Chiral anomaly at finite temperature

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Based on a simplified derivation of the Abelian chiral anomaly, we prove the temperature independence of the anomaly in 3+1 dimensions.

INTRODUCTION

The chiral anomaly has been studied extensively by many authors.[1-4] In particular, the nonperturbative approach, with a regularization scheme for manipulating the divergence in the phase factor, arrives at a neat expression for the chiral anomaly in $2n$ dimensions.[4] However, it seems that such a method is not suitable for dealing with the finite-temperature case. Recently, the method of the derivative expansion has been used to provide the proof of the temperature independence of the anomaly in the Schwinger model.[5]

In this Brief Report, we try to propose a new method for studying the temperature dependence of the chiral anomaly in four dimensions. Every step carried out below is well defined; no ambiguous divergence occurs. The whole calculation seems quite elegant, but we have to confine ourselves to the Abelian case.

FOUR-DIMENSIONAL MODEL, $T=0$

In four-dimensional Euclidean space, the fermion part of the action is defined as

$$S_f' = \int d^4x\, \bar{\psi}(\not{p} - e\not{V})\psi \tag{1}$$

with V_μ being the Hermitian external Abelian vector field. The Dirac matrices γ_μ and γ_5 are all Hermitian such that

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}, \quad \mathrm{tr}\gamma_\mu\gamma_\nu\gamma_\alpha\gamma_\beta = 4\epsilon_{\mu\nu\alpha\beta} . \tag{2}$$

To consider the chiral anomaly of model (1), we add an axial vector which can be understood as a parameter arising from the chiral transformation of ψ, i.e.,

$$S_f = \int d^4x\, \bar{\psi}(\not{p} - e\not{V} - i\not{A}\gamma_5)\psi . \tag{3}$$

Then the fermion part of the generating functional reads

$$Z = \exp(-S_{\text{eff}}) = \int D\bar{\psi} D\psi\, e^{-S_f} . \tag{4}$$

Thus we have

$$\left.\frac{\delta S_{\text{eff}}}{\delta A_\mu}\right|_{A_\mu=0} = -\frac{1}{Z}\left.\frac{\delta Z}{\delta A_\mu}\right|_{A_\mu=0} = -i\langle j_{5\mu}\rangle , \tag{5}$$

$$j_{5\mu} = \bar{\psi}\gamma_\mu\gamma_5\psi . \tag{6}$$

Substituting (3) into (4) and performing the functional integration, one obtains

$$Z = \det(\not{p} - e\not{V} - i\not{A}\gamma_5) = \det{}^{1/2} H , \tag{7}$$

where

$$H = (\not{p} - e\not{V} - i\not{A}\gamma_5)^2 = H_0 + H_1 \tag{8a}$$

with

$$\begin{aligned} &H_0 = p^2, \quad H_1 = B - i(\not{D}\not{A} - \not{A}\not{D})\gamma_5 , \\ &B = \not{V}^2 + \not{A}^2 - (\not{p}\not{V} + \not{V}\not{p}), \quad \not{D} = \not{p} - e\not{V} . \end{aligned} \tag{8b}$$

By using the formula

$$\det H = \exp(\mathrm{Tr}\ln H) \tag{9}$$

and noting that

$$\ln A/B = -\int_0^\infty dt\, t^{-1}(e^{-At} - e^{-Bt}) \tag{10}$$

we will evaluate the renormalized $\ln H$ by the expression

$$(\ln H)_{\text{ren}} = -\int_0^\infty dt\, t^{-1} e^{-Ht} . \tag{11}$$

The reasonableness of this prescription lies in the fact that the divergence arising from the integral in (11) will be absorbed into the renormalized coupling constant e (see below).

Now let us expand $\mathrm{Tr}\exp(-Ht)$ as follows:[6]

$$\begin{aligned} \mathrm{Tr}\exp[-(H_0+H_1)t] = \mathrm{Tr}\Big[& e^{-H_0 t} + (-t)e^{-H_0 t}H_1 + \tfrac{1}{2}(-t)^2\int_0^1 du\, e^{-(1-u)H_0 t}H_1 e^{-uH_0 t}H_1 \\ & + \tfrac{1}{3}(-t)^3\int_0^1 du\, u\int_0^1 dv\, e^{-(1-u)H_0 t}H_1 e^{-u(1-v)H_0 t}H_1 e^{-uvH_0 t}H_1 + \cdots \Big] . \end{aligned} \tag{12}$$

For dealing with the chiral anomaly, the terms in the perturbative expansion (12) without the totally antisymmetric tensor $\epsilon_{\mu\nu\alpha\beta}$ called "normal-parity naive anomalies"[3] will be irrelevant. It is because these terms could be canceled that we add some counterterms which may be chosen appropriately while preserving the vector Ward identity. On the other hand, one cannot eliminate those terms including $\epsilon_{\mu\nu\alpha\beta}$ by whatever choice in counterterms while preserving the vector Ward identity.

Furthermore, as we shall set A_μ to zero at the final stage, only the terms linear in A_μ will survive in calculation. Thus one can drop all the uninteresting terms and write

$$\begin{aligned}
\mathrm{Tr}\exp(-Ht) &= \mathrm{Tr}\left[\tfrac{1}{2}(-t)^2\int_0^1 du\, e^{-(1-u)H_0 t}H_1 e^{-uH_0 t}H_1\right] \\
&= \mathrm{Tr}\left[\frac{t^2}{2}\int_0^1 du\, e^{-(1-u)H_0 t}[Be^{-uH_0 t}(-i)(\not{D}A - A\not{D})\gamma_5 - i(\not{D}A - A\not{D})\gamma_5 e^{-uH_0 t}B]\right] \\
&= -i\frac{t^2}{2}\int_0^1 du \frac{1}{(2\pi)^4}\int d^4p\, d^4q\, e^{-(1-u)p^2 t - uq^2 t}4\epsilon_{\mu\nu\alpha\beta} \\
&\qquad \times\{[ep_\mu V_\nu(p-q)+eV_\mu(p-q)q_\nu] \\
&\qquad\quad \times[q_\alpha A_\beta(q-p)-A_\alpha(q-p)p_\beta+e(A_\alpha V_\beta)(q-p)-e(V_\alpha A_\beta)(q-p)] \\
&\qquad\quad +[p_\mu A_\nu(p-q)-A_\mu(p-q)q_\nu+e(A_\mu V_\nu)(p-q) \\
&\qquad\qquad -e(V_\mu A_\nu)(p-q)][eq_\alpha V_\beta(q-p)+eV_\alpha(q-p)p_\beta]\} \\
&= -i\frac{4t^2}{(2\pi)^4}e^2\epsilon_{\mu\nu\alpha\beta}\int_0^1 du\int d^4p\, d^4q\, e^{-p^2 t - u(1-u)q^2 t} \\
&\qquad \times q_\mu V_\nu(q)[(A_\alpha V_\beta)(-q)-(V_\alpha A_\beta)(-q)]\,, \qquad (13)
\end{aligned}$$

with, e.g., $V_\nu(q)$ being the vector potential V_ν in momentum q representation, while $(A_\alpha V_\beta)(q)=\int A_\alpha(p)V_\beta(q-p)dp$. In the last step of (13) a change in variables $q\to q-p$, $(p-qu)\to p$ has been made. Substituting (13) into (4), (7), (9), and (11), and performing the integration with respect to t one finds

$$\begin{aligned}
S_{\mathrm{eff}} &= i\frac{e_R^2}{2(2\pi)^2}\epsilon_{\mu\nu\alpha\beta}\int d^4q\, q_\mu V_\nu(q)[(A_\alpha V_\beta)(-q)-(V_\alpha A_\beta)(-q)] \\
&= \frac{e_R^2}{2(2\pi)^2}\epsilon_{\mu\nu\alpha\beta}\int d^4x\, \partial_\mu V_\nu(x)[A_\alpha(x)V_\beta(x)-V_\alpha(x)A_\beta(x)]\,. \qquad (14)
\end{aligned}$$

Here, in accompanying the limiting procedure (11), we bring the bare coupling constant e^2 to its renormalized value

$$e_R^2 = \lim_{s\to 0}\Gamma(s)e^2\,.$$

Thus we get

$$\begin{aligned}
\langle\partial_\mu j_{5\mu}\rangle &= i\partial_\mu\left[\left.\frac{\delta S_{\mathrm{eff}}}{\delta A_\mu}\right|_{A_\mu=0}\right] \\
&= \frac{i}{4(2\pi)^2}e_R^2\epsilon_{\mu\nu\alpha\beta}F_{\mu\nu}(x)F_{\alpha\beta}(x) \qquad (15)
\end{aligned}$$

with $F_{\mu\nu}(x)=\partial_\mu V_\nu(x)-\partial_\nu V_\mu(x)$.

This is the well-known result of the Abelian chiral anomaly. Though the above method seems rather simple, it does work, so we proceed to discuss the chiral anomaly at a finite temperature.

FOUR-DIMENSIONAL MODEL $T\neq 0$

The standard (imaginary-time) method for manipulating the temperature field theory ascribes to the following substitutions:

$$\int\frac{d^4p}{(2\pi)^4}\to\frac{1}{\beta}\sum_{n=-\infty}^{\infty}\int\frac{d^3\mathbf{p}}{(2\pi)^3}\,, \qquad (16)$$

where $\beta=1/kT$, and the fermion energy

$$E_n=(2n+1)\pi/\beta\,. \qquad (17)$$

Therefore, one obtains the effective action at finite temperature T as

$$\begin{aligned}
S_{\mathrm{eff}}^T &= \frac{2ie}{(2\pi)^3}\epsilon_{\mu\nu\alpha\beta}\int_0^\infty dt\, t\int_0^1 du\int d^4q\left[\frac{1}{\beta}\sum_{n=-\infty}^{\infty}\int d^3\mathbf{p}\exp\left[-\frac{\pi^2}{\beta^2}(2n+1)^2 t-\mathbf{p}^2 t-u(1-u)q^2 t\right]\right] \\
&\qquad \times q_\mu V_\nu(q)[(A_\alpha V_\beta)(-q)-(V_\alpha A_\beta)(-q)]\,. \qquad (18)
\end{aligned}$$

For evaluating the summation over n, we resort to the Plana summation formula:[7]

$$\sum_{n=1}^{\infty} f(n)+\tfrac{1}{2}f(0)=\int_0^{\infty} f(x)dx + i\int_0^{\infty} \frac{f(it)-f(-it)}{e^{2\pi t}-1}dt \qquad (19)$$

under the condition of $f(z)$ being an analytic function on the complex plane $\mathrm{Re}z \geq 0$.

Now our function

$$f(z)=\exp\left[-\frac{\pi^2}{\beta^2}(2z+1)^2 t\right]$$

does satisfy the above condition and the contributions of positive and negative n to the second term in (19) just cancel each other, so we have

$$\sum_{n=-\infty}^{\infty} \exp\left[-\frac{\pi^2}{\beta^2}(2n+1)^2 t\right] = \int_{-\infty}^{\infty} \exp\left[-\frac{\pi^2}{\beta^2}(2x+1)^2 t\right]dx \ . \qquad (20)$$

This equality can be understood by inspecting the area under the continuous function $f(z)$ and that under the zigzag curve $f(n)$. Then integrating with respect to $\mathbf{p}$ next before accomplishing the integration with respect to t, we reach the simple result

$$S_{\mathrm{eff}}^{T}=S_{\mathrm{eff}}^{(T=0)} \ . \qquad (21)$$

In summary, in the case of the four-dimensional Abelian gauge field case for the model (1), we have proved that the chiral anomaly is temperature independent. This conclusion is in conformity with that in Ref. 8 where it is argued that the dynamical symmetry breaking will not be influenced by the temperature change.

Unfortunately, we failed to generalize the method to the non-Abelian case of chiral anomaly, so further investigation is needed.

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