} .$$

The ratio of the time constant to its maximum value will be named normalized time constant t; it obeys

$$t_m < t < 1 \quad \text{with } t_m \equiv 1-\sqrt{(1-p_1)R_2} .$$

In the general case, the source is eventually detuned from $\Delta\nu_{\text{opt}}$ from a resonance ν_0 which leads us to introduce the normalized detuning defined by $\Delta f = 2\pi\Delta\nu_{\text{opt}}\tau''$. With these notations, the ordinary reflectance of the cavity has the exact expression

$$|F_{00}|^2 = \frac{1}{R_2}\frac{(1-2t+tt_m)^2+(1-t_m)^2(1-t_m/t)\Delta f^2 t^2 \text{sinc}^2(\Delta f t_m/2)}{1+\Delta f^2 t^2(1-t_m/t)\text{sinc}^2(\Delta f t_m/2)} ,$$

where the notation $\text{sinc}(x)$ denotes the function $\sin(x)/x$. Fortunately a simple approximate form can be given when Δf is much smaller than the free spectral range and when $t_m \ll 1$ which is satisfied when gravitational frequencies are restricted to a range of values less than a few kilohertz:

$$|F_{00}| = \left[1-\frac{4t(1-t)}{1+\Delta f^2 t^2}\right]^{1/2} .$$

The minimum value of $|F_{00}|$ is reached at resonance where

$$|F_{00}|_{\text{res}} = |1-2t| .$$

Within the same approximation, we have

$$\text{Arg}F_{00} = \pi + \arctan\left[\frac{2\Delta f t(1-t)}{1-2t-\Delta f^2 t^2}\right] .$$

We have further

$$|F_{10}| = \frac{Qt(1-t)}{\sqrt{1+\Delta f^2 t^2}\sqrt{1+(\Delta f - f)^2 t^2}} ,$$

$$|F_{20}| = \frac{Qt(1-t)}{\sqrt{1+\Delta f^2 t^2}\sqrt{1+(\Delta f + f)^2 t^2}} .$$

In particular when the source is resonant ($\Delta f = 0$), we have the special case

$$|F_{00}| = |1-2t| ,$$

$$|F_{10}| = |F_{20}| = \frac{Qt(1-t)}{\sqrt{1+f^2t^2}} .$$

We have consequently, for the NSNR,

$$S(f) = \frac{2Qt(1-t)}{\sqrt{1+f^2t^2}} . \quad (5)$$

When the normalized gravitational frequency $f_0 = 2\pi\nu_g^{(0)}\tau''$ tends to zero, the optimal value of τ'_S has the limiting value $\tau''/2$ and the limiting value of the SNR turns out to be $Q/2$. When $f_0 \gg 1$, the function $S(t)$ begins to saturate as soon as

$$t_0 = \frac{2}{f_0}, \quad \text{i.e.,} \quad \tau'^{(0)}_S = \frac{1}{\pi\nu_g^{(0)}} .$$

This value will be taken as a reasonable choice, for the true optimal value is much higher but irrelevant, giving only a slightly better value of S. In this case, we have, for the NSNR,

$$S(f) = \frac{4Q}{(f_0^2 + 4f^2)^{1/2}} ;$$

in particular, (6)

$$S(f_0) = 1.78\frac{Q}{f_0} .$$

In dimensional expression, this is

$$S(\nu_g) = (4\nu_{\text{opt}}/\nu_g)[1 + (2\nu_g/\nu_g^{(0)})^2]^{-1/2}$$

and

$$S(\nu_g^{(0)}) = 1.78\nu_{\text{opt}}/\nu_g^{(0)} .$$

Let us point out an important feature—with Fabry-Perot cavities, it is possible to use a detuned source with respect to the cavity eigenfrequency of an amount $\Delta f = f_0$ so that the sideband generated by the gravitational wave is resonant

$$\nu_{\text{opt}} = \nu_0 + \nu_g^{(0)} ,$$

leading to

$$|F_{00}| = \left[1 - \frac{4t(1-t)}{1 + f_0^2 t^2}\right]^{1/2}$$

and

$$|F_{10}| = \frac{Qt(1-t)}{(1 + f_0^2 t^2)^{1/2}[1 + (f - f_0)^2 t^2]^{1/2}} .$$

When $f_0 \gg 1$, a reasonable choice of τ'_S is again

$$\tau'_S = \frac{1}{\pi\nu_g^{(0)}}$$

and with only one resonant sideband, the optimized NSNR becomes

$$S(f) = 0.89\frac{Q}{f_0}\frac{1}{[1 + 4(1 - f/f_0)^2]^{1/2}} \quad (7)$$

or

$$S(\nu_g) = (0.89\nu_{\text{opt}}/\nu_g^{(0)}) \times [1 + 4(1 - \nu_g/\nu_g^{(0)})^2]^{-1/2} .$$

Figure 4 gives a comparison of the sensitivities versus ν_g for a delay line, for a Fabry-Perot both at resonance and with detuning, in the conditions we have described above.

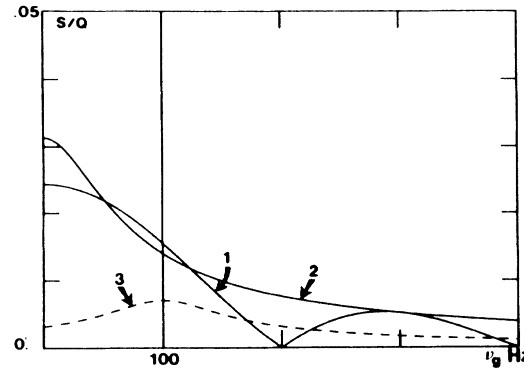

FIG. 4. Transfer function of a Michelson interferometer with multipass arms (optimized at 100 Hz): (1) delay lines; (2) resonant FP cavities; (3) detuned FP cavities.

The detuned Fabry-Perot is less sensitive than the other configurations, but the fact that it brings a higher reflectance makes it interesting when recycling is applied, as we will see in the next part.

III. STANDARD RECYCLING

A. Principles of standard recycling

A classical Michelson interferometer tuned at a dark fringe behaves just like a mirror—most of the power incoming from the source is reflected back. We can use it as the second mirror of a cavity, the front mirror of it is called the recycling mirror. It will be shown that this configuration increases the SNR by allowing more efficient use of the available power. Figure 5 shows the principle of operation. Let R_r, T_r, p_r be the parameters (reflectivity and transmittivity coefficients, losses) of the recycling mirror and R_S, T_S, p_S those of the splitter. It is easy to show that at a dark fringe we have an operator **S** for the whole system:

$$\mathbf{A}' = \mathbf{S} \cdot \mathbf{A} ,$$

where the relevant coefficients of **S** are

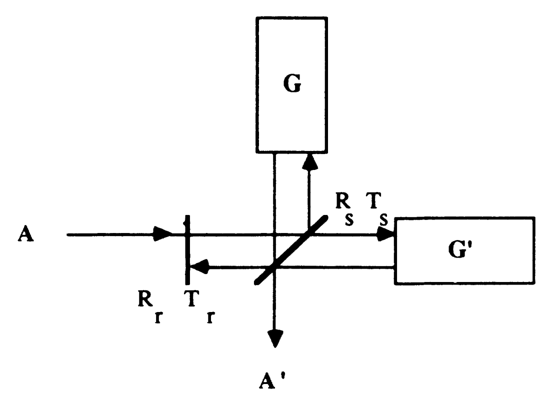

FIG. 5. Sketch of the standard recycling setup.

$$|S_{10}|=(1-p_S)\sqrt{T_r}\frac{|G_{10}|}{1-(1-p_S)\sqrt{R_r}\,|G_{00}|}\,,$$

$$|S_{20}|=(1-p_S)\sqrt{T_r}\frac{|G_{20}|}{1-(1-p_S)\sqrt{R_r}\,|G_{00}|}\,.$$

G is the operator associated with a gravito-optic transducer, either a delay line or a Fabry-Perot cavity, directed along the y axis, and **G**′ the operator associated with the same transducer, directed along the x axis; both have the same coefficients G_{ii} but opposite coefficients G_{01} and G_{02}. One sees already that the recycling rate can be optimized for given losses and **G**, we find

$$(\sqrt{R_r})_{\text{opt}}=(1-p_r)(1-p_S)\,|G_{00}|\,.$$

So that it is possible to give optimized values of the components of the NSNR:

$$|S_{10}|=|G_{10}|\left[\frac{(1-p_r)(1-p_S)^2}{1-(1-p_r)(1-p_S)^2|G_{00}|^2}\right]^{1/2},$$

$$|S_{20}|=|G_{20}|\left[\frac{(1-p_r)(1-p_S)^2}{1-(1-p_r)(1-p_S)^2|G_{00}|^2}\right]^{1/2}.$$

By assuming low extra cavity losses $p=p_r+2p_S$ we may write simply

$$|S_{10}|=\frac{|G_{10}|}{\sqrt{1-(1-p)|G_{00}|^2}}\,,\qquad |S_{20}|=\frac{|G_{20}|}{\sqrt{1-(1-p)|G_{00}|^2}}\,. \tag{8}$$

In the reference antenna we shall take $p_r=10^{-4}$ and $p_S=10^{-3}$. The problem to be discussed below is the optimization of either the storage time for delay lines, or the decay time for cavities, when recycling is applied. In the general situation, the optimal value of that time constant will depend on the gravitational frequency f_0 at which one wants to optimize, and on the recycling losses denoted by p. Two frequency ranges will appear: the low-frequency range and the high-frequency range. In the low-frequency range, long storage times are required, the recycling losses are therefore dominated by the reflectivity losses in the arms, and the optimal storage time is almost independent of p. In the high-frequency range, the required storage times are relatively short, so that the losses in the arms may happen to be comparable with the recycling losses p, and the optimal storage time will depend on both p and f_0. The effective value of p will be determined not only by the losses of the recycling mirror and of the beam splitter, but also by the fact that the interference on the beam splitter may be affected by small misalignments or by a slight asymmetry between the two arms of the Michelson interferometer.

B. Standard recycling with delay lines

In the case when **G** represents a delay line, as shown earlier, we have

$$|G_{00}|=e^{-t}\,,$$

$$|G_{10}|=|G_{20}|=\frac{Q}{f}\left|\sin\left[\frac{ft}{2}\right]\right|e^{-t}\,.$$

So that, assuming low losses, the phase relations denote pure amplitude modulation, and the NSNR of the global system is given (for extracavity losses $p=p_r+2p_S$) by

$$S(f)=\frac{2Q}{f}\frac{\left|\sin\left[\frac{ft}{2}\right]\right|}{\sqrt{1-(1-p)e^{-2t}}}e^{-t}\,. \tag{9}$$

For a given gravitational frequency denoted by f_0, the corresponding optimal normalized storage time is given by the implicit equation

$$t_0=\frac{2}{f_0}\arctan\left[\frac{f_0}{2}[1-(1-p)e^{-2t_0}]\right]$$

which is easily solved by iterations. For values of f_0 small compared to $1/p$ (say $\nu_g^{(0)}<50$ Hz in the reference interferometer), t_0 is seen to be almost independent of p and takes a value near 0.8 for zero frequency. The corresponding limit for the NSNR is $0.4Q$. A convenient interpolation formula valid within this range is

$$t_0=(1.56+0.18f_0^2)^{-1/2}\,.$$

Now, in the case when the normalized gravitational frequency is large enough (say $n_g^{(0)}>50$ Hz in the reference interferometer), we can put $t=x/f_0\ll 1$ and write

$$S(f_0)=Q\left[\frac{2}{f_0}\right]^{1/2}\frac{|\sin(x/2)|}{\sqrt{x+pf_0/2}}\,.$$

The optimal value of x is solution of

$$x+\frac{pf_0}{2}=\tan\left[\frac{x}{2}\right].$$

The solution x_0 takes values in the range $[2.33,\pi]$ for values of $pf_0/2$ in the range $[0,\infty]$. A good interpolation formula valid except for low frequencies is

$$x_0=2.33+\frac{0.81pf_0}{4.25+pf_0}\quad\text{then } \tau_S=\frac{x_0}{2\pi\nu_g^{(0)}}\,.$$

The optimized frequency response of the recycling setup is now

$$S(f)=\frac{Q}{f}\left[\frac{2f_0}{x_0}\right]^{1/2}\left|\sin\left[\frac{x_0 f}{2f_0}\right]\right|. \tag{10}$$

In particular, if the frequency f_0 is within the especially interesting band $1\ll f_0\ll 1/p$ say 50 Hz to 500 Hz in the reference antenna, we can write

$$S(f)=0.92\frac{Q}{f}\sqrt{f_0}\left|\sin\left[\frac{1.17f}{f_0}\right]\right|,$$

in particular $S(f_0)=0.85(Q/\sqrt{f_0})$ or, in ordinary notation,

$$S(\nu_g)=0.92\frac{\nu_{\text{opt}}}{\nu_g}(2\pi\nu_g^{(0)}\tau'')^{1/2}\,|\sin(1.17\nu_g/\nu_g^{(0)})|\ .$$

Such an optimized transfer function (for 100 Hz) is plotted in Fig. 6.

C. Case of resonant Fabry-Perot cavities

The relevant operator **G** is now **F**:

$$|F_{00}|=|1-2t|\ ,$$

$$|F_{10}|=|F_{20}|=Qt(1-t)/\sqrt{1+f^2t^2}\ .$$

The phases are such that the NSNR with the optimal recycling rate takes the form

$$S(f)=\frac{2Qt(1-t)}{\sqrt{1+f^2t^2}\sqrt{1-(1-p)(1-2t)^2}}\ . \qquad (11)$$

In the low-frequency domain, when the extra cavity losses are small compared to the reflectivity losses of the cavities, i.e., $p \ll t$, which corresponds typically to the frequency range $0\rightarrow 50$ Hz for the reference antenna, $S(f)$ becomes independent of p:

$$S(f)=Q\left[\frac{t(1-t)}{1+f^2t^2}\right]^{1/2}\ .$$

The optimal value of t at $f=f_0$ is then

$$t_0=\frac{1}{1+(1+f_0^2)^{1/2}}\ .$$

The corresponding optimized transfer function is

$$S(f)=\frac{Q}{\sqrt{2}}\frac{1}{\left[1+\sqrt{1+f_0^2}+\dfrac{f^2-f_0^2}{2\sqrt{1+f_0^2}}\right]^{1/2}}\ .$$

If now f is large enough (>50 Hz in the reference antenna), we can set $t=x/f_0\ll 1$ so that the NSNR becomes

$$S(f_0)=\frac{Q}{\sqrt{f_0}}\frac{x}{\sqrt{1+x^2}\left[x+\dfrac{pf_0}{4}\right]^{1/2}}\ .$$

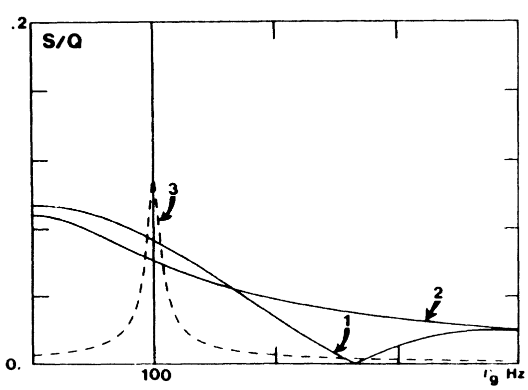

FIG. 6. Transfer function of a Michelson interferometer with multipass arms and standard recycling (optimized at 100 Hz): (1) delay lines; (2) resonant FP cavities; (3) detuned FP cavities.

The value of x which makes S optimal is solution of

$$x^3-x-\frac{pf_0}{2}=0\ .$$

The exact solution is somewhat cumbersome but the following interpolation formula is quite sufficient for our purposes:

$$t_0=\frac{1}{f_0}\left[1+\frac{pf_0}{2}\right]^{-1/3}\ .$$

In fact, if f_0 is not too high (within the band $50\rightarrow 500$ Hz in the reference antenna), the solution differs little from $x_0=1$, so that we can take $t_0=1/f_0$; i.e.,

$$\tau'^{(0)}_S=\frac{1}{2\pi\nu_g^{(0)}}\ .$$

The optimized frequency response of the recycling setup is now

$$S(f)=Q\left[\frac{f_0}{f^2+f_0^2}\right]^{1/2}$$

in particular (12)

$$S(f_0)=\frac{Q}{\sqrt{2f_0}}\ .$$

In ordinary notation we have

$$S(\nu_g)=\nu_{\text{opt}}\left[\frac{2\pi\tau''}{\nu_g^{(0)}}\right]^{1/2}\frac{1}{[1+(\nu_g/\nu_g^{(0)})]^{1/2}}\ .$$

An optimized transfer function (for 100 Hz) is represented on Fig. 6 so as to be compared with the case of delay lines.

D. Case of detuned Fabry-Perot cavities

As already noted, Fabry-Perot cavities can be driven out of resonance, leading to a different response to gravitational frequencies, to a slightly worse signal amplitude, but a higher reflectivity. This mode of operation is expected to give interesting results in a recycling configuration. Let us discuss this idea.

Assume the optical frequency to be detuned with respect to an eigenfrequency of the cavity by an amount equal to the gravitational frequency to be detected:

$$\nu_{\text{opt}}=\nu_0+\nu_g^{(0)}\ .$$

The detuned cavity operator, as shown earlier, contains the following elements:

$$|F_{00}|^2=1-\frac{4t(1-t)}{1+\Delta f^2t^2}\ ,$$

$$|F_{10}|=\frac{Qt(1-t)}{\sqrt{1+\Delta f^2t^2}\sqrt{1+(\Delta f-f)^2t^2}}\ .$$

With an optimal recycling rate for $\nu_g=\nu_g^{(0)}$ and with

$\Delta f = f_0$, we find the NSNR as

$$S(f) = \frac{S(f_0)}{[1+(f-f_0)^2 t^2]^{1/2}} .$$

The peak value, $S(f_0)$ is

$$S(f_0) = \frac{Qt(1-t)}{[p(1+f_0^2 t^2)+4t(1-t)(1-p)]^{1/2}} . \qquad (13)$$

We shall consider that $p \ll t$, for it will be seen that this approximation holds even for relatively high frequencies (up to the kilohertz in the reference antenna) due to the fact that high values of the optimal time constant τ_S' are required in the detuned system. The resulting expression for the peak value of the NSNR is

$$S(f_0) = Q \left[\frac{t(1-t)^2}{(pf_0^2-4)t+4} \right]^{1/2} .$$

The optimal value of t is

$$t_0 = \frac{2}{3+(1+2pf_0^2)^{1/2}}$$

$$\Longrightarrow S(f_0) = Q \left[\frac{2(1+\sqrt{1+2pf_0^2})}{(3+\sqrt{1+2pf_0^2})^3} \right]^{1/2} . \qquad (14)$$

The NSNR has a narrow-band-type behavior characterized by a bandwidth of

$$\delta f = [3+(1+2pf_0^2)^{1/2}]\sqrt{3} .$$

The transfer functions of delay line and resonant FP interferometers are not essentially changed by standard recycling apart from a gain factor, but the transfer function of the detuned FP interferometer becomes resonant, due to the long time constant required. It is easily seen that, in the limit $pf_0 \to 0$, $S(f_0) \to Q/4$, and $\delta f \to 4\sqrt{3}$. It is more interesting to compare these characteristics to those of the following section concerning synchronous recycling. With a choice of x_0 two or three times less than the optimal value, the linewidth is seen to be increased while the peak value is only slightly decreased, giving interesting transfer functions, with a finite-band response localized in the gravitational spectrum. Examples of transfer functions corresponding to that mode of operation are plotted in Fig. 7. Figure 8 summarizes the discussion of standard recycling systems by a plot of the optimal NSNR value for the different systems.

IV. SYNCHRONOUS RECYCLING

A. General principles of synchronous recycling

The basis of synchronous recycling is to include two mutually orthogonal gravito-optic transducers in a ring cavity of high finesse. If the effective storage time in each arm is equal to half the gravitational period, the phase lag between the perturbed and nonperturbed light waves, or better, between two counterpropagating waves, is expected to increase with time up to a limiting value imposed by the finite losses of the recycling cavity. We consider a ring cavity (see Fig. 9) with a recycling mirror of parameters R_r, T_r, p_r as in the previous section, and a transfer mirror of parameter R_t. In this ring cavity, two orthogonal gravito-optic transducers, G and G' are included. Let **G** and **G**$'$ be the two corresponding operators—both have the same G_{ii} coefficients, but opposite G_{01}, G_{02} coefficients. The global operator associated with the ring cavity included in the recycling setup is **S** with

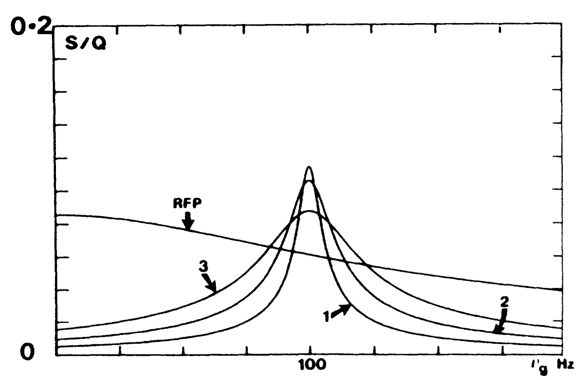

FIG. 7. Transfer function for standard recycling with detuned FP cavities and nonoptimal time constant. (1) $t = t_{\text{opt}}$; (2) $t = t_{\text{opt}}/2$; (3) $t = t_{\text{opt}}/4$. RFP: standard recycling interferometer with resonant FP cavities (for comparison).

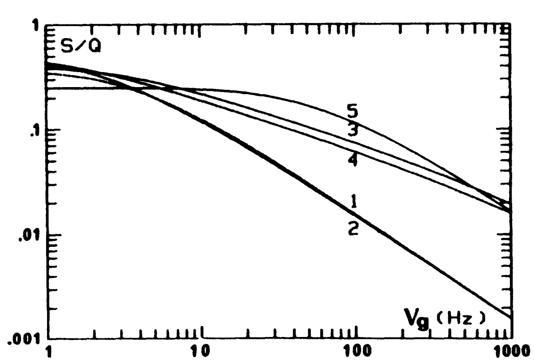

FIG. 8. Optimal values of the NSNR versus gravitational frequency for standard recycling systems: (1) delay lines—no recycling (for comparison); (2) resonant FP—no recycling (for comparison); (3) delay lines—standard recycling; (4) resonant FP—standard recycling; (5) detuned FP—standard recycling (peak value).

$$S=[\sqrt{R_r}-(1-p_r)\sqrt{R_t}e^{2i\omega a/c}\mathbf{G}'\cdot\mathbf{G}](1-\sqrt{R_rR_t}e^{2i\omega a/c}\mathbf{G}'\cdot\mathbf{G})^{-1} ,$$

where $2a$ is the total length of the transfer paths. A direct computation of the coefficients of $\mathbf{S}$ gives

$$S_{0i}=\frac{-T_r\sqrt{R_t}e^{2i\omega a/c}G_{0i}(G_{ii}-G_{00})}{(1-\sqrt{R_rR_t}e^{2i\omega a/c}G_{00}^2)(1-\sqrt{R_rR_t}e^{2i\omega a/c}G_{ii}^2)} \quad (i=1,2) .$$

B. Synchronous recycling with delay lines

In the case of delay lines, the preceding setup may be regarded as a very long ring cavity of length of $4nL+2a$. It follows that the free spectral range between two eigenfrequencies is $c/4nL$ (a being very small compared to L) and that a gravitational wave of frequency $c/4nL$ will be able to transfer light power from a carrier at resonance to two resonant sidebands. Let us develop this idea.

By replacing $\mathbf{G}$ by the delay line operator in the recycling formula, we obtain, assuming $n \gg 1$,

$$S_{10}=\frac{-2T_r\sqrt{R_t}e^{iz}R^{2n}e^{4in\omega L/c}e^{-2in\Omega L/c}\frac{\nu_{\rm opt}}{\nu_g}\sin\left[\frac{n\Omega L}{c}\right]^2}{(1-\sqrt{R_rR_t}e^{iz}R^{2n}e^{4in\omega L/c})(1-\sqrt{R_rR_t}e^{iz}R^{2n}e^{4in\omega L/c}e^{-4in\Omega L/c})} ,$$

where $z=2\omega a/c$, S_{20} has a similar expression with Ω replaced by $-\Omega$. It is always possible to choose z in such a way that the carrier frequency is a resonance of the global ring cavity:

$$e^{iz}e^{4i\omega nL/c}=1 .$$

Then,

$$S(f)=\frac{4T_r\sqrt{R_t}e^{-2t}\frac{Q}{f}\sin^2\left[\frac{ft}{2}\right]}{(1-\sqrt{R_rR_t}e^{-2t})\,|\,1-\sqrt{R_rR_t}e^{-2t}e^{-2ift}\,|} . \quad (15)$$

We intend to optimize the maximum of this function of Ω which is reached for $f_0t=\pi$, i.e., $\nu_g^{(0)}=1/2\tau_s$. This means that for that particular gravitational frequency, the optical carrier and its two sidebands are resonant in the ring cavity:

$$\nu_{\rm opt}-\nu_g^{(0)},\nu_{\rm opt},\nu_{\rm opt}+\nu_g^{(0)}$$

are successive eigenfrequencies. At

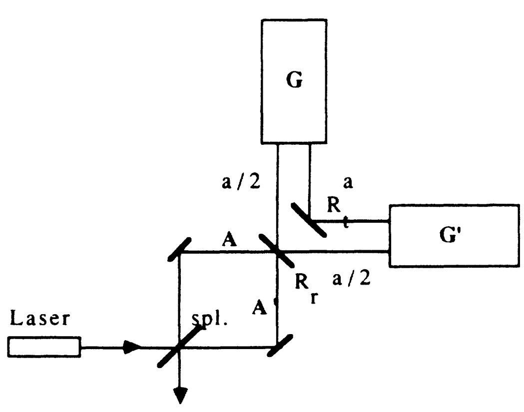

FIG. 9. Sketch of the whole synchronous recycling setup.

$$\nu_g=\nu_g^{(0)}$$

we have

$$S(f_0)=\frac{4T_r\sqrt{R_t}e^{-2\pi/f_0}\frac{Q}{f_0}}{(1-\sqrt{R_rR_t}e^{-2\pi/f_0})^2} .$$

The optimal value of the recycling rate is therefore

$$\sqrt{R_r}=(1-p_r)\sqrt{R_t}e^{-2\pi/f_0}$$

yielding an optimal peak value ($R_t\approx 1$):

$$S(f_0)=\frac{4Q}{f_0}\frac{e^{-2\pi/f_0}}{1-(1-p)e^{-4\pi/f_0}}$$

with (16)

$$p=1-(1-p_r)R_t .$$

A plot of $S(\nu_g)$ is given in Fig. 14. The limiting value for very low frequencies is therefore $S(0)=0$; for f_0 not too low we have

$$S(f_0)=\frac{Q}{\pi+pf_0/4} .$$

When extra cavity losses p are weak (in the reference antenna, p has been given the same value as in the case of standard recycling, namely, $p=2.1\times10^{-3}$), we have a flat maximum of the function $S(f_0)$: within the region $1\ll f_0\ll 4/p$, $S(f_0)\approx Q/\pi$. In the general case, the transfer function can be expressed as

$$S(f)=S(f_0)\times\left[1+\left[\frac{2(\pi/f_0)(1-p)e^{-4\pi/f_0}}{1-(1-p)e^{-4\pi/f_0}}(f-f_0)\right]^2\right]^{-1/2} . \quad (17)$$

This is a resonant type response characterized by a bandwidth [full width at half maximum (FWHM)] of

$$\delta f=\frac{\sqrt{3}[1-(1-p)e^{-4\pi/f_0}]}{t_0(1-p)e^{-4\pi/f_0}}$$

for the central band $1 \ll f_0 \ll 4\pi/p$, we have the very simple result:

$$S(f)=\frac{Q}{\pi}\frac{1}{[1+\frac{1}{4}(f-f_0)^2]^{1/2}}, \quad \delta f=4\sqrt{3}\ . \tag{18}$$

C. Synchronous recycling with Fabry-Perot cavities

1. Classical properties of coupled cavities

When the gravito-optic transducers are Fabry-Perot cavities, the recycling setup may be viewed as a system of three cavities: two long cavities of length L coupled by means of a third short one of length a (see Fig. 10). If we ignore losses and external coupling, we can see that such an optical device has a system of eigenfrequencies obtained by duplication from the spectrum of a single isolated cavity—each eigenfrequency ν_0 of the isolated FP cavity is split into two new eigenfrequencies: ν_A, ν_S corresponding to symmetric and antisymmetric eigenmodes. The values of ν_A, ν_S depend on the tuning of the coupling short cavity. When the coupling cavity is at a maximum of transmission, the coupling is strong and the difference

$$|\nu_A-\nu_S|$$

is of the same order of magnitude as the free spectral range. On the contrary, if the coupling cavity is at a maximum of reflection, the coupling is weak, and the frequency gap becomes small. Assuming extremely high reflectivities of the rear mirrors and finite reflectivities of the front mirrors, it can be shown that

$$2\pi(\nu_A-\nu_0)L/c=\arctan\left[\frac{1-\sqrt{R_1}}{1+\sqrt{R_1}}\cot(z/4)\right],$$

$$2\pi(\nu_s-\nu_0)L/c=-\arctan\left[\frac{1-\sqrt{R_1}}{1+\sqrt{R_1}}\tan(z/4)\right],$$

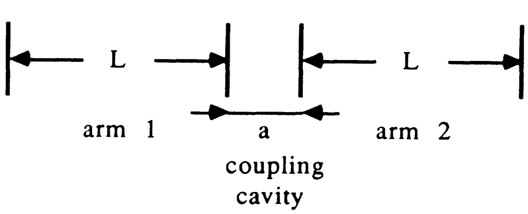

FIG. 10. Coupled cavities.

where z is the propagation phase over the length $2a$, i.e., the tuning of the short cavity (see Fig. 11). We have, furthermore,

$$\nu_A-\nu_S=\frac{c}{2\pi L}\arctan\left[\frac{1-R_1}{2\sqrt{R_1}\sin(z/2)}\right]. \tag{19}$$

For z near 0 or 2π, that is, for a high transmission coefficient of the coupling cavity, we have

$$|\nu_A-\nu_S|=\frac{c}{4L},$$

which is the free spectral range of a ring cavity of length $4L$. Because of the strong coupling, the front mirrors are ignored.

For $z=\pi$, at the maximum of reflectivity of the coupling cavity, we have

$$|\nu_S-\nu_A|=\frac{cR_1^*}{4\pi L}=\frac{1}{\pi\tau_S'}, \quad R_1^*=1-R_1 \ll 1\ ,$$

where τ_S' is the common time constant of both long cavities. For high values of τ_S', values of $|\nu_A-\nu_S|$ comparable with gravitational frequencies can be attained, and power can be transferred from the carrier to one sideband provided that their frequencies coincide with ν_A and ν_S, respectively. We are going to discuss this idea below.

2. Synchronous recycling with FP cavities

Figure 12 summarizes the notation involved. The optical paths were separated for more clarity. We have assumed a separation between the two counterpropagating waves so that we can apply the preceding formula for synchronous recycling which is valid for a ring cavity. This can be practically done by suitable elements which are not taken into account here, for their losses can be included in the transfer losses. By using the matrix algebra presented in Secs. II A and II B, it can be shown that the operator associated with the whole system is such that

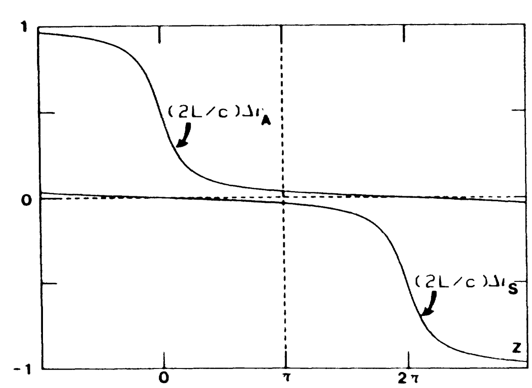

FIG. 11. Eigenfrequencies of a system of coupled cavities.

$$|S_{10}|=\frac{2T_r\sqrt{R_t}T_1^2R_2\frac{\nu_{\rm opt}}{\nu_g}\sin\left[\frac{\Omega L}{c}\right]^2}{|1+Re^{2i\omega L/c}|^2|1+Re^{2i(\omega-\Omega)L/c}|^2|-\sqrt{R_rR_t}e^{iz}F^2||1-\sqrt{R_rR_t}e^{iz}F_-^2|}\ .$$

We have used the following notation:

$$z=\frac{2\omega a}{c},\quad R=\sqrt{R_1R_2},\quad F=\frac{(1-p_1)\sqrt{R_2}e^{2i\omega L/c}+\sqrt{R_1}}{1+Re^{2i\omega L/c}},\quad F_-=\frac{(1-p_1)\sqrt{R_2}e^{2i(\omega-\Omega)L/c}+\sqrt{R_1}}{1+Re^{2i(\omega-\Omega)L/c}}\ .$$

Only one sideband can be made resonant at time. We shall confine our attention on S_{10}. The discussion for S_{20} is quite analogous. The NSNR is therefore $S(f)=|S_{10}|$.

Let Δf be the normalized detuning of the source with respect to an eigenfrequency of an isolated cavity: $\Delta f\equiv 2\pi(\nu_{\rm opt}-\nu_0)\tau''$, and let f be the normalized gravitational frequency: $f\equiv 2\pi\nu_g\tau''$. With this notation we obtain

$$S(f)=\frac{2T_r\sqrt{R_t}T_1^2R_2\frac{Q}{f}\sin^2\left[\frac{ft_m}{2}\right]}{|1-\sqrt{R_1R_2}e^{ift_m}|^2|1-\sqrt{R_1R_2}e^{i(\Delta f-f)t_m}|^2AB} \tag{20}$$

with

$$A=|1-\sqrt{R_rR_t}e^{iz}F^2|,\quad B=|1-\sqrt{R_rR_t}e^{iz}F_-^2|\ .$$

We assume the frequency of the source (carrier) to coincide with the antisymmetric eigenfrequency of the system:

$$\frac{z}{2}+\mathrm{Arg}F\equiv\pi\ \ (\mathrm{mod}2\pi)\ .$$

In order to find the peak value of $S(f)$ we assume further the lower sideband frequency to coincide with that of the symmetric eigenmode:

$$\frac{z}{2}+\mathrm{Arg}F_-\equiv 0\ \ (\mathrm{mod}2\pi)\ .$$

Let us see at which value of f the preceding coincidence takes place. We must have

$$\tan(\mathrm{Arg}F)=\tan(\mathrm{Arg}F_-)=-\tan(z/2)\ .$$

A general form of the phase of a detuned cavity was given in Sec. II B 2 yielding

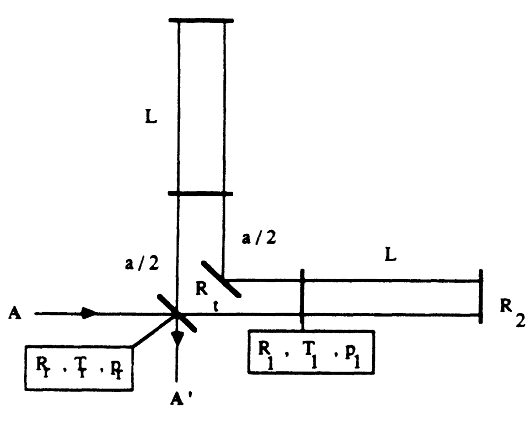

FIG. 12. Synchronous recycling with FP cavities: notation.

$$\mathrm{Arg}F=\pi+\arctan\left[\frac{2\Delta ft(1-t)}{1-2t-\Delta f^2t^2}\right],$$

$$\mathrm{Arg}F_-=\pi+\arctan\left[\frac{2(\Delta f-f)t(1-t)}{1-2t-(\Delta f-f)^2t^2}\right].$$

By solving in f the preceding equations we obtain the gravitational resonant frequency as a function of the carrier detuning:

$$f_0t=\frac{1-2t+\Delta f^2t^2}{\Delta ft}\ . \tag{21}$$

The requirement that Δf makes the carrier to coincide with the antisymmetric eigenmode is now

$$\frac{2\Delta ft(1-t)}{1-2t-\Delta f^2t^2}=-\tan\left[\frac{z}{2}\right]$$

the solution of which relates the antisymmetric detuning Δf_A to the tuning of the coupling cavity:

$$\Delta f_At=(1-t)\cot\left[\frac{z}{2}\right]+\left[(1-t)^2\cot^2\left[\frac{z}{2}\right]+1-2t\right]^{1/2}. \tag{22}$$

The same equation would give the symmetric detuning:

$$\Delta f_St=(1-t)\cot\left[\frac{z}{2}\right]-\left[(1-t)^2\cot^2\left[\frac{z}{2}\right]+1-2t\right]^{1/2}. \tag{23}$$

The resonant gravitational frequency can be related to z by

$$f_0t=(\Delta f_A-\Delta f_S)t$$
$$=2\left[(1-t)^2\cot^2\left[\frac{z}{2}\right]+1.-2t\right]^{1/2}.$$

The laser source can be properly tuned to coincide with the antisymmetric eigenfrequency, by locking it on the minimum of reflection of the ring cavity corresponding to the antisymmetric resonance. We intend now to optimize $S(f_0)$ when f_0 has its minimum value, namely, when $z\equiv\pi$; then

$$\Delta f_A t=\sqrt{1-2t}\ ,$$
$$\Delta f_S t=-\sqrt{1-2t}\ ,$$
$$f_0 t=2\sqrt{1-2t}\ .$$

In this same special case, the reflectivities of the cavities are the same for the carrier and the sideband and we have

$$|F_-^2|=|F^2|=1-2t\ .$$

If $f_0>>1$, we have $t\approx 2/f_0$ and consequently $\tau_S'\approx 1/\pi v_g^{(0)}$ as found in Sec. IV C 1.

Taking into account that $\Delta f_A,\Delta f_S,f_0$ are small compared with the free spectral interval of the cavities, we find an approximate form for the peak value of the NSNR:

$$S(f_0)=\tfrac{1}{2}Qf_0t^2\frac{T_r}{[1-\sqrt{R_rR_t}(1-2t)]^2}\ .$$

The optimal value of R_r is $\sqrt{R_r}=(1-p_r)\sqrt{R_t}(1-2t)$, so that the optimal peak value is finally

$$S(f_0)=\tfrac{1}{2}Qf_0t^2\frac{1}{1-(1-p_r)R_t(1-2t)^2} \tag{24}$$

(see Fig. 14). The relation giving f_0t as a function of t can be inverted giving

$$t=\frac{4}{f_0^2}(\sqrt{1+f_0^2/4}-1) \tag{25}$$

so that, if $f_0>>1$, the approximation $t\approx 2/f_0$ holds. Then

$$S(f_0)=\frac{Q}{4}\frac{1}{1+pf_0/8}\ ,$$

where p has the same definition and value as in Sec. IV B. If further f_0 is not too high ($f_0<<8/p$) we have an estimation of the optimal value of $S(f_0)$: $S_{\max}\approx Q/4$. In fact, $S(f_0)$ has a flat maximum of about that value in the range $1<<f_0<<8/p$ (50 Hz to 500 Hz in the reference antenna, and falls to zero when f_0 becomes either very small or very large.

Let us return now to the general case, when z denotes an arbitrary tuning of the coupling cavity, not near $z\equiv 0$ (mod 2π) however, and let us study the transfer function $S(f)$. It is possible to give a very simple approximate form of $S(f)$ when f is near f_0 and f_0 in the optimal range defined above: $1<<f_0<<8/p$. Let us set

$$\kappa=\frac{f-f_0}{f_0}<<1$$

neglecting second-order terms in t or κ, only the phase term in the quantity named B will change and with

$$e^{i(2\text{Arg}F_-+z)}=e^{-4i\kappa\tan(z/4)}$$

we obtain

$$S\approx\tfrac{1}{2}Qf_0t\frac{\sin^2(z/2)}{[1+2\sin^2(z/4)][1+2\cos^2(z/4)]}\times\frac{1}{\left|1+\dfrac{2i\kappa\cot(z/4)}{2t[1+2\sin^2(z/4)]}\right|}$$

which yields, owing to the relation $f_0t\approx 2/\sin(z/2)$,

$$S(f)=\frac{Q\,|\sin(z/2)|}{3+\sin^2(z/2)}\left\{1+\frac{4(f-f_0)^2\cos^4(z/4)}{[1+2\sin^2(z/4)]^2}\right\}^{-1/2}\ . \tag{26}$$

The gravitational bandwidth is, therefore,

$$\delta f=\sqrt{3}\frac{1+2\sin^2(z/4)}{\cos^2(z/4)}\quad (\text{FWHM})\ .$$

In the special case $z\equiv\pi$ we have simply

$$S(f)=\frac{Q}{4}\frac{1}{[1+(f-f_0)^2/4]^{1/2}},\quad \delta f=4\sqrt{3}\ . \tag{27}$$

The preceding form is quite similar to that found for delay lines. The synchronous recycling system using Perot-Fabry cavities is however continuously tunable in gravitational frequency by adjusting the optical path in the coupling cavity and the corresponding reflectance phase of the cavities (see Fig. 13) by tuning the frequency of the laser, instead of discretely (by changing n) in the case of delay lines. Figure 14 summarizes the results obtained for the sensitivities of the two types of synchronous recycling systems.

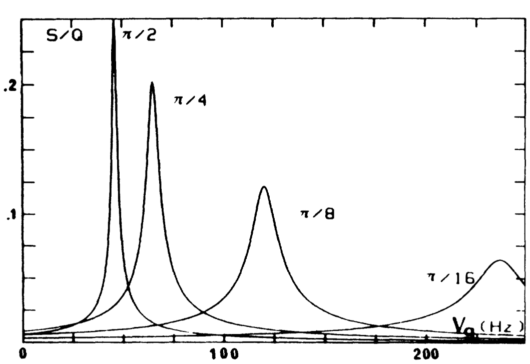

FIG. 13. Transfer functions of a synchronous recycling interferometer with FP cavities at various reflectivity phase ($\text{Arg}F=\pi/2$, $\pi/4$, $\pi/8$, $\pi/16$): tunability.

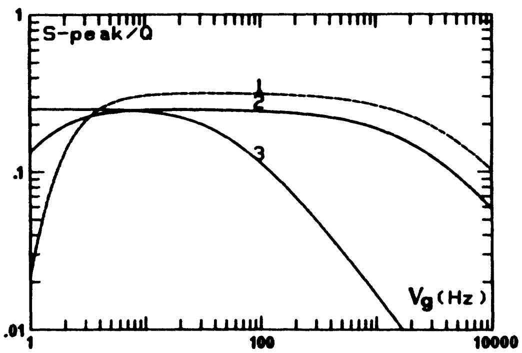

FIG. 14. Synchronous recycling interferometers: peak value of the NSNR: (1) case of delay line; (2) case of FP cavities; (3) standard recycling with detuned FP and optimal time constant (for comparison).

V. WIDEBAND AND NARROW-BAND ANTENNAS

The discussion of the different recycling schemes has shown that the gravitational frequency response of the corresponding interferometers can be deeply different from that of the initial nonrecycled transducers. As a first approximation we can distinguish between the cases of wideband responses, and the cases of resonant, or narrow-band responses. In all the following cases we will give numerical estimations of the shot-noise limited sensitivity based on the reference antenna and on an effective laser power ηP of 10 W at a wavelength of 0.5 μm, which yields

$$\left[\frac{\hbar\omega_{\text{opt}}}{\eta P}\right]^{1/2} \approx 2\times 10^{-10}\ \text{Hz}^{-1/2} .$$

A. Wideband antennas

Apart from the ordinary nonrecycling interferometers involving delay lines or FP cavities, the standard recycling scheme provides us three new wideband systems that are to be compared. Recall the essential features of each.

1. Michelson interferometer with delay lines and no recycling

Zero-frequency limit of the NSNR amplitude:

$$S(0)=Q/e .$$

Optimal storage time for $f_0 \gg 1$, corresponding optimal response and minimum detectable, photon noise limited h:

$$t_0=\frac{\pi}{f_0},\quad S(f)=\frac{2Q}{f}\left|\sin\left[\frac{\pi}{2}\frac{f}{f_0}\right]\right|,\quad S(f_0)=\frac{2Q}{f_0} ,$$

at $\nu_g^{(0)}=100$ Hz, we have $h_{\text{PN}}=1.7\times 10^{-23}\ \text{Hz}^{-1/2}$.

2. Michelson with FP cavities and no recycling

Zero-frequency limit of the NSNR amplitude: $S(0)=Q/2$.

Optimal time constant and optimized response:

$$t_0=\frac{2}{f_0} ,$$

$$S(f)=\frac{4Q}{f_0}\frac{1}{(1+4f^2/f_0^2)^{1/2}} ,$$

$$S(f_0)=\frac{4}{\sqrt{5}}\frac{Q}{f_0} ,$$

for $\nu_g^{(0)}=100$ Hz we obtain $h_{\text{PN}}=1.9\times 10^{-23}\ \text{Hz}^{-1/2}$.

3. Michelson with delay line and standard recycling

Zero-frequency limit of the NSNR amplitude: $S(0)=0.4Q$.

Near-optimal time constant, optimized response in the band $1 \ll f_0, pf_0 \ll 1$,

$$t_0=\frac{2.3}{f_0} ,$$

$$S(f)=0.92\frac{Q}{f}\sqrt{f_0}\left|\sin\left[\frac{1.17f}{f_0}\right]\right| ,$$

$$S(f_0)=0.85\frac{Q}{\sqrt{f_0}} ,$$

for $\nu_g^{(0)}=100$ Hz, we have $h_{\text{PN}}=3.5\times 10^{-24}\ \text{Hz}^{-1/2}$.

4. Michelson with resonant FP cavities and standard recycling

Zero-frequency limit of the NSNR: $S(0)=Q/2$.

Optimal time constant, optimized response in the band $f_0 \gg 1, pf_0 \ll 1$:

$$t_0=\frac{1}{f_0} ,$$

$$S(f)=\frac{Q}{\sqrt{f_0}}\frac{1}{(1+f^2/f_0^2)^{1/2}} ,$$

$$S(f_0)=\frac{Q}{\sqrt{2f_0}} ,$$

for $\nu_g^{(0)}=100$ Hz, we have $h_{\text{PN}}=4.2\times 10^{-24}\ \text{Hz}^{-1/2}$.

We can conclude that delay lines and Fabry-Perot systems are almost equivalent from this theoretical point of view, either in conventional or recycling antennas. Furthermore we note that standard recycling provides a gain of $0.4\sqrt{f_0}$ within the preceding range. When no recycling is applied, the optimal NSNR is proportional (for $f_0 \gg 1$) to Q/f_0, i.e., to $\nu_{\text{opt}}/\nu_g^{(0)}$ and thus, is independent of the interferometer arm length, provided that the suitable time constant is achieved: Whether it has been obtained by many reflections over a short distance or by few reflections over a long distance does not matter. On the other hand, when recycling is applied, we see that the NSNR becomes proportional to $Q/\sqrt{f_0}$ and that the interferometer size is now important. A larger size allows fewer reflections in achieving the optimal time constant thus lowers the reflectivity losses of the arms, which permits a higher power buildup in the system and finally a better SNR. If we now examine the very-low-frequency limit, all systems are limited by their upper bound on the possible storage times: this is why the zero-frequency limits for all wideband systems is a fraction of Q. In that very-low-frequency part of the gravitational spectrum, the photon noise limited sensitivity will improve linearly with the length of the detector.

B. Narrow-band antennas

Let us recall briefly the essential features of the three types of narrow-band receivers for GW that we have encountered up to now.

1. The standard recycling setup with detuned FP cavities

Zero-frequency limit of the peak value of the SNR: $S(0)=Q/4$.

Optimal time constant, optimized response in the band $1 \ll f_0, pf_0 < 1$:

$$t_0=\frac{2}{3+(1+2pf_0^2)^{1/2}} ,$$

$$S(f_0)=Q\left[\frac{2(1+\sqrt{1+2pf_0^2}\,)}{(3+\sqrt{1+2pf_0^2}\,)^3}\right]^{1/2} ,$$

$$S(f)=\frac{S(f_0)}{[1+(f-f_0)^2t_0^2]^{1/2}} .$$

Bandwidth: $\delta f=[3+(1+2pf_0^2)^{1/2}]\sqrt{3}$; for the reference antenna at $\nu_g^{(0)}=100$ Hz, we get $h_{\rm PN}=2.3\times10^{-24}$ Hz$^{-1/2}$, $\delta\nu_g=15.4$ Hz.

The features of this kind of recycling become identical to that of synchronous recycling when p tends to zero.

2. The synchronous recycling setup with delay lines

Zero-frequency limit of the peak value of the NSNR: $S(0)=0$.

Optimal storage time, optimized response in the band $f_0 \gg 1, pf_0 \ll 1$:

$$t_0=\frac{\pi}{f_0},\quad S(f)=\frac{Q}{\pi}\frac{1}{[1+\frac{1}{4}(f-f_0)^2]^{1/2}} .$$

Bandwidth: $\delta f=4\sqrt{3}$; for the reference antenna we have at $\nu_g^{(0)}=100$ Hz, we get $h_{\rm PN}=8.3\times10^{-25}$ Hz$^{-1/2}$, $\delta\nu_g=5.5$ Hz.

3. The synchronous recycling setup with FP cavities

Zero-frequency limit of the peak value of the NSNR: $S(0)=0$.

Optimal time constant, optimized response in the range $f_0 \gg 1$, $pf_0 \ll 1$ assuming an antiresonant coupling cavity ($z\equiv\pi$)

$$t_0=\frac{2}{f_0},\quad S(f)=\frac{Q}{4}\frac{1}{[1+\frac{1}{4}(f-f_0)^2]^{1/2}} .$$

Bandwidth: $\delta f=4\sqrt{3}$; for the reference antenna: at $\nu_g^{(0)}=100$ Hz, we get $h_{\rm PN}=10^{-24}$ Hz$^{-1/2}$, $\delta\nu_g=5.5$ Hz.

Delay lines or Fabry-Perot cavities in synchronous recycling systems are thus almost equivalent. We can say that the gain obtained at the peak value of the NSNR by synchronous recycling is roughly a factor of $0.15f_0$ with respect to no recycling, and a factor of $0.4\sqrt{f_0}$ with respect to standard recycling, when the optimal decay time is achieved, we also note that standard recycling with detuned Fabry-Perot cavities is characterized in the realistic part of the gravitational frequency spectrum by the same characteristics as the synchronous recycling. For shorter values of the decay time, a smaller peak value of the NSNR, but a larger bandwidth are obtained. Moreover, the product (peak value)$\times$(bandwidth) is larger in the standard recycling system with detuned cavities, which means that it should be especially interesting in the case of not purely monochromatic sources.

The scaling factor Q shows the importance of the interferometer arm length. The synchronous recycling system is very sensitive to intracavity losses: an increase of the apparatus size results in fewer reflections to reach the suitable time constant, therefore, in a higher finesse of the ring cavity, which increases the SNR.

VI. CONCLUSION AND PERSPECTIVE

We have presented here a unified formalism for the study of all the kinds of passive interferometers which have been proposed so far for the detection of gravitational waves. This allowed us to compare directly the relative shot-noise limited sensitivities of these interferometers. The important results are the following.

(i) The sensitivity gain brought by the use of recycling techniques varies with the gravitational frequency: for the reference antenna, in the frequency range between 50 and 500 Hz, it is roughly equal to the square root of this frequency (expressed in Hz) in the case of a wideband antenna (standard recycling), and to the frequency in the case of a narrow-band antenna (synchronous recycling). It lies between these two values in the intermediate case of detuned recycling.

(ii) The use of a recycling technique calls for very long arm lengths: the sensitivity is proportional to the length in the case of a narrow-band recycling system, and to the square root of the length, in the case of standard recycling.

(iii) Delay-line or Fabry-Perot gravito-optic transducers show essentially the same sensitivity in all cases, but the Fabry-Perot systems are much more versatile: while any modification of the transfer function of a delay-line system requires a major change of the apparatus (moving or changing mirrors), the response of a Fabry-Perot system can be adapted rapidly with just a slight change of the laser frequency or a micrometric movement of one mirror.

(iv) The new technique of standard recycling with detuned cavities gives the possibility of finding a compromise between bandwidth and peak sensitivity, which should prove to be very useful, specially at the time of the detection of the first signals, when the sensitivity of wideband systems will still be marginal.

(v) The smallest detectable gravitational-wave amplitude h_{pn} obtainable with a realistic laser ($\eta P = 10$ W) and the use of recycling techniques should guarantee the observation of a few events per year, since the present theoretical estimations for strong extragalactic sources in the local cluster give amplitudes h around 3×10^{-23}.

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